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ON DUALITY IN RATIONAL APPROXIMATION

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1. Introduction. Let K be a compact Hausdorff space and $C(K)$ be the space of real-valued continuous functions on K. For a given pair of subspaces $G, H \subset C(K)$, let $R(G,H) = \{g/h : g \in G, h \in H$ and $h(t) > 0$ for all $t \in K\}$. In [**3**] we gave a necessary and sufficient condition for a function $f \in C(K)$ to belong to $R(G,G)$. The characterization was given in terms of the measures orthogonal to G.

In this paper we generalize this result in three different ways:

1) We consider the case when $G \neq H$.

2) We give a formula for the distance between a function $f \in C(K)$ and $R(G,H)$.

3) For a given sequence of continuous functions $f_1, \ldots, f_n$ we study the existence of functions $r_i \in R(G_i, H)$ such that $r_i = g_i/h$ with $g_i \in G_i$, $h \in H$ and $||f_i - r_i|| < \varepsilon$.

The last problem can be called a simultaneous rational approximation with common denominator. This problem turns out to be relevant to multi-variable rational approximation.

The second part of this paper is dedicated to various applications to cases when the spaces G_i and H are spanned by algebraic or trigonometric polynomials with gaps. In particular, some generalizations of the results of J. Bak and D.J. Newman [**1**] and G. Somorjai [**5**] are given.

2. The main theorem. Let $G_1, \ldots, G_n, H \subset C(K)$ be subspaces of $C(K)$. Let $\overline{f} = (f_1, \ldots, f_n)$ be an n-tuple of functions from $C(K)$. Consider the set

$$R(G_1, \ldots, G_n; H) = \left\{\left(\frac{g_1}{h}, \ldots, \frac{g_n}{h}\right) : g_i \in G_i, h \in H, h(t) > 0 \, \forall \, t \in K\right\}.$$

Clearly $R(G_1, \ldots, G_n; H) \subset \times_{i=1}^n C(K); R(G_i, H) = \emptyset$ if and only if H does not contain strictly positive functions. For $\overline{f} = (f_1, \ldots, f_n)$ we introduce a notion of distances to $R(G_i, H)$ by

$$d(\overline{f}, R(G_1,\dots,G_n;H)) =$$
$$\inf\left\{\max\left\{\left\|f_i - \frac{g_i}{h}\right\| : i=1,\dots,n\right\} : \left(\frac{g_i}{h},\dots,\frac{g_n}{h}\right) \in R(G_1,\dots,G_n;H)\right\}.$$

We also use the standard identification of the dual space to $C(K)$ with the space $\mathcal{M}(K)$ of regular Borel measures on K.

THEOREM 1. *Let $\overline{f}, R(G_i,H)$ be as before and $R(G_i,H) \neq \emptyset$. Then*

$$\begin{aligned} &d(\overline{f}, R(G_1\dots,G_n;H)) \\ &= \sup\{d(\overline{f}, R(G_1',\dots,G_n';H')) : G_i', H' \supset H \end{aligned}$$

and $\dim(C(K)/G_i')$

$$\begin{aligned} &= \dim(C(K)/H') = 1\} \\ &= \sup\left\{\varepsilon : \sum \tilde{f}_i\mu_i + \nu \geq 0\right. \\ &\text{for } \mu_i \perp G_i, \nu\perp H, \sum \|\mu_i\| + \|\nu\| \neq 0, \\ &\qquad \left. \|\tilde{f}_i - f_i\| < \varepsilon\right\} \\ &= \sup\left\{\varepsilon : \sum f_i\mu_i + \nu \geq \varepsilon \sum |\mu_i|,\right. \\ &\qquad \left. \mu_i\perp G_i, \nu\perp H, \sum \|\mu_i\| + \|\nu\| \neq 0\right\}. \end{aligned}$$

PROOF. It is convenient to denote the above quantities as d_1, d_2, d_3, and d_4 respectively. Clearly $d_1 \geq d_2$.

We next show that $d_3 \geq d_1$. Indeed choose $d < d_1$. Consider the set

$$W(\overline{f},\varepsilon) = \left\{(g_1,\cdots,g_n,h) \in \times_{i=1}^{n+1} C(K), h(t) > 0 \right.$$
$$\left. \forall t \in K : \left\|fi - \frac{g_i}{h}\right\| < \varepsilon\right\} \subset \times_{i=1}^{n+1} C(K).$$

Then our assumption is equivalent to

$$W(f,d) \cap G_1 \times G_2 \times \cdots \times G_n \times H = \emptyset.$$

It is easy to observe that W is an open convex set in $\times_{i=1}^{n+1} C(K)$, hence, by the Hahn-Banach theorem there exists a functional φ on $[\times_{i=1}^{n+1} C(K)]$ such that

$$(1) \qquad \varphi \perp G_1 \times G_2 \times \cdots \times G_n \times H \text{ and } \varphi(g_1, \ldots, g_n, h) > 0$$

for every $(g_1, \cdots, g_n, h) \in W(\overline{f}, d)$. Since every φ is of the form $\varphi = (\mu_1, \cdots, \mu_n, \nu) \in \times_{i=1}^{n+1} \mathcal{M}(K)$, it follows from (1) that there exist $\mu_i \perp G_i, \nu \perp H$ so that $\sum \mu_i(g_i) + \nu(h) \geq 0$ as soon as $||f_i - \frac{g_i}{h}|| < d$. Let $||\tilde{f}_i - f_i|| < d$. Then, for any strictly positive h, choose $g_i = \tilde{f}_i h$. We have, for any positive h,

$$\sum \mu_i(\tilde{f}_i h) + \nu(h) \geq 0$$

or equivalently $(\sum \tilde{f}_i \mu + \nu) \geq 0$. To summarize, we have shown that, for every $d < d_1, d_3 \geq d$. Hence $d_3 \geq d_1$.

Next we show $d_4 \geq d_3$. Suppose that $\sum \tilde{f}_i \mu_i + \nu \geq 0$. Then, for any sequence of functions $F_i \in C(K)$ with $||F_i|| \leq 1$, we have

$$0 \leq \sum \tilde{f}_i \mu_i + \nu = \nu + \sum (f_i - \varepsilon F_i) \mu_i = \nu + \sum f_i \mu_i - \varepsilon \sum F_i \mu_i.$$

Hence $\nu + \sum f_i \mu_i \geq \varepsilon \sum F_i \mu_i$. Taking the supremum over all choices of F_i, we have $\nu + \sum f_i \mu_i \geq \varepsilon \sum |\mu_i|$.

Finally $d_4 \leq d_2$. Let $d < d_4$ and let $\mu_i \perp G_i, \nu \perp H$ so that $\sum f_i \mu_i + \nu > d \sum |\mu_i|$. Then, reversing the previous step, we see that $\sum f_i \mu + \nu \geq d \sum F_i \mu_i$ for any sequence of F_i with $||F_i|| \leq 1$. Hence

$$(2) \qquad \sum \tilde{f}_i \mu_i + \nu > 0 \text{ for all } \tilde{f}_i ||f_i - \tilde{f}i|| < d.$$

Choose $G_i' = \ker \mu_i, H' = \ker \nu$. We want to show that there is no $g_i' \in G_i', h \in H'$ so that $h > 0$ and $||fi - \frac{g_i'}{h}|| < d$ for all $i = 1, \ldots, n$. Indeed if there were, we would have $g_i' = \tilde{f}_i h$ and $(\sum \tilde{f} i \mu_i + \nu)(h) = 0$. That contradicts (2). □

3. Examples and applications. We start with some applications of the technique described above to the density of rational functions with respect to Müntz polynomials.

Let $\Lambda = (\lambda_j)$ be an ordered infinite sequence of positive real numbers. Let

$$E = \operatorname{span}\{1, t^\lambda\}_{\lambda \in \Lambda} \subset C_{[0,1]}.$$

It was proved (cf. [**1**, **3-5**]) that, for every $\varepsilon > 0$ and every $f \in C_{[0,1]}$, there exist $q, h \in E, h > 0$ so that $||f - \frac{g}{h}|| < \varepsilon$. Here we consider the problem of simultaneous approximation with a common denominator. We will need the result proved in [**3**].

PROPOSITION 1. *Let* $E = \operatorname{span}\{1, t^\lambda\}_{\lambda \in \Lambda}$. *Let* $\mu \perp E, \mu \neq 0$. *Then*

a) *if* $\lambda_j \to \infty$, *then* $\sup \operatorname{supp} \mu^+ \cap \operatorname{supp} \mu^-$;

b) *if* $\lambda_j \to 0$, *then* $0 \in \operatorname{supp} \mu^+ \cap \operatorname{supp} \mu^-$.

THEOREM 2. *Given* $f_1, \ldots, f_n \in C([0,1])$ *and* $\varepsilon > 0$, *there exist* $g_1, \ldots, g_n, h \in E$ *with* $h(t) > 0$ *for all* $t \in [0,1]$ *so that*

$$\left|\left| fi - \frac{g_i}{h} \right|\right| < \varepsilon \quad \textit{for } i = 1, \ldots, n.$$

PROOF. By Theorem 1 it is enough to show that, for any set of measures $\mu_1, \cdots, \mu_n, \nu \perp E$ and for any $\varepsilon > 0$, we can find $\tilde{f}_i$ with $||\tilde{f}_i - f|| < \varepsilon$ and $\sum \tilde{f}_i \mu_i + \nu$ is not a positive measure. We first consider the case when $\lambda_j \to 0$. Then there exist $\alpha > 0$ and $\tilde{f}_i \in C_{[0,1]}$ so that $||\tilde{f}_i - f_i|| < \varepsilon$ and

$$\tilde{f}_i|_{[0,\alpha]} = \beta_i = \text{constant}.$$

Then $(\sum \tilde{f}_i \mu_i + \nu)|_{[0,\alpha]} = (\sum \beta_i \mu_i + \nu)|_{[0,\alpha]}$. Since $(\sum \beta_i \mu_i + \nu) \perp E$ we have from Proposition 1 that $(\sum \beta_i \mu_i + \nu)|_{[0,\alpha]}$ is neither a positive nor negative measure. Hence $(\sum \tilde{f}_i \mu_i + \nu)$ cannot be positive.

Similarly, if $\lambda_j \to \infty$, we choose $\tilde{f}_i$ so that $||\tilde{f}_i - f_i|| < \varepsilon$ and $\tilde{f}_i$ is a constant near the right hand side end point of the $\operatorname{supp} \mu_i$. Then, again near the $\sup(\operatorname{supp}(\sum \tilde{f}_i \mu_i + \nu))$, the measure $\sum \tilde{f}_i \mu_i + \nu$ coincides with the linear combination of measures $\mu_i \nu$ and thus by Proposition 1, the measure $\sum \tilde{f}_i \mu_i + \nu$ cannot be a positive measure. $\square$

COROLLARY. *Given an infinite sequence* λ *of positive real numbers, let*

$$E = \operatorname{span}\{1, t_1^{\lambda_1}, t_2^{\lambda_2}, \cdots, t_n^{\lambda_n}\}_{(\lambda_1, \ldots, \lambda_n) \in \times_{i=1}^n \Lambda} \subset C([0,1]^n).$$

Then, for every $f \in C([0,1]^n)$ *and* $\varepsilon > 0$, *there exist* $g, h \in E$ *with* $h(t_1, \cdots, t_n) > 0$ *for all* $(t_1, \ldots, t_n) \in [0,1]^n$ *such that* $\|f - \frac{g}{h}\| < \varepsilon$.

PROOF. We first approximate $f(t_1, \cdots, t_n)$ by the tensor product of functions of one variable

$$f(t_1, \ldots, t_n) = \sum_{j=1}^{m} \prod_{i=1}^{n} f_{ij}(t_i).$$

We next approximate the functions $\{f_j(t_i)\}$ by the expressions of the form $g_j(t_i)$ over $h(t)$ where $g_i(t_i), h(t_i) \in \operatorname{span}\{1, t_i^{\lambda}\}_{\lambda \in \Lambda}$. Since we can accomplish this process with the fixed denominator for all j, we then can add the rational approximation to $\prod f_{ij}(t_i)$ together and still remain in the class E, since only the numerators were added and the denominator remains the same.

That Corollary leads to the following interesting problem. Let $E \subset C(K)$ be a subspace. Let

$$R(E) = \left\{ \frac{g}{h} : g, h \in E, h(t) > 0 \forall t \in K \right\}.$$

PROBLEM 1. Suppose that $R(E)$ is dense in $C(K)$. Does it imply that $R(E \otimes E)$ is dense in $C(K \times K)$?

While we do not know the answer to this problem (we suspect it is negative), we can construct a subspace $E \subset C(K)$ so that $R(E)$ is dense in $C(K)$, yet there are functions $f_1, f_2 \in C(K)$ that do not have a simultaneous approximation with a common denominator from $R(E)$: Let $K = [0, 2\pi]$; let $E \subset C(K)$ given by

$$E = \{f \in C(K) : \int_0^{2\pi} f(x) \sin x dx = \int_0^{2\pi} f(x) \cos x dx = 0\}.$$

Then it was shown in **[3]** that $R(E)$ is dense in $C(K)$. However if we choose $f_1 = \sin x, f_2 = \cos x,\ \mu_1 = \sin x dx, \mu_2 = \cos x dx$ and $\nu = 0$, we have

$$f_1\mu_1 + f_2\mu_2 + \nu = (\sin^2 x + \cos^2 x) dx = dx \geq \frac{1}{\sqrt{2}} |\sin x + \cos x| dx.$$

Hence, by Theorem 1, if we choose $\overline{f} = (f_1, f_2), G_1 = G_2 = H = E$ we have

$$d(\overline{f}, G_i, H) \geq \frac{1}{\sqrt{2}}.$$

The next problem deals with the approximation by means of rationals of the form $\frac{g}{h}$ where g and h are chosen from different subspaces. Let $\Lambda = (\lambda_j)$ and $\Lambda' = (\lambda'_j)$ be two different sequences of positive real numbers; define

$$G = \operatorname{span}\{1, t^\lambda\}_{\lambda \in \Lambda}, \qquad H = \operatorname{span}\{1, t^\lambda\}_{\lambda \in \Lambda'} \subset C_{[0,1]}.$$

PROBLEM 2. Are functions of the form $\frac{g}{h}, g \in G, h \in H$ dense in $C_{[0,1]}$?

We will give two results in this direction here that can be viewed as an extension of the theorems proved in [**2**] and [**3**].

THEOREM 3. *Let G, H be a pair of subspaces of $C(K)$ such that $R(G,H)$ is dense in $C(K)$. Then every measure $\mu \perp G$ has the property that*

$$\operatorname{supp} \mu^+ \cap \operatorname{supp} \mu^- \neq \emptyset.$$

PROOF. Suppose that $\operatorname{supp} \mu^+ \cap \operatorname{supp} \mu^- = \emptyset$. Then there exists a function $f \in C(K)$ such that

$$f|\operatorname{supp}_{\mu^+} \equiv 1; \qquad f|\operatorname{supp}_{\mu^-} \equiv -1.$$

Choose $\nu \perp H$ to be identically zero. Then $f\mu + \nu = f\mu = |\mu|$ which by Theorem 1 implies that $f \notin \overline{R(G,H)}$. □

THEOREM 4. *Let (λ_i) be an infinite sequence of positive reals with $\lim \lambda_j = 0$. Let $H = \operatorname{span}\{1, t^{\lambda_j}\} \subset C_{[0,1]}$. Let G be a subspace of $C_{[0,1]}$ such that $\mu \perp G$ implies $\operatorname{supp} \mu^+ \cap \operatorname{supp} \mu^- \neq \emptyset$. Then every $f \in C_{[0,1]}$ with the property $f(0) = 0$ can be uniformly approximated by elements of $R(G,H)$.*

PROOF. Given $\varepsilon > 0$ and $f \in C_{[0,1]}$ with $f(0) = 0$, choose $\alpha > 0$ so that there exists $\tilde{f} \in C_{[0,1]}$ with $||f - \tilde{f}|| < \varepsilon$ and $\tilde{f}|_{[0,\alpha]} \equiv 0$. Choose

$\mu \perp G$ and $\nu \perp H$. If $\nu \neq 0$ we have $\tilde{f}\mu + \nu|_{[0,\alpha]}$. By Proposition 1 we know that $\nu|_{[0,\alpha]}$ cannot be a nonnegative measure.

Suppose now that $\nu = 0$. Then we have to prove that the inequality $f\mu \geq \varepsilon|\mu|$ cannot hold for any $\varepsilon > 0$. If it does, then $(\frac{f}{\varepsilon})\mu \geq |\mu|$ and hence the functional μ attains its norm on the function$(-1) \vee (\frac{f}{\varepsilon} \wedge 1)$ which implies that $\operatorname{supp}\mu^+ \cap \operatorname{supp}\mu^- = \emptyset$, a contradiction.

Examples of subspaces G satisfying the conditions of Theorem 4 are given in [**2**] and [**3**].

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