ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 1, Winter 1988

GLOBAL UNIFORMLY EXPANDING FINITE AMPLITUDE WAVES

W.D. CURTIS

1. Introduction. If a solution of wave equation has the form

$$f(r,t) = F(\frac{r}{c_0 t})$$

we say f is a uniformly expanding wave, since for various values of t, the spatial profile is always the same except for a change of scale. The simple centered rarefaction wave of gas dynamics is of this type [**2**]. In [**4**] Taylor obtains such a solution to the problem of the gas flow outside a spherical surface expanding at a constant rate. In this article we consider globally defined, uniformly expanding wave solutions of certain finite-amplitude wave equations.

The propagation of finite amplitude waves in a channel of cross-sectional area $A = A(r)$ is governed by the partial differential equation [**3**]

$$u_t + (c_0 + \beta u)u_r + \frac{c_0 u A'(r)}{2A} = 0. \tag{1.1}$$

Here, u is particle velocity, r is range measured along the channel, and c_0 is the ambient sound speed. The constant β is the non-linearity parameter which, for γ-law gas has the value $\beta = \frac{\gamma+1}{2}$. If we let $f = \frac{\beta}{c_0}u$ and $\sigma = c_0 t$, the equation (1.1) becomes

$$f_\sigma + (t + f)f_r + \frac{fA'}{2A} = 0. \tag{1.2}$$

We are interested in discussing solutions of (1.2) of the form

$$f(r,\sigma) = F(x); x = r/\sigma = r/c_0 t. \tag{1.3}$$

If we substitute (1.3) into (1.2) we get

$$x^2 F'(x) - (1+F)xF'(x) - \frac{rFA'(r)}{2A} = 0 \tag{1.4}$$

Received by the editors on March 10, 1986

In order that the equation (1.4) hold we need

$$\frac{rF(x)A'(r)}{2A}$$

to be a function of x only. Unless F vanishes identically, we need

$$\frac{rA'(r)}{A(r)} = k = \text{ constant.}$$

Thus,

$$A'(r) = \frac{kA(r)}{r},$$

so that

$$A(r) = A(1)r^k. \tag{1.5}$$

We have therefore proved

THEOREM 1.1. *The equation* (1.1) *admits non-trivial uniformly expanding wave solutions only when the cross-sectional area is a power law, as in* (1.5)

If we let $\alpha = k/2$, then the equation satisfied by F becomes

$$x(x - 1 - F)F' - \alpha F = 0, \tag{1.6}$$

or

$$F' = \frac{\alpha F}{x(x - 1 - F)}. \tag{1.7}$$

We are interested in solutions of (1.6) or (1.7) on the interval $(0, \infty)$. For $\alpha > 0$, (1.7) has a critical point at $(x, F) = (1, 0)$. Thus if $x = 1$ and $F(x) = 0$, then $F'(x)$ is not determined by (1.7).

When we speak of a solution (or smooth solution) of (1.7) on $(0, \infty)$ we mean F is C^1 on $(0, \infty)$ and satisfies the ODE (1.6) at all $x \in (0, \infty)$. In terms of (1.7), F must satisfy (1.7) whenever $F(x) \neq x - 1$ and at any x where $F(x) = x - 1$ we must have $F(x) = 0$ so $x = 1$. We speak of a weak solution of (1.7) on $(0, \infty)$ if F is continuous on $(0, \infty)$, F satisfies (1.7) and is C^1 for $x \neq 1$, and $\lim_{x \to 1^-} F'(x)$ and $\lim_{x \to 1^+} F'(x)$ exist (but are, perhaps, unequal). If F is a solution of (1.7) on $(0, \infty)$, then

$u(r,t) = \frac{\beta}{c_0} F(r/c_0 t)$ is a global, uniformly expanding wave solution of (1.1). If F is a weak solution as described above, then u may have a weak discontinuity, that is, a jump in the partial derivatives u_r and u_t, along the characteristic $r = c_0 t$.

2. The plane wave ($\alpha = 0$). For $\alpha = 0$ we must refer to(1.6), which becomes

$$x(x - 1 - F)F' = 0. \tag{2.1}$$

THEOREM 2.1. *There are only two smooth solutions of* (2.1) *namely*

(a) $F(x) = c_1$, *c constant*

(b) $F(x) = x - 1$.

PROOF. Clearly (a) and (b) are solutions. Suppose $F : (0, \infty) \to R$ is any solution of (2.1). Then, for each $x \in (0, \infty)$, either

$$F(x) = x - 1 \text{ or } F'(x) = 0.$$

Suppose there is an x where $F(x) \neq x - 1$. We show that F must be constant. Consider

$$A = \{x | x > 0 \text{ and } F(x) \neq x - 1\}.$$

Then A is a non-empty open set. Let (a_1, b_1) be a component of A. Suppose $a_1 > 0, b_1 < \infty$. Then $a_1, b_1 \notin A$ so $F(a_1) = a_1 - 1, F(b_1) = b_1 - 1$. But then $F(b_1) - F(a_1) = b_1 - a_1$ is zero by the Mean value Theorem, because if $a_1 < c < b_1$, then $c \in A$ so $F'(c) = 0$. Thus A has no component with both ends in $(0, \infty)$. So any component is of the forms $(0, b_1), (a_1, \infty)$ or $(0, \infty)$. In the case when $(0, \infty)$ is a component, we are done. Suppose there is a component $(0, b_1)$ with $b_1 < \infty$. If there is no other component, then F is constant on $(0, b_1)$ while $F(x) = x - 1$ for $b_1 \leq x$. Then $F'(b_1)$ does not exist. If another component exists, it is of the form (a_1, ∞) with $a_1 \geq b_1$. If $a_1 = b_1$, then F is constant. If $a_1 > b_1$, then $F(x) = x - 1$ for $b_1 \leq x \leq a - 1$ and F' does not exist at a_1 or b_1. The remaining cases are similar. This completes the proof.

REMARK. The two solution obtained above are expected. They are the constant amplitude plane wave solutions and the centered simple wave solution

$$f(r,t) = \frac{r}{c_0 t} - 1.$$

Note that if we relax the requirement of smoothness we get more solutions. For $0 < a_1 < b_1 < \infty$ there is a weak solution of (1.7) whose derivative jumps at a_1 and b_1 as shown in fig. (2.1) below. The resulting solution of (1.1) has weak discontinuities along the characteristics $r = a_1 c_0 t, r = b_1 c_0 t$.

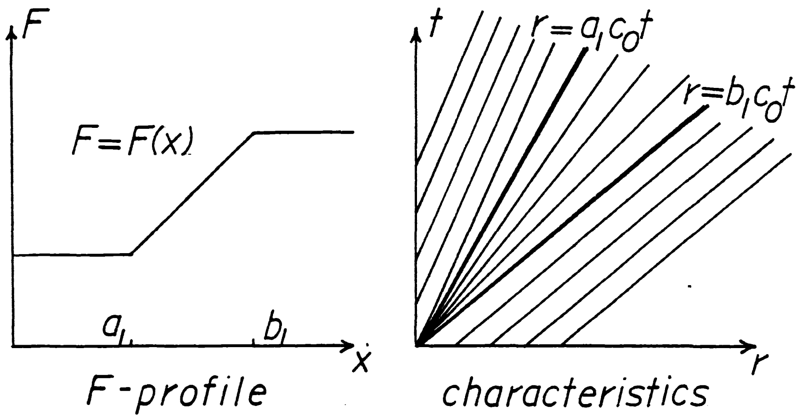

FIGURE 2.1

3. The Cases $0 < \alpha \leq 1$**.** This includes the two important cases of cylindrical waves ($\alpha = \frac{1}{2}$) and spherical waves ($\alpha = 1$). We prove that for $0 < \alpha \leq 1$, smooth solutions exist only in the case $\alpha = 1$, that is, for spherical waves. If we allow weak discontinuities we find solutions for $0 < \alpha < 1$.

Suppose F is a solution of (1.7) on an interval (a, b) and F' is never 0 on (a, b); then F maps (a, b) monotonically onto an interval (c, d). Let $x = x(F), x : (c, d) \to (a, b)$, be the inverse of F and define $v : (c, d) \to R$ by $v(F) = 1/x(F)$. Then v is a solution of the linear first-order differential equation,

$$\frac{dv}{dF} - \left(\frac{1+F}{\alpha F}\right) = -\frac{1}{\alpha F}. \tag{3.1}$$

We shall frequently be in the above situation and shall use the notations just described. We remark that the change of variable $v = x^{-1}$ is just the standard transformation of a Bernoulli equation to a linear equation; the ODE satisfied by $x = x(F)$ is

$$\frac{dx}{dF} = \frac{x(x-1-F)}{\alpha F},$$

which is of Bernoulli type.

We shall make use of the fact that a solution of a linear equation like (3.1) can have singularities only at a singular point of the differential equation. Thus, for example, if $0 \le a < b$ and v is a solution of (3.1) on (a, b), then if we know $\lim_{F\to b^-} v(F) = \infty$, we can conclude $b = \infty$. Similarly, if $\lim_{F\to a^+} v(F) = \infty$, then $a = 0$.

Recall that a solution of (1.7) on an interval (a, b) is required to be differentiable at all points of (a, b). On the other hand, by a weak solution F of (1.7) on (a, b) we mean that if $1 \in (a, b)$ we require F be continuous at $x = 1$ and have continuous left and right derivatives at $x = 1$, but the left and right derivatives at $x = 1$ may be different.

LEMMA 3.1. *Let F be a weak solution of* (1.7) *on* $(0, \infty)$. *Then* $F(1) = 0$.

PROOF. We prove $F(x_1) = x_1 - 1$ for some $x_1 \in (0, \infty)$. If so then, since F is differentiable for $x \ne 1$, we see from (1.7) that we must have $x_1 = 1$. But then $F(1) = 0$.

Suppose $F(x) > x - 1$ for all x. Then by (1.7) we conclude $F'(x) < 0$ for $x > 1$. Then clearly $F(x)$ can't be greater than $x - 1$ for all $x > 1$.

Suppose $F(x) < x - 1$ for all $x \in (0, \infty)$. Then $F(x) < 0$ for $0 < x < 1$. But whenever $F(x) < 0$ and $F(x) < x - 1$ we have $F'(x) < 0$, so we conclude $F(x) < 0$ for all $x \in (0, \infty)$. Thus, F maps $(0, \infty)$ monotonically onto an interval (ρ, σ) where $-\infty \le \rho < \sigma \le 0$. Then we get

$$v : (\rho, \sigma) \to (0, \infty),$$

and we see $\lim_{F\to\sigma^-} v(F) = \infty$. Hence $\sigma = 0$. But then $\lim_{x\to 0^+} F(x) = 0$, contradicting $F(x) < x - 1$ for all x. This proves the lemma.

The following result will be useful for the further analysis to be given.

LEMMA 3.2. *Let $0 < \alpha \le 1$ and let v be a solution of* (3.1) *on either* $(-\infty, 0)$ *or* $(0, \infty)$. *Then*

(a) $\lim_{F\to 0} v(F) = 1$.

(b) *If* $0 < \alpha < 1$, *then* $\lim_{F\to 0} v'(F) = \frac{1}{a-1}$.

(c) *If* $\alpha = 1$, *then* $\lim_{F\to 0} v'(F) = -\infty$.

PROOF. (a) Consider the case where v is defined on $(0,\infty)$. Any solution of (3.1) is of the form

$$v(F) = F^{\frac{1}{\alpha}} e^{F/\alpha}\left[A - \frac{1}{\alpha}\int_1^F \frac{1}{z}\left(\frac{1}{ze^z}\right)^{1/\alpha} dz\right] \tag{3.2}$$

for some constant A. Rewrite (3.2) as

$$v(F) = \frac{[A - \frac{1}{\alpha}\int_1^F \frac{1}{z}\left(\frac{1}{ze^z}\right)^{1/\alpha} dz]}{\frac{e^{-F/\alpha}}{F^{1/\alpha}}}$$

and apply l'Hospital's rule to get (a).

(b) This time we give the proof for the case when v is defined on $(0,\infty)$. Let $\xi = -F$. Then (3.1) becomes

$$\frac{dv}{d\xi} - \left(\frac{1-\xi}{\alpha\xi}\right)v = -\frac{1}{\alpha\xi}, 0 < \xi < \infty. \tag{3.3}$$

Then $v(\xi)$ is of the form

$$v(\xi) = \xi^{\frac{1}{\alpha}} e^{-\frac{\xi}{\alpha}}\left[A - \frac{1}{\alpha}\int_{\xi 0}^{\xi} z^{-\frac{1}{\alpha}-1} e^{z/\alpha} dz\right].$$

We have

$$v'(\xi) =$$

$$\frac{(1-\xi)v(\xi) - 1}{\alpha\xi} = \frac{1}{\alpha\xi}\left\{(1-\xi)\xi^{\frac{1}{\alpha}} e^{-\xi/\alpha}\left[A - \frac{1}{\alpha}\int_{\xi 0}^{\xi} z^{-\frac{1}{\alpha}-1} e^{z/\alpha} dz\right] - 1\right\}.$$

By l'Hospital's rule, the desired limit is obtained by differentiating the expression in braces, letting $\xi \to 0$, and then dividing the result by α.

The derivative of the expression in braces, which we denote by N is (3.4)

$$N = \frac{\xi - 1}{\alpha\xi} + \frac{\xi^{\frac{1}{\alpha}-1} e^{-\xi/\alpha}}{\alpha}(1 - (\alpha+2)\xi + \xi^2)\left[A - \frac{1}{\alpha}\int_{\xi 0}^{\xi} z^{-\frac{1}{\alpha}-1} e^{z/\alpha} dz\right]$$

Now we represent the expression in brackets in a convenient asymptotic form (for small ξ). In fact, it is easy to see that (since $0 < \alpha < 1$),

$$A - \frac{1}{\alpha}\int_{\xi 0}^{\xi} z^{-\frac{1}{\alpha}-1} e^{z/\alpha} dz = \xi^{-\frac{1}{\alpha}} - \frac{1}{\alpha(\alpha-1)}\xi^{-\frac{1}{\alpha}+1} + \phi(\xi), \tag{3.5}$$

where $\phi(\xi)\xi^{\frac{1}{\alpha}-1} \to 0$ as $\xi \to 0^+$.

Substitute (3.5) into (3.4) and replace $e^{-\xi/\alpha}$ in (3.4) by

$$1 - \frac{\xi}{\alpha} + \psi(\xi), \text{ where } \psi(\xi) = 0(\xi^2).$$

We then get

$$\begin{aligned} N &= \frac{\xi - 1}{\alpha\xi} + \\ &\frac{\xi^{\frac{1}{\alpha}-1}}{\alpha}(1 - \frac{\xi}{\alpha} + \psi(\xi))(1 - (\alpha+2)\xi + \xi^2)(\xi^{-\frac{1}{\alpha}} - \frac{1}{\alpha(\alpha-1)}\xi^{-\frac{1}{\alpha}+1} + \phi(\xi)) \\ &= \frac{\xi - 1}{\alpha\xi} + \\ &\frac{1}{\alpha\xi}(1 - \frac{\xi}{\alpha} + \psi(\xi))(1 - (\alpha+2)\xi + \xi^2)(1 - \frac{\xi}{\alpha(\alpha-1)} + \xi^{1/\alpha}\phi(\xi)). \end{aligned}$$

If we multiply this out and make use of the properties of ϕ and ψ we get

$$\lim_{\xi \to 0^+} N = \frac{\alpha}{1-\alpha}.$$

Thus

$$\lim_{\xi \to 0^+} v'(\xi) = \frac{1}{1-\alpha}.$$

We then conclude, since $dv/dF = -dv/d\xi$,

$$\lim_{F \to 0^-} v'(F) = \frac{1}{\alpha - 1}.$$

A similar calculation can be made for a solution of (3.1) on $(0, \infty)$.

(c) The argument in this case is similar to that given in (b). If we consider v on $(0, \infty)$ we get

$$v'(F) = (1 + F)e^F\left[A - \int_1^F \frac{e^{-z}}{z^2} dz\right] - \frac{1}{F}.$$

An asymptotic analysis gives

$$v'(F) = \ln F + 0(1) \text{ as } F \to 0^+.$$

Therefore (c) holds. This completes the proof.

LEMMA 3.3. *Let $0 < \alpha \le 1$ and let F be a solution of* (1.7) *on* $(0,1)$ *such that* $\lim_{x\to 1^-} F(x) = 0$. *Then* $F(x) = 0$ *for all* $x \in (0,1)$.

PROOF. If $F(x) = 0$ for some x_1, the uniqueness of solutions requires F identically zero. Suppose $F < 0$ on $(0,1)$. Now we can never have $F(x) = x - 1$. If $F(x) < x - 1$ on $(0,1)$, then from (1.7) we see $F' < 0$ on $(0,1)$ so we cannot have $\lim_{x\to 1^-} F(x) = 0$. Suppose

$$x - 1 < F(x) < 0 \quad \text{for all } x \in (0,1).$$

Then $F' > 0$ on $(0,1)$ so F maps $(0,1)$ onto some interval (a,b) with $-1 \le a < b < 0$. Then the function $v = v(F)$ maps (a,b) onto $(1,\infty)$ and $v' < 0$ on (a,b). Since $\lim_{F\to a^+} v(F) = +\infty$, we conclude $a = -\infty$. But $a \ge -1$, so we have a contradiction.

Suppose $F(x) > 0$ for all $x \in (0,1)$. Then, from (1.7), $F' < 0$ on $(0,1)$. Since $\lim_{x\to 0^-} F(x) = 0$, F must map $(0,1)$ onto some interval $(0,A)$. As before, consider the inverse function

$$x : (0,A) \to (0,1)$$

and its reciprocal,

$$v : (0,A) \to (1,\infty).$$

By Lemma (3.2), we have $\lim_{F\to 0^+} v'(F) < 0$. Thus $F > 0$ and, near $0, v'(F) < 0$. But $x'(F) = -v'(F)/(v(F))^2$ and so $x'(F) > 0$ for F near 0. Then $F'(x) = 1/x'(F(x)) > 0$ for x just to the left of 1. But we observed above that in the present case $F' < 0$ on $(0,1)$ so we have a contradiction. This proves the lemma.

Combining Lemma (3.1) and Lemma (3.3) we get

THEOREM 3.4. *Let $0 < \alpha \le 1$ and let $F : (0,\infty) \to R$ be a (weak) solution of* (1.7). *Then* $F(x) = 0$ *for* $0 < x \le 1$.

LEMMA 3.5. *Let $0 < \alpha \le 1$ and let F be a solution of* (1.7) *on*$(1,\infty)$ *such that* $\lim_{x\to 1^+} F(x) = 0$ *and such that* $F(x) \le 0$ *for some* $x \in (1,\infty)$. *Then* $F(x) = 0$ *for all* $x \in (1,\infty)$.

PROOF. If $F(x) \leq 0$ for some $x \in (1,\infty)$ but F is not identically zero, then F is never 0. Thus, $F < 0$ on $(1,\infty)$. Then $F' < 0$ on $(1,\infty)$, and so F maps $(1,\infty)$ onto an interval $(\lambda, 0)$, where $\lambda < 0$. Let $v : (\lambda, 0) \to (0,1)$ be the reciprocal of the inverse of F. Then, by Lemma (3.2)

$$\lim_{F\to 0^-} v'(F) < 0.$$

Then $\lim_{x\to 1^+} F'(x) > 0$, which is impossible since $\lim_{x\to 1^+} F(x) = 0$ and $F(x) < 0$ for all $x > 1$. This proves the lemma.

We now see that, for $0 < \alpha \leq 1$, any non-trivial weak solution must satisfy $0 < F(x) < x - 1$ on $(1,\infty)$. For any such F, we have $F' > 0$ on $(1,\infty)$. Then F maps $(1,\infty)$ onto some interval $(0, A)$, where $0 < A \leq \infty$. Then, as usual, we have the function $v : (0, A) \to (0,1)$, with $v' < 0$ on $(0, A)$. Now if $\alpha = 1$, then $\lim_{F\to 0^+} v'(F) = -\infty$ from which it follows that $\lim_{x\to 1^+} F'(x) = 0$. For $0 < \alpha < 1$ we get $\lim_{F\to 0^+} v'(F) = \frac{1}{\alpha - 1}$ and hence $\lim_{x\to 0^+} F'(x) = 1 - \alpha$.

This latter fact proves

THEOREM 3.6. *For $0 < \alpha < 1$ there is no smooth solution of (1.7) on $(0,\infty)$.*

However, we have

THEOREM (3.7). *Let $0 < \alpha$. Given A with $0 < A \leq \infty$, there is a unique F defined on $(1,\infty)$ satisfying (1.7) and such that $\lim_{x\to\infty} F(x) = A$.*

PROOF. By the remarks in the preceeding paragraph, we need to show that, given A, there is a unique $v : (0, A) \to (0,1)$ which satisfies (3.1) and such that

(a) $v' < 0$ on $(0, A)$

(b) $\lim_{F\to A^-} v(F) = 0$.

Consider first the case $A = \infty$. Now any solution of (3.1) on $(0,\infty)$ is of the form

$$v(F) = F^{1/\alpha} e^{F/\alpha} \Big[\frac{v_0}{e^{\frac{1}{\alpha}}} - \frac{1}{\alpha} \int_1^F \frac{1}{z} \big(\frac{1}{ze^z}\big)^{1/\alpha} dz \Big]. \tag{3.6}$$

Let

$$v^* = \frac{e^{1/\alpha}}{\alpha} \int_1^\infty \frac{1}{z}(\frac{1}{ze^z})^{1/\alpha} dz.$$

We claim the solution given in (3.6) satisfies $\lim_{F\to\infty} v(F) = 0$ if and only if $v_0 = v^*$. Indeed, if $v_0 > v^*$ then $v(F) \to \infty$ as $F \to \infty$. If $v_0 < v^*$, then $v(F) = 0$ at some point and is negative thereafter. Since $v'(F) < 0$ if $v < 1/(1+F)$, we see in this case we cannot have $v(F) \to 0$ as $F \to \infty$. If $v_0 = v^*$, then l'Hospital's rule gives $\lim_{F\to\infty} v(F) = 0$.

For $v_0 = v^*$ we must have $v' < 0$ on $(0,\infty)$ since if $v'(F) \geq 0$ for some F then $v(F) \geq 1/(1+F)$. Then v would be increasing for large F and v would not approach 0 as $F \to \infty$.

For $0 < A < \infty$, let v be the solution of (3.1) on $(0,\infty)$ satisfying the initial condition $v(A) = 0$. Consider v on $(0, A)$. Then v is clearly the only function satisfying the desired conditions. This completes the proof.

The following result summarizes what we have proved.

THEOREM (3.8). *For $\alpha = 1$ there is a family of smooth solutions of (1.7) on $(0,\infty)$. Given A with $0 \leq A \leq \infty$, there is a unique solution F of (1.7) on $(0,\infty)$ such that $\lim_{x\to\infty} F(x) = A$. These are the only (weak) solutions for $\alpha = 1$. For $0 < \alpha < 1$ there are no smooth solutions. There is a family of weak solutions as follows: Given $A, 0 < A \leq \infty$, there is a unique weak solution F on $(0,\infty)$ with $\lim_{x\to\infty} F(x) = A$. Each of these has a jump in the derivative at $x = 1$ in that $\lim_{x\to 1^-} F'(x) = 0, \lim_{x\to 1^+} F'(x) = 1-\alpha$. There are no other non-trivial weak solutions.*

4. The case $\alpha > 1$. Note that Lemma (3.1) holds for $\alpha > 1$. Thus, we must determine solutions of (1.7) on $(0,1)$ and $(1,\infty)$ which approach 0 as $x \to 1^-$ and $x \to 1^+$, respectively. For the case of $(0,1)$ we note that the proof of Lemma (3.3) shows any F on $(0,1)$ which satisfies $\lim_{x\to 1^-} F(x) = 0$ must either be identically 0 or everywhere positive. In case $\alpha > 1$, however, there are solutions which are positive on $(0,1)$.

Consider a solution F of (1.7) on $(0,1)$ with $F(x) \to 0$ as $x \to 1^-$. Then $F' < 0$ on $(0,1)$ so F maps $(0,1)$ onto $(0,A)$ where $0 < A \leq \infty$. Then $v : (0,A) \to (1,\infty)$ is an increasing solution of (3.1) so that we must have $A = \infty$. Then we conclude.

$$\lim_{x \to 0^+} F(x) = +\infty.$$

In order to discuss behavior of $F'(x)$ as $x \to 1^+$ or 1^- we consider the nature of the critical point $(0, 1)$ of the autonomous system in (F, v)-space,

$$\frac{dF}{dt} = \alpha F, \quad \frac{dv}{dt} = (1 + F)v - 1. \tag{4.1}$$

If we let $w = v - 1$ to translate the critical point to the origin, the linearized system is

$$\frac{d}{dt}\begin{bmatrix} F \\ w \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} F \\ w \end{bmatrix}.$$

The eigenvalues and eigenvectors of this matrix are:

$$\lambda = 1, \text{ eigenvector } \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \lambda = \alpha, \text{ eigenvector } \begin{bmatrix} \alpha - 1 \\ 1 \end{bmatrix}.$$

We now apply Theorem (5.1) of [**1**, p. 384]. For $\alpha > 1$ it follows that any solution of (4.1) which approaches $(0, 1)$ as $t \to -\infty$ does so with a limiting direction tangent either to the line $F = 0$ or the line $v - 1 = F/(\alpha - 1)$. There will be exactly two trajectories approaching tangent to $v - 1 = F/(\alpha - 1)$, one in each direction. There will be infinitely many trajectories approaching $(0, 1)$ in each of the two directions tangent to $F = 0$. It follows that there are solutions of (3.1)

$$v : (0, \infty) \to (1, \infty)$$

and that of these, exactly one satisfies

$$\lim_{F \to 0^+} v'(F) = \frac{1}{\alpha - 1}$$

while all others satisfies

$$\lim_{F \to 0^+} v'(F) = \infty.$$

Therefore, there are solutions F of (1.7) on $(0,1)$ with $F' < 0$ on $(0,1)$ and

$$\lim_{x\to 1^-} F(x) = 0, \quad \lim_{x\to 0^+} F(x) = \infty.$$

One of these satisfies $\lim_{x\to 1^-} F(x) = 1 - \alpha$, while the others satisfy $\lim_{x\to 1^+} F'(x) = 0$.

Now consider solutions F of (1.7) on $(1,\infty)$ with $\lim_{x\to 1^+} F(x) = 0$. By Theorem (3.7) there is a family of such solutions; given A with $0 \le A \le \infty$, there is a unique such F with $\lim_{x\to\infty} F(x) = A$.

Thus, there are global smooth solutions for any $\alpha > 1$ and they need not vanish on $(0,1)$.

5. Conclusion. We have calculated all global uniformly expanding wave solutions of the finite amplitude wave equation for cross sectional area

$$A(r) = A_1 r^k$$

for $0 \le k \le 2$. For $0 < k < 2$, all have a weak discontinuity along the characteristic $r = c_0 t$ and all vanish for $r \le c_0 t$.

For $k = 2$ (spherical case), everywhere smooth solutions exist, but these too vanish in the region $r < c_0 t$.

For $k > 2$ smooth solutions exist and they may or may not vanish for $r < c_0 t$.

References

1. Coddington, E.A. and Levinson, N., *Theory of Ordinary Differential Equations,* McGraw-Hill, New York, 1955.

2. Courant,R. and Friedrichs, K.O., *Supersonic Flow and Shock Waves,* Springer-Verlag, New York, 1976.

3. Pelinovskiy, Y., Petukhov, Y.V. and Fridman, V.Y., *Approximate Equations for the Propagation of Strong Acoustic Signals in the Ocean,* Izvestia, Atmospheric and Oceanic Physics **15** (1979), 299-304.

4. Taylor, G.I., *The Air Wave Surrounding an Expanding Sphere,* Proc. Roy. Soc. A**186** (1946), 273-292.

Department of Mathematics, Kansas State University, Manhattan, KS 66506