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IDENTITY-PRESERVING EMBEDDINGS OF COUNTABLE RINGS INTO 2-GENERATOR RINGS

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ABSTRACT. A technique is presented for embedding countable rings with identity into 2-generator rings with identity so that the embedding respects the identity elements and the centers. As applications we provide a number of examples of finitely generated rings with interesting pathology.

1. Introduction. A common theme in the study of algebraic structures is the embedding of a given structure into a less complicated one. In this note we consider the problem of embedding countable rings into 2-generator rings so that the identity element is preserved. Embeddings not preserving the identity have been constructed by several authors. Each of the papers [**1**, **4** and **5**] presents a method for embedding countable rings into 2-generator rings, but none of the methods respects the identity. We will use a modification of the ideas in [**5**] to solve the more difficult identity-preserving problem. Our embedding has the added advantage of respecting the centers. The payoff is a variety of interesting consequences, some known, but others which we were unable to find in the literature. For example:

(1) Embedding $\mathbf{Q}$ into a 2-generator ring A provides an example of a countable-dimensional $\mathbf{Q}$-algebra A which cannot be decomposed as $A \cong \mathbf{Q} \otimes_{\mathbf{Z}} R$, for R a ring which is free as a $\mathbf{Z}$-module.

(2) A slight modification of the embedding technique permits the construction of a finitely generated primitive ring R with non-zero socle such that eRe is not finitely generated for some primitive idempotent e, and R^*, the group of units of R, is not finitely generated. Thus, although "being finitely generated" is a Morita invariant, associated structures do not, in general, inherit this "finitely generated" property.

(3) Any countable commutative ring can be made the center of a 2-generator ring.

(4) There exists a 2-generator simple ring of characteristic zero.

It is perhaps worth noting that a group-theoretic analogue of (4) is the existence of a finitely generated infinite simple group. Such a group was constructed by Higman in [**2**].

Our techniques also apply to embedding countably-generated algebras over a field F into 2-generator F-algebras. In this setting we can duplicate some of the results of [**3**].

We employ the following conventions and notation. All rings R contain an identity element 1_R and all ring embeddings $\theta : R \to S$ preserve the identity, that is, $\theta(1_R) = 1_S$. For any subset $X \subseteq R$, $\langle X \rangle$ denotes the subring of R generated by X (1_R need not be in $\langle X \rangle$). An *n-generator* ring R is one for which there exists a subset $X \subseteq R$ with $|X| = n$ and $R = \langle X \rangle$. The group of units of R is denoted by R^*.

We shall frequently use the following simple observation: a countable ring R is not finitely generated if and only if there exists an ascending chain $R_1 \subseteq R_2 \subseteq \cdots$ of proper subrings of R with $\cup R_n = R$.

If α is a countably-infinite ordinal, then $M_\alpha(R)$ denotes the ring of all $\aleph_0 \times \aleph_0$ column-finite matrices over R with the rows and columns ordered according to α. In particular, $M_\omega(R)$ is the usual ring of $\aleph_0 \times \aleph_0$ column-finite matrices over R, whereas $M_{\omega^2}(R)$ is the ring of $\aleph_0 \times \aleph_0$ column-finite matrices containing $\aleph_0 \times \aleph_0$ blocks, where each block is an element of $M_\omega(R)$. Notice that if V is the free right R-module on $\aleph_0$ generators, then $M_\alpha(R)$ is simply the matrix representation of $\mathrm{End}_R(V)$ with respect to an ordered basis for V of order type α. For a positive integer n, $M_n(R)$ is the usual ring of $n \times n$ matrices over R.

2. The main result.

We start with a lemma which reduces the problem of embedding into 2-generator rings to that of embedding into finitely generated rings.

LEMMA. *If R is an n-generator ring, then $M_{n+2}(R)$ is a 2-generator ring.*

PROOF. Let R be generated by $r_1, \ldots, r_n$. Let $S = M_{n+2}(R)$ and define $a, b \in S$ by

$$a = \begin{pmatrix} 0 & r_1 & r_2 & \cdots & r_n & 0 \\ 1 & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \end{pmatrix},$$

where all non-displayed entries are zero. Note that $b^{-1} = b^{n+1}$, and that

$$(b^{-1}a)^2 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix} = e \in \langle a, b \rangle \subseteq S,$$

whence

$$ab^k e = \begin{pmatrix} r_k & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix} = r_k e \in \langle a, b \rangle \quad \text{for } 1 \leq k \leq n.$$

It follows that $re \in \langle a, b \rangle$ for each $r \in R$. Then $b^{i-1} r e b^{-(j-1)}$ is the matrix with r in the ij position and 0 elsewhere, $1 \leq i,\ j \leq n$. Therefore $\langle a, b \rangle = M_{n+2}(R)$. □

REMARK. Let $\varphi : R \to M_{n+2}(R)$ be given by

$$\varphi(r) = \begin{pmatrix} r & & & \\ & r & & \\ & & \ddots & \\ & & & r \end{pmatrix}.$$

Then φ is an identity-preserving ring monomorphism which maps the center of R onto the center of $M_{n+2}(R)$. □

We now present the main result.

THEOREM. *Any countable (respectively, finite) ring with identity can be embedded in a 2-generator (respectively, finite 2-generator) ring with identity, the embedding preserving the identity and respecting the centers.*

PROOF. Let R be a countable ring with identity, and let $r_1, r_2, \ldots$ be a listing of the elements of R. We produce a sequence of three identity-preserving ring embeddings, the composition of which embeds R in a 5-generator ring whose center contains the center of R. The result then follows from the Lemma and subsequent remark.

Denote $S = M_\omega(R)$ and define $\varphi_1 : R \to eSe$ by $\varphi_1(r) = r \cdot e$, where

$$e = e_{11} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}$$

is the matrix unit of S with 1_R in the first row, first column and 0's elsewhere. Let

$$a = \begin{pmatrix} r_1 & r_2 & \cdots \\ & & \\ & & \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & & & & \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \end{pmatrix}$$

in S. Note that

$$(*) \qquad ab^k e = r_{k+1} e = \varphi_1(r_{k+1}), \qquad \text{for } 0 \le k < \infty.$$

In particular, $\varphi_1(R) \subseteq \langle a, b, e\rangle \subseteq S$.

A deficiency of our container S is that the top corner eSe (the codomain of the embedding φ_1) is too small relative to the bottom corner $(1-e)S(1-e)$. The idea of our next step is to embed S into a ring T so that the corresponding top and bottom corners are isomorphic.

Let $T = M_{\omega^2}(R)$ and define $\varphi : S \to T$ by

$$\varphi(s) = \begin{pmatrix} s & & & & & & \\ & s & & & & & \\ & & s & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{pmatrix}.$$

Denote

$$f = \varphi(e) = \left(\begin{array}{c|c|c} \begin{matrix}1&&&\\&0&&\\&&\ddots&\\&&&\ddots\end{matrix} & & \\ \hline & \begin{matrix}1&&&\\&0&&\\&&\ddots&\\&&&\ddots\end{matrix} & \\ \hline & & \ddots \end{array}\right)$$

and let $\varphi_2 = \varphi|_{eSe} : eSe \to fTf$. Then $\varphi_2\varphi_1(R) \subseteq \langle \varphi(a), \varphi(b), \varphi(e) = f\rangle \subseteq T$. Let

$$f_{11} = \left(\begin{array}{c|c|c} \begin{matrix}1&&&\\&0&&\\&&\ddots&\\&&&\ddots\end{matrix} & & \\ \hline & \begin{matrix}0&&&\\&0&&\\&&\ddots&\\&&&\ddots\end{matrix} & \\ \hline & & \ddots \end{array}\right),$$

a matrix unit of T. Then there are T-module isomorphisms $Tf \cong \prod_{i=1}^{\infty} Tf_{11} \cong T(1-f)$, since each of these modules is isomorphic to a countable product of columns of T. (Note that we could not have claimed that $Tf \cong T(1-f)$ if we had taken $T = M_\omega(S)$.) It follows that there exist $v \in fT(1-f)$ and $w \in (1-f)Tf$ such that $vw = f$ and $wv = 1-f$. Define $\varphi_3 : fTf \to T$ by $\varphi_3(x) = x + wxv$. Then φ_3 is an identity-preserving ring embedding. Moreover, $\theta = \varphi_3\varphi_2\varphi_1 : R \to T$ is an identity-preserving ring embedding with $\theta(R) \subseteq \langle \varphi(a), \varphi(b), f, v, w\rangle$. It is easy to check that, for $r \in$ center R,

$$\theta(r) = rf + w(rf)v = rf + w(r1)v = rf + (r1)(wv) = rf + r(1-f)$$

$$= r \cdot 1 = \begin{pmatrix} r &&& \\ & r && \\ && \ddots & \\ &&& \ddots \end{pmatrix},$$

so that θ maps the center of R onto the center of T. This completes the proof. □

3. Consequences.

COROLLARY 1. *For any countable field F, there exists a simple F-algebra A which is generated as a ring by two elements. In particular, there exists a simple, finitely generated ring of characteristic zero.*

PROOF. By the Theorem, there exists a 2-generator ring S containing F in its center. Choose a maximal ideal M of S and let $A = S/M$. Since $F \cap M = 0$, we have that F embeds in the center of A, whence A is the desired algebra. □

REMARK. Corollary 1 supplies, for any countable field F, examples of simple finitely generated F-algebras A whose centers have infinite transcendency degree over F. For we may take any countable field extension K of F of infinite transcendency degree, and then embed K in the center of a finitely generated simple ring A. For a more definitive result on this topic, the reader should consult [**3**, Theorem 1]. □

If A is a finite-dimensional $\mathbf{Q}$-algebra it is well-known (and elementary) that $A \cong \mathbf{Q} \otimes_{\mathbf{Z}} R$, where R is a subring of A such that $(R, +)$ is a free abelian group. However, the Theorem provides an example to show that this decomposition does not, in general, extend to the infinite dimensional case (equivalently, infinite dimensional $\mathbf{Q}$-algebras need not have a basis relative to which the structure constants are integers).

COROLLARY 2. *There exists a countable dimensional $\mathbf{Q}$-algebra A which is not of the form $\mathbf{Q} \otimes_{\mathbf{Z}} R$, where R is a ring with $(R, +)$ a free abelian group.*

PROOF. Let $S_1 \subseteq S_2 \cdots \subseteq S_n \subseteq \cdots$ be a chain of proper subrings of $\mathbf{Q}$ with $\mathbf{Q} = \cup S_n$ and let A be a $\mathbf{Q}$-algebra. Note that if A is of the form $\mathbf{Q} \otimes R$, with R as above, then $A = \cup A_n$, where each $A_n = S_n \otimes R$ is a proper subring of A, hence A cannot be finitely generated as a ring.

By the Theorem we can produce a ring embedding $\theta : \mathbf{Q} \to A$, where A is a finitely generated ring. Since θ preserves the identity, A is a $\mathbf{Q}$-algebra in the natural way. By our earlier remarks, A cannot be of the form $\mathbf{Q} \otimes R$. □

The following proposition is presumably well-known, but for the sake of completeness we include its proof. Unlike some other authors, we use the expression "finite ring" to describe a ring containing only a finite number of elements. We shall use Proposition 1 to show (Corollary 3) that the Theorem fails in the setting of commutative rings — even the ring $\mathbf{Q}$ of rational numbers cannot be embedded in a finitely generated commutative ring.

PROPOSITION 1. *Let A be a finite-dimensional algebra over a field F. If A is finitely generated as a ring, then A must be finite.*

PROOF. We first show that if F is a field which is not finitely generated as a ring, and $A \neq 0$ is a finite-dimensional algebra over F, then A is not finitely generated (as a ring). Clearly we may assume F is countable. Then $F = \cup S_n$ for some chain $S_1 \subseteq S_2 \subseteq \cdots$ of finitely generated proper subrings of F. Let $B = \{b_1, \dots, b_m\}$ be an F-basis for A. Then there exist $c_{ijk} \in F$ (the structure constants) such that

$$b_i b_j = \sum_{k=1}^{m} c_{ijk} b_k.$$

For each n, let

$$T_n = \langle S_n, \text{ all the } c_{ijk} \rangle \subseteq F.$$

Then the T_n form a chain, $F = \cup T_n$, and since F is not finitely generated, $T_n \neq F$ for all n. Now let

$$A_n = \langle T_n, B \rangle \subseteq A \quad \text{for } n = 1, 2, \dots .$$

Since all the c_{ijk} lie in T_n, we have $A_n = \sum_{i=1}^{m} T_n b_i \neq A$ for all n. Thus the A_n form a chain of proper subrings of A, and their union is A, which shows A is not finitely generated. It follows that if $A \neq 0$ is finitely generated, then F must be finitely generated.

To complete the proof, it will suffice to show that an infinite field F cannot be finitely generated as a ring. Let P be the prime subfield of F. We prove by induction on r that an infinite field of the form $F = P(a_1, \dots, a_r)$ is not finitely generated (as a ring). When $r = 0$, $F = P = \mathbf{Q}$ is not finitely generated. Suppose $r > 0$. Let $K = P(a_1, \dots, a_{r-1})$. If a_r is transcendental over K, then $K[a_r]$ has an infinite number of primes. Let

$$R_n = \langle K, p_1, \dots, p_n, 1/p_1, \dots, 1/p_n \rangle, \quad n = 1, 2, \dots,$$

where $p_1, p_2, \dots$ is an enumeration of the primes of $K[a_r]$. Then $R_1 \subseteq R_2 \subseteq \dots$ with $F = \cup R_n$ and, since $K[a_r]$ is a unique factorization domain, $R_n \neq F$ for all n. Thus F is not finitely generated in this case. On the other hand, if a_r is algebraic over K, then K is an infinite field and hence not finitely generated by induction. But now F is a finite-dimensional K-algebra, whence F is not finitely generated by the result established in the first half of the proof. □

COROLLARY 3. *A non-zero commutative algebra A over an infinite field F cannot be embedded in a finitely generated commutative ring.*

PROOF. Suppose A can be embedded in a finitely generated commutative ring R. Choose a maximal ideal M of R. Then R/M is a finitely generated field, hence finite by Proposition 1. This contradicts the fact that F can be embedded in R/M. □

A construction similar to that employed in the proof of the Theorem provides the following example.

EXAMPLE. There exists a primitive, finitely generated ring R with non-zero socle such that:

(1) For some primitive idempotent $e \in R$, eRe is not a finitely generated ring, and

(2) R^*, the group of units of R, is not a finitely generated group.

PROOF. We construct R as a finitely generated subring of $M_\omega(\mathbf{Q})$. Let $\{r_i \mid 1 \leq i < \infty\}$ be an enumeration of $\mathbf{Q}$ and, as in the proof of the Theorem, let

$$a = \begin{pmatrix} r_1 & r_2 & \cdots & r_n & \cdots \\ & & & & \\ & & & & \end{pmatrix}, \quad b = \begin{pmatrix} 0 & & & \\ 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}, \quad e = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}.$$

Let

$$c = \begin{pmatrix} 0 & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & & \ddots \end{pmatrix}$$

and define $R = \langle e, a, b, c\rangle$. Note $1 = e + bc \in R$.

As in the proof of Theorem, $\mathbf{Q}e \subseteq R$, whence $eRe \cong \mathbf{Q}$ is not finitely generated. It is easy to check that, for any $n \times n$ rational matrix A, the matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is an element of R rational vector space of countable dimension, the natural action of R on V makes V a faithful simple R-module. Hence R is primitive, and plainly has non-zero socle.

Let $N = \{r \in R \mid \operatorname{rank} r < \infty\} = \operatorname{socle} R$ and let $\overline{R} = R/N$. Note that $a, e \in N$ and that $\bar{c} = (\bar{b})^{-1}$, where $\bar{r} = r + N$ for any $r \in R$. It follows that $\overline{R} = \mathbf{Z}[\bar{b}, \bar{b}^{-1}]$, with $\bar{b}$-transcendental over $\mathbf{Z}$; in fact $\overline{R}$ is just the group ring over $\mathbf{Z}$ of the infinite cyclic group generated by $\bar{b}$. A simple computation shows that the only units of $\overline{R}$ are $\pm(\bar{b})^j$, $j \in \mathbf{Z}$.

Let $r \in R^*$. Then $\bar{r} \in (\overline{R})^*$, so r is of the form $\pm\, b^j + x$ or $\pm\, c^j + x$ for some $j \geq 0, x \in N$. First we show that the possibility $r = b^j + x, j > 0$, cannot occur. To see this, note that any such r will have the form

$$r = \begin{pmatrix} U & W \\ 0 & I \end{pmatrix}.$$

Here U is a $k \times l$ rational matrix, W is a $k \times \omega$ rational matrix, I is the $\omega \times \omega$ identity matrix, and $k = l + j$ is a positive integer chosen such that any row below the k-th consists of all 0's except for one 1 contributed by the matrix b^j. (Without loss of generality, choose k such that $l > 0$.) To see that r is not a unit we construct a non-zero $s \in M_\omega(\mathbf{Q})$ such that $sr = 0$. The matrix s will be of the form

$$s = \begin{pmatrix} s_1 & s_2 & \cdots \\ & & \\ & & \end{pmatrix}.$$

Let $U_1, \ldots, U_l$ be the column vectors of U and, since $l < k$, choose $(s_1, \ldots, s_k) \neq (0, \ldots, 0)$ such that $(s_1, \ldots, s_k) \cdot U_i = 0, \quad 1 \leq i \leq l$. If $\{W_i \mid k+1 \leq i < \infty\}$ is the set of column vectors of W, let, for $i > k$, $s_i = -(s_1, \ldots, s_k) \cdot W_i$. Then it is easy to check that $sr = 0$.

Thus, if r is a unit of R, $r = \pm c^j + x$, where $j \geq 0, \quad x \in N$. We show that, in this case also, we cannot have $j > 0$. This follows since any $r = c^j + x$ with $j > 0$ can be put into matrix form $r = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ where A is a $t \times t$ matrix for some positive integer t such that the last row of A is zero. Let $s = \begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix}$ where $A' \neq 0$ is a $t \times t$ right annihilator of A. Then $0 \neq s \in M_\omega(\mathbf{Q})$ and $rs = 0$, again contradicting the fact that r is a unit. Thus a unit must have the form $x \pm 1$ for some $x \in N$. It follows that $R^* = \cup R_t$ where, for $t \geq 1$, R_t is the set of all matrices in $M_\omega(\mathbf{Q})$ of the form $\begin{pmatrix} A & B \\ 0 & \pm I \end{pmatrix}$ such that $A \in \mathrm{GL}(t, \mathbf{Q})$ and B is any $t \times \omega$ rational matrix for which $\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right) \in R$. Inasmuch as the R_t's form a chain of proper subgroups of R^*, we infer that R^* is not finitely generated. □

Our final result, on centers, utilizes much of the machinery developed to this point.

PROPOSITION 2. *The center of a 2-generator ring can be an arbitrary countable commutative ring.*

PROOF. Let R be a countable commutative ring and let $T = M_\omega(R)$. By the proof of the Theorem (note $T \cong M_{\omega^2}(R)$), there exists a finitely generated subring $B \subseteq T$ with

$$B \supseteq \text{ center } T = \left\{ \begin{pmatrix} r & & & \\ & r & & \\ & & r & \\ & & & \ddots \end{pmatrix} : r \in R \right\}.$$

Employing the construction of the first and second paragraphs of the Example, we can produce a finitely generated subring C of T which contains the standard matrix units of T. Let $A = \langle B \cup C \rangle \subseteq T$. Then center $A =$ center $T \cong R$. In view of the Lemma, we can replace A by the appropriate $M_{n+2}(A)$, to obtain the desired 2-generator ring with center exactly R. □

REMARK. Proposition 2 provides a quick proof of the fact that there are exactly $2^{\aleph_0}$ non-isomorphic 2-generator rings (and hence "most" 2-generator rings are not finitely presented). For there are certainly $2^{\aleph_0}$ non-isomorphic countable commutative rings (e.g., localizations of $\mathbf{Z}$), each of which can be made the center of a 2-generator ring. A further consequence is that there is no "universal" 2-generator ring R which contains copies of all countable rings, because such an R could have only countably many 2-generator subrings.

Added in proof. Dr. Peter Neumann has kindly pointed out that the group-theoretic analogue of Proposition 2 (i.e. the centre of a 2-generator group can be an arbitrary countable abelian group) was established by Phillip Hall in "Finiteness conditions for soluble groups", Proc. London Math. Soc. **4** (1954), 419-436.

We are also grateful to Jan Okninski for pointing out how Example contrasts with the commutative case: for any finitely generated commutative ring R, its group of units R^* is finitely generated if and only if the additive group of the Jacobson radical of R is finitely generated. This is shown by H. Bass in "Introduction to some methods of algebraic K-theory", Conference Board of the Mathematical Sciences 20, AMS, 1973.

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