ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 4, 2009

# K-HOMOLOGY CLASSES OF DIRAC OPERATORS ON SMOOTH SUBSETS OF SINGULAR SPACES

PETER HASKELL AND CHARLOTTE WAHL

ABSTRACT. We identify conditions under which Dirac operators, constructed using different metrics on a smooth dense submanifold $M$ of a compact singular space $X$, represent the same class in the $K$-homology of $X$. This result clarifies the sense in which the invariants defined by Dirac operators can be regarded as intrinsic to $X$.

**1. Introduction.** Elliptic differential operators on smooth closed manifolds represent many of the geometric and topological invariants of those manifolds. Analytic manifestations of topological and geometric invariants of a singular space can sometimes be found among elliptic differential operators on a smooth dense submanifold of the singular space. The isomorphism, which holds in many interesting cases [**5, 8**], between the $L^2$ de Rham cohomology of the dense submanifold and the intersection cohomology of the singular space, is one important and interesting example of this relationship between analysis and topology. This isomorphism suggests an approach to Hodge theory on singular algebraic varieties, in part through its relation to the associated $L^2$ Dolbeault cohomology, see e.g., [**19**]. The references in [**13**] list selected papers by many of the researchers who have worked on these aspects of de Rham and Dolbeault cohomology.

In situations where there is no known topological analogue, such as intersection cohomology, it is less obvious which analytically defined invariants should be regarded as intrinsic to the singular spaces on whose dense submanifolds they are represented. For example, spin Dirac operators often define invariants without previously known singular-space counterparts. The operators most likely to provide intrinsic invariants

---

2000 AMS *Mathematics subject classification.* Primary 58J20, 19K56, 19K35.
*Keywords and phrases.* Dirac operator, $K$-homology, singular space.
The work of the second author was supported by the U.S. National Science Foundation through the ADVANCE Institutional Transformation Award SBE-0244916.
Received by the editors on December 27, 2006, and accepted on January 28, 2008.
DOI:10.1216/RMJ-2009-39-4-1245 

are those that represent classes in the $K$-homology of the singular space. Because the dense submanifold is noncompact, it is not always the case that an elliptic differential operator on the submanifold represents a class in the $K$-homology of the space. When an operator does represent such a class, the class may depend on the metric used to define the operator. In this paper, after recalling many examples of operators that represent classes in the $K$-homology of singular spaces, we identify conditions under which the classes represented by spin Dirac operators are independent of the metrics used to define the operators.

**2. Analytic cycles for $K$-homology.** Throughout this paper we let $X$ denote a metrizable compact singular space with open dense submanifold $M$. Let $Y$ denote $X \setminus M$. In this section we discuss a representative sample of the ways in which a Dirac operator (in the sense of [**17**]) $D$ on $M$ can represent or fail to represent a class in the $K$-homology of $X$. In discussing $K$-homology, we rely on the $KK$ theory for algebras of continuous functions both because Dirac operators naturally define $KK$ cycles and because $KK_*(C_0(M), \mathbf{C})$, the $KK$ theory of the algebra of continuous functions, vanishing at infinity, on $M$ is probably the clearest, most efficient notation for the groups we associate to the noncompact manifold $M$. Because there is only one notion of the $K$-homology for a compact metrizable space, we will often use the short notation $K_*(X)$, respectively $K_*(Y)$, for the groups $KK_*(C(X), \mathbf{C})$, respectively $KK_*(C(Y), \mathbf{C})$.

**Definition 2.1.** (See [**16**], the original source, or [**4**].) For $A$, a $C^*$-algebra of continuous functions, a $KK(A, \mathbf{C})$ cycle consists of a Hilbert space $H$, a $C^*$-representation $f \mapsto m_f$ of $A$ in the $C^*$-algebra of bounded operators on $H$ and a bounded operator $T$ on $H$ satisfying the following conditions for every $f \in A$.

(1) $[T, m_f]$ is compact.

(2) $(T^2 - 1) \circ m_f$ is compact.

(3) $(T - T^*) \circ m_f$ is compact.

When $H$ is ungraded, the cycle is a $KK_1$ cycle. When $H$ is $\mathbf{Z}/2$-graded, $T$ must be of degree one, and the cycle is a $KK_0$ cycle. The $KK$ groups can be defined in much more generality, but we do not need that generality. In particular, we have chosen the notation $m_f$ for the representation to emphasize that our Hilbert spaces will always be spaces of $L^2$ sections of vector bundles and our algebras of continuous functions will always act by pointwise multiplication on these sections. When this representation is clear from the context, we use the notation $(H, T)$ for the $KK$ cycle.

The passage from a Dirac operator $D$ to a $KK$ cycle often follows the path described in more generality in [**2**].

**Definition 2.2** [**2**]. A closed, densely defined operator $D$ on $H$ is called regular if it satisfies the following two conditions.

(1) The domain of $D^*$ is dense in $H$.

(2) The operator $1 + D^*D$ has image dense in $H$.

**Theorem 2.3.** (See [**2**], the original source, or [**4**].) *Suppose that the algebra $A$ of continuous functions acts on the Hilbert space $H$ as described above and that the regular operator $D$ on $H$ satisfies the following three conditions.*

(1) $D = D^*$.

(2) *For each $f \in A$, $m_f \circ (1 + D^2)^{-1}$ is compact.*

(3) *$A$ contains a dense subset of elements $f$ for which $[D, m_f]$ extends to define a bounded operator on $H$.*

*Then $(H, D \circ (1 + D^2)^{-1/2})$ defines a $KK_*(A, \mathbf{C})$ cycle, which we will usually denote $(H, D)$.*

The short exact sequence of $C^*$-algebras

$$0 \longrightarrow C_0(M) \longrightarrow C(X) \longrightarrow C(Y) \longrightarrow 0$$

induces a six-term exact sequence in $K$-homology. ($K$-homology is periodic with period two.) Of particular interest to us are the pieces

$$\cdots \to KK_i(C(X), \mathbf{C}) \to KK_i(C_0(M), \mathbf{C}) \to KK_{i-1}(C(Y), \mathbf{C}) \to \cdots$$

for $i = 0$ and $i = 1$. The map $KK_i(C(X), \mathbf{C}) \to KK_i(C_0(M), \mathbf{C})$ is induced by the inclusion of $C_0(M)$ in $C(X)$. The map $KK_i(C_0(M), \mathbf{C}) \to$

$KK_{i-1}(C(Y), \mathbf{C})$ is the connecting morphism (or boundary map) $\partial$ in $K$-homology.

Let $D$ be a Dirac operator on $M$. $D$ acts on sections of a vector bundle $E$. (In the graded case $E$ decomposes as $E_0 \oplus E_1$, and $D$ maps sections of each summand to sections of the other summand.) At the least $D$ is defined on smooth compactly supported sections. In choosing a Riemannian metric on $M$ and a Hermitian structure on $E$, we define Hilbert spaces of sections $L^2(E)$ on which $C_0(M)$ acts by pointwise multiplication. Let $D$ denote a closed extension of $D$'s restriction to smooth compactly supported sections. As proven in [**3**], $(L^2(E), D(1+D^*D)^{-1/2})$ represents a class, which we will denote $[D]_M$, in $KK_*(C_0(M), \mathbf{C})$.

**Theorem 2.4.** (See [**3, 15**].) *The class $[D]_M$ is independent of the metric and Hermitian structure used in its definition. If the restriction of $D$ to smooth compactly supported sections has more than one closed extension, $[D]_M$ is independent of the extension chosen.*

In asking whether $D$ represents a class in the $K$-homology of $X$, we are asking whether constructions like those above, but with $f \in C(X)$ acting by pointwise multiplication by its restriction to $M$, define a cycle for $KK_*(C(X), \mathbf{C})$. Because $C(X)$ is unital, the Baaj-Julg [**2**] approach to using $D$ to define a cycle for $KK_*(C(X), \mathbf{C})$ requires that $D$ be self-adjoint with compact $(1 + D^2)^{-1}$. Also, when $Y$ is bigger than a union of isolated points, the requirement that $[D, m_f]$ be compact may be satisfied for $f$ in a dense subset of $C_0(M)$ without necessarily being satisfied for $f$ in a dense subset of $C(X)$.

The combination of Theorem 2.4 and the $K$-homology exact sequence can be used to identify pairs $M \subset X$ for which there is no hope that a Dirac operator on $M$ will define a class in $K_*(X)$. We will call this observation a proposition merely to draw attention to it.

**Proposition 2.5.** *Let $D$ be a Dirac operator acting on sections of the vector bundle $E$ over $M$. Suppose that there is a metric on $M$ and Hermitian structure on $E$ for which the construction of $[D]_M$ yields a class whose image $\partial([D]_M)$ is nonzero in $KK_{i-1}(C(Y), \mathbf{C})$. Then* *there is no choice of metric and Hermitian structure in which $D$ represents a cycle for $KK_i(C(X), \mathbf{C})$.*

*Proof.* The class $[D]_M$ is independent of the metric and Hermitian structure used in its construction. If $[D]_M$ were the image of a class in $K_i(X)$, it would be in the kernel of $\partial$. $\square$

One consequence of the preceding proposition is that if the topological space $X$ can be given the structure of a smooth compact manifold with boundary, and if $M$ is the interior of $X$, then a spin Dirac, spin$^c$ Dirac, or signature operator $D$ on $M$ fails to define a class in $K_*(X)$ because the image of $[D]_M$ in $K_{*-1}(\partial X)$ is the nonzero class represented by the analogous operator on $\partial X$ [**3**]. Another consequence is due to [**18**]. Let $X$ admit the structure of an odd-dimensional spin manifold $M$ with isolated conical singularities $Y = \{y_1, \dots, y_n\}$. Let $D$ be the spin Dirac operator on $M$. $K_0(Y)$ is a direct sum of copies of $\mathbf{Z}$, with one summand for each point in $Y$, $\partial([D]_M) = (k_1, \dots, k_n) \in K_0(Y)$, where $k_i$ is the index of the spin Dirac operator on the cross-section of the cone associated with $y_i$. Hence, if any of these indices is nonzero, $D$ does not define a class in $K_1(X)$. As is emphasized in [**18**], this phenomenon is directly related to the absence of a locally defined self-adjoint extension of $D$.

Not all of the results associated with incomplete metrics on $M$ are negative. Let $X$ be a complex projective algebraic variety with singular locus $Y$, and let $\pi\colon \widetilde{X} \to X$ be a desingularization of $X$. The Dolbeault operator (or any Dirac operator on $\widetilde{X}$) defines a class in $K_0(\widetilde{X})$ whose image under $\pi_*\colon K_0(\widetilde{X}) \to K_0(X)$ maps to the class of the analogous operator on $M$. If $X$ has isolated conical singularities and $M$ is even-dimensional, then choices of closed signature, de Rham and, in the case of spin $M$, spin Dirac operators, define classes in $K_0(X)$. The analysis underlying this assertion appears in [**6, 7, 10**]. Functions that are smooth on $M$ and constant on some neighborhood of each singular point of $X$ provide a dense subset of $C(X)$ whose commutators with the geometric differential operators are bounded. Similarly, [**6, 7, 11**] show that if $X$ is a piecewise linear even-dimensional pseudomanifold with $M$, the complement of its codimension-two skeleton, assigned a piecewise flat metric, then geometric differential operators as above define classes in $K_0(X)$. The dense subset of $C(X)$ with which the operators have bounded commutators is the collection of functions that are locally constant in directions normal to skeleta. Under conditions identified by [**6, 7, 10, 11**] that guarantee self-adjointness of the geometric differential operators, the same results hold for odd-dimensional $M$.

The preceding paragraph fails to give sufficient emphasis to one subtlety. For a given incomplete metric of the type described above, there can be more than one closed extension of a Dirac operator's restriction to smooth vector-bundle sections compactly supported on $M$. The different extensions can have different indices and so they can define different classes in $K_0(X)$. For details see, e.g., [**10, 11**].

The above subtlety is not an issue when the metric on $M$ is complete. On complete $M$ a Dirac operator is essentially self-adjoint [**9**]. *Henceforth, we focus on spin Dirac operators $D$ and on complete spin manifolds $M$.* A natural condition implying that a spin Dirac operator $D$ is a $KK$ cycle is the condition that $M$ have properly positive scalar curvature.

**Definition 2.6.** The manifold $M$ with scalar curvature $\kappa$ is said to have properly positive scalar curvature if, for every real $K$, there is a compact subset of $M$ off of which $\kappa > K$.

**Proposition 2.7.** *If the complete spin manifold $M$ has properly positive scalar curvature, then, for the spin Dirac operator $D$, $(1+D^2)^{-1}$ is compact.*

*Proof.* As noted, e.g., in [**17**], the Bochner-Lichnerowicz formula states that $D^2 = \nabla^*\nabla + \kappa/4$. The proof of the proposition appears many places, including in [**20**] and, in a slightly different context, in [**12**]. □

Manifolds with ends that are warped products or multiply warped products provide a variety of examples of complete manifolds with properly positive scalar curvature. Let $P$ be a closed manifold with Riemannian metric $g_P$ and positive scalar curvature $\kappa_P$, and let $M$ be formed by using the identity on $P$ to attach to a compact manifold with boundary $P$ a cylinder $[0,\infty) \times P$. Letting $r$ denote the variable parametrizing $[0,\infty)$, assign $M$ to be a Riemannian metric that, for

$r$ greater than some large enough $R$, takes the form $dr \otimes dr + r^{2k} g_P$, where $k < 0$. This metric makes the end $[R, \infty) \times P$ what is called a warped product. By a calculation that can be found in [**10**], if $P$ is $m$-dimensional, a warped product $[R, \infty) \times P$ with metric $dr \otimes dr + f^2(r) \cdot g_P$ has scalar curvature

$$\kappa = -m(m-1) \cdot (f'(r)/f(r))^2 - 2m \cdot f''(r)/f(r) + f^{-2}(r) \cdot \kappa_P.$$

In our case, with $f(r) = r^k$, $\kappa \sim r^{-2k} \cdot \kappa_P$. Under our assumptions, $M$ has a properly positive scalar curvature.

Ends that are multiply warped products arise from a similar construction. Let $N$ also be a closed manifold, assign the product $P \times N$ a Riemannian metric $g_P + g_N$, and assign the product $[0, \infty) \times P \times N$ the Riemannian metric $dr \otimes dr + r^{2k} g_P + r^{2c} g_N$. Again, by the calculations in [**10**], if $P$ has positive scalar curvature and if $k < 0$, $P$ contributes to the scalar curvature of $M$ a term asymptotic to $r^{-2k} \kappa_P$. If $c > k$, this term dominates the scalar curvature of $M$ as $r \to \infty$.

Among the compactifications $X$ of manifolds $M$ with ends as discussed above are the one point compactification and the compactification as a manifold with boundary. In the multiply warped case, the latter compactification makes $X$ a manifold $M$ with boundary $P \times N$. If we apply the projection $P \times N \to N$, respectively $P \times N \to P$, to the boundary, the resulting quotient is a compactification $X_N$, respectively $X_P$, in which $X_N \setminus M = N$, respectively $X_P \setminus M = P$.

In all of these examples, if $X$ is the one-point compactification of $M$, then $C(X)$ admits a dense subset of functions $f$, each of which is smooth on $M$ and constant off some compact subset of $M$, and each of which has bounded $df$. Similarly, even if $k < 0$, if $c \geq 0$, then $C(X_N)$ admits a dense subset of functions whose restrictions to $M$ are smooth with bounded $df$. These functions take values that, eventually in $r$, are independent of the $[0, \infty)$ and $P$ coordinates.

**Theorem 2.8.** *Let the spin manifold $M$ be an open dense subset of a metrizable compact singular space $X$. Assume that $M$ has a complete metric with properly positive scalar curvature. Assume that $C(X)$ contains a dense subset of functions $f$ satisfying: each function $f$'s restriction to $M$ is smooth with bounded $df$. Let $D$ be the spin Dirac operator acting on sections of the spinor bundle $S$ over $M$. Then*

*$(L^2(S), D)$ defines a cycle representing a class in $K_i(X)$, where $i$ is (the parity of) the dimension of $M$.*

*Proof.* This result is a consequence of Proposition 2.7 and Theorem 2.3. □

Although a complete Riemannian metric on $M$ does not extend to give $X$ the structure of a metric space, the use of complete metrics in this context is common. A complete metric has the advantage that Dirac operators are essentially self-adjoint. Also, although it is not our focus here, it is worth noting that invariant metrics on symmetric spaces define complete metrics on locally symmetric spaces. The various compactifications of locally symmetric spaces provide, by analogy, some motivation for looking at compactifications whose algebras of continuous functions do not admit dense subsets of functions with bounded exterior derivatives on $M$.

**Theorem 2.9.** *Let the spin manifold $M$ be an open dense subset of a metrizable compact singular space $X$. Assume that $M$ has a complete metric with properly positive scalar curvature $\kappa$. Assume that $C(X)$ contains a dense subset of functions $f$ satisfying: each function $f$'s restriction to $M$ is smooth with the pointwise norm of $df$ bounded above, off some compact subset of $M$, by some constant multiple of $\kappa^{1/4}$. Let $D$ be the spin Dirac operator acting on sections of the spinor bundle $S$ over $M$. Then $(L^2(S), D)$ defines a cycle representing a class in $K_i(X)$, where $i$ is (the parity of) the dimension of $M$.*

*Proof.* Because we are not assuming condition (3) of Theorem 2.3, we need a slightly different proof of condition (1) of Definition 2.1. The reasoning underlying the proof appears, in a slightly different context, in [**12**]. We will recall details in the next section when we prove that the next section's construction defines a homotopy in the sense appropriate for $KK$ theory. □

Manifolds with ends that are multiply warped products provide examples to which this theorem applies. In our previous notation, $C(X_N)$ contains a dense subset of functions $f$ for which the pointwise norm of $df$ has behavior $\sim r^{-c}$. If $k < 0$ and $c \geq k/2$, the hypotheses of the theorem are satisfied. On the other hand, note that if the Dirac operator on $N$ has nonzero index, the Dirac operator on $M$ cannot represent a class in $K_*(X_P)$ because $\partial([D]_M)$ is a nonzero multiple of the nonzero class in $K_{*-1}(P)$ represented by the Dirac operator on $P$. The negative results notwithstanding, we have seen that there are often many metrics on $M$ for which Dirac operators on $M$ represent classes in $K_*(X)$. The next section identifies conditions under which the $K_*(X)$ class represented in this way is independent of the metric used in the construction of its representative.

**3. The homotopy.** If $X$ is a compact metrizable singular space with an open dense submanifold $M$, we have seen that a Dirac operator on $M$ can represent a $K$-homology class for $X$. The construction of the $K$-homology cycle depends on the choice of a Riemannian metric on $M$. The significance of the construction as a source of invariants for $X$ depends in part on having the $K$-homology class be independent of the metric used in the construction of the cycle. In this section we identify, for spin Dirac operators and for complete Riemannian metrics with properly positive scalar curvature, some conditions that imply such independence. Our approach is based on using the Baaj-Julg [**2**] (or unbounded Kasparov bimodule) approach to constructing a $KK$-theoretic homotopy between the representatives constructed using different metrics.

Throughout this section $M$ is a *spin submanifold* of the singular space $X$, and the Dirac operators are *spin Dirac operators*. On $M$, let $g_0$ and $g_1$ be *complete* Riemannian metrics with properly positive scalar curvatures, $\kappa_0$ and $\kappa_1$, and assume that the associated spin Dirac operators $D_0$ and $D_1$, acting on sections of the spinor bundle $S$, define cycles for $K_*(X)$ for the reasons given in Theorems 2.8 or 2.9.

Suppose that $\{(\alpha_t, \beta_t = 1 - \alpha_t) \colon t \in (0,1]\}$ and $\{(\phi_t, \psi_t = 1 - \phi_t) \colon t \in (0,1]\}$ are smooth families of $C^\infty$ partitions of unity on $M$, with $[D_0, \alpha_t]$ (and hence $[D_0, \beta_t]$) bounded, independent of $t$, as operators on $L^2_0(S)$, the Hilbert space of spinors that are $L^2$ with respect to the metric $g_0$, and with $[D_0, [D_0, \alpha_t]]$ (and hence $[D_0, [D_0, \beta_t]]$) bounded, independent of $t$, as operators from the domain of $D_0$ to $L^2_0(S)$. (Here we use the notation for a function to denote the operator of pointwise multiplication by the function.) Suppose further that $\{\gamma_t : t \in (0,1]\}$

is a smooth family of $C^\infty$ functions on $M$ satisfying: for all $x \in M$ and for all $t$, $0 \leq \gamma_t(x) \leq 1$; for each $t$, $\gamma_t$ is identically 1 on some neighborhood of the support of $\alpha_t$; for each $t$ the support of $\gamma_t$ has empty intersection with the support of $\psi_t$; and the commutators of $\gamma_t$ and $D_0$ satisfy uniform bounds of the type described above for $\alpha_t$. Suppose that associated with $\{(\alpha_t, \beta_t)\}$ and $\{(\phi_t, \psi_t)\}$ is a family of triples of compact subsets of $M$, $\{(B_t, C_t, E_t)\}$, for which:

(1) for each $t$, $B_t$ is contained in the interior of $C_t$ and $C_t$ is contained in the interior of $E_t$;

(2) as $t \to 0$, $B_t$ exhausts $M$;

(3) the exhaustion is monotone in the sense that, for $t > s$, $B_t \subset B_s$, and the analogous monotonicity holds for the sets $C_t$ and the sets $E_t$; and

(4) for each $t$, the support of $\alpha_t$ is contained in the interior of $C_t$, the support of $\beta_t$ is contained in the complement of $B_t$, the support of $\phi_t$ is contained in the interior of $E_t$ and the support of $\psi_t$ is contained in the complement of $C_t$.

Finally suppose that $\{\Gamma_t : t \in (0,1]\}$ is a smooth family of $C^\infty$ functions on $M$ satisfying: for all $x \in M$ and for all $t$, $0 \leq \Gamma_t(x) \leq 1$; for each $t$, $\Gamma_t$ is identically 1 on some neighborhood of the support of $\beta_t$; for each $t$ the support of $\Gamma_t$ has empty intersection with $B_t$; and the commutators of $\Gamma_t$ and $D_0$ satisfy uniform bounds of the type described above for $\alpha_t$.

For $t \in (0,1]$, we let $g_t = \phi_t \cdot g_0 + \psi_t \cdot g_1$ define a family of metrics on $M$. (At this point it is convenient for the notation to assume that $g_0 = g_1$ on $E_1$. This possible compactly supported change of metric has no effect on later reasoning.) We assume that, off some compact set, the scalar curvature $\kappa_t$ associated with $g_t$ is uniformly bounded below by some positive constant multiple of the minimum of $\kappa_0$ and $\kappa_1$. We assume that the $L^2$ norms, with respect to the metrics $g_t$ of sections $w$ of the spinor bundle $S$ are uniformly bounded below by some positive constant multiple of the $L^2$ norm of $w$ with respect to the metric $g_0$. (Here and throughout this section we assume that the Hermitian metric on the spinor bundles is independent of $t$.)

*Remark* 3.1. We express our assumptions in terms of the hypotheses that are essential to our reasoning, but manifolds with ends that are warped or multiply warped products provide examples in which all of these hypotheses follow from a few natural choices. In manifolds with such ends, $B_t$, $C_t$, $E_t$, $\alpha_t$, $\beta_t$, $\gamma_t$, $\Gamma_t$, $\phi_t$ and $\psi_t$ can be chosen to depend only on $r$, with $B_t$ corresponding to $r \leq 1/t$, $C_t$ corresponding to $r \leq (1/t)+1$, and $E_t$ corresponding to $r \leq (1/t)+2$, for example.

To use the preceding choices to show that $(L^2_0(S), D_0)$ and $(L^2_1(S), D_1)$, the cycles defined using the metrics $g_0$ and $g_1$, represent the same class in $K_*(X)$, we construct a homotopy between the cycles, i.e., we construct a cycle for $KK_*(C(X), C([0,1]))$ whose "evaluations" at endpoints 0 and 1 give $(L^2_0(S), D_0)$ and $(L^2_1(S), D_1)$. The definition of such cycles is analogous to the definition of the cycles for $KK_*(C(X), \mathbf{C})$ except that the Hilbert space is replaced by a Hilbert $C^*$-module whose inner product takes values in the $C^*$-algebra $C([0,1])$. For details see, e.g., [**4**].

The construction of the homotopy proceeds as follows. Let $\widetilde{S}$ denote $\pi^*(S)$, where $\pi$ is the projection $M \times (0,1] \to M$. Begin with the collection $\mathcal{E}$ of smooth sections $w$ of $\widetilde{S}$ that satisfy:

(1) for each $t \in (0,1]$, the restriction $w_t$ of $w$ to $M \times \{t\}$ is compactly supported;

(2) for each $w$ there is a positive $t_0$ such that, for $t < t_0$, the support of $w_t$ is contained in $C_t$; and

(3) as $t \to 0$, $w_t$ converges in $L^2_0(S)$. (Note that, by condition 2, the $L^2_0(S)$ and $L^2_t(S)$ norms of $w_t$ agree eventually as $t \to 0$.)

The $L^2_t(S)$ inner products associated with the metrics $g_t$ on $M$ define a $C((0,1])$-valued inner product. Conditions 2 and 3 above permit us to use limits as $t \to 0$ to extend the sections to $t = 0$. The inner product extends also to take values in $C([0,1])$. Let $H$ denote the completion of $\mathcal{E}$ to a Hilbert $C^*$-module over $C([0,1])$. Note that a straightforward calculation shows that we get the same result if we start with a collection $\mathcal{E}_0$ for which condition (2) is replaced by the statement: for each $w$ there is a positive $t_0$ such that, for $t < t_0$, $w_t$ is independent of $t$.

The concept of homotopy in $KK$-theory depends on the notion of evaluation of a Hilbert $C([0,1])$-module $H$ at each of its endpoints. For example, evaluation at 0 is defined by letting $C([0,1])$ act on $\mathbf{C}$ by the formula $f \cdot \lambda = f(0)\lambda$ and then forming the tensor product $H \otimes_{C([0,1])} \mathbf{C}$. For details, see [**4**]. Intuitively this process forms what could be called a quotient in a sense appropriate to Hilbert $C^*$-modules of $H$ by the elements of $H$ whose inner products with themselves take the value zero at 0.

**Lemma 3.2.** *Each $\tau \in (0,1]$ has a neighborhood $N_\tau$ over which the identity map defines a quasi-isometry* (*with respect to* sup *norms on bounded continuous functions on $N_\tau$*) *between the restriction of $H$ to $N_\tau$ and the Hilbert $C^*$-module of bounded continuous functions on $N_\tau$ with values in $L^2_\tau(S)$. The evaluation of $H$ at $\tau = 0$, respectively $\tau = 1$, gives $L^2_0(S)$, respectively $L^2_1(S)$.*

*Proof.* The assertion about $\tau \in (0,1]$ follows from estimates based on the continuity of $g_t$ and the observation that each $\tau \in (0,1]$ has a neighborhood over which the $g_t$'s agree off some compact subset of $M$. The assertions about evaluation follow directly from the construction of $H$. In particular, the assertion for $\tau = 0$ is a consequence of our assumption about the relationship between the $L^2_t(S)$ norms and the $L^2_0(S)$ norm, our definition of $\mathcal{E}$, and our construction of $H$ from $\mathcal{E}$. $\square$

Let $\widetilde{D}$ denote the operator on $\mathcal{E}$ defined by $\widetilde{D}(w) = D_t(w_t)$. $\widetilde{D}$ is defined only on those sections $w$ for which, as $t \to 0$, $D_t(w_t)$ converges in $L^2_0(S)$. Let $D$ denote the operator-norm closure of $\widetilde{D}$ as an operator on the Hilbert $C^*$-module $H$. Because the smooth compactly supported sections of $S$ are dense in $L^2_0(S)$, $D$ is densely defined. In fact, the dense subspace $\mathcal{E}_0$ is a core for both $D$ and $D^2$.

**Lemma 3.3.** *$D$ is self-adjoint* (*and so $D^*$ is densely defined*).

*Proof.* By the essential self-adjointness of the restriction of each $D_t$ to smooth compactly supported sections, $\widetilde{D}$ is symmetric. For any $w \in H$, our characterization of $H$ allows us to rely on sections $\widetilde{w}$ that are smooth, supported locally in $t$ and have $\widetilde{w}_t$ compactly supported in

$M$ to show that $\langle D\widetilde{w}, w\rangle = \langle \widetilde{w}, D^*w\rangle$ can be satisfied only if, for each $t$, $w_t$ is in the domain of $D_t$ and $(D^*w)_t = D_t(w_t)$. □

**Lemma 3.4.** *$1 + D^2$ has dense image, and $(1 + D^2)^{-1}$ is compact.*

*Proof.* Our characterization of $H$ permits the reduction of the proof that $1+D^2$ has dense image to calculations that are local in $t$. It suffices to observe that, for each $t$, the set of smooth compactly supported sections forms a core for $D_t^2$ and that the spectrum of $D_t^2$ is discrete, with each value having finite multiplicity and with infinity as the only accumulation point. This observation implies that the image under $1 + D_t^2$ of the set of smooth compactly supported sections is dense in $L_t^2(S)$.

To show that $(1 + D^2)^{-1}$ is compact, it suffices to proceed locally in $t$ to approximate this operator in norm by compact operators. The local results can be patched together by a partition of unity in $t$. Local approximation in some neighborhood of $\tau \in (0, 1]$ and local approximation in some neighborhood of $\tau = 0$ require different arguments.

Suppose that $\tau \in (0, 1]$. For $N_\tau$ as in Lemma 3.2, let $U$ denote the quasi-isometry mapping the space of bounded continuous $H_\tau$-valued functions on $N_\tau$ to the restriction of $H$ to $N_\tau$. ($U$ leaves sections unchanged, but its inclusion in the notation helps keep track of the Hilbert spaces involved.) By undoing the conjugation, it suffices to show that, over $N_\tau$, $U^{-1}(1 + D^2)^{-1}U$ is a compact operator on the Hilbert $C^*$-module of bounded continuous $H_\tau$-valued functions on $N_\tau$. We proceed by comparing $U^{-1}(1 + D^2)^{-1}U$ to the constant compact operator-valued function on $N_\tau$ with value $(1 + D_\tau^2)^{-1}$. The difference

$$(1 + D_\tau^2)^{-1} - U^{-1}(1 + D^2)^{-1}U$$

is equal to

$$(1 + D_\tau^2)^{-1}(U^{-1}(1 + D^2)U - (1 + D_\tau^2))U^{-1}(1 + D^2)^{-1}U.$$

We can make the norm of this operator as small as we wish by choosing $N_\tau$ small enough to keep the coefficients in the middle factor, which is a differential operator, as small as necessary.

Suppose that $\tau = 0$. Let $C > 0$ be a constant. Let $\eta = \zeta^*\zeta$ be an arbitrary nonnegative vector-bundle map on $\widetilde{S}$ with pointwise norm bounded above by $C$. For any $s \in [0,1]$ and any $w_s$ in the domain of $D_s^2$,

$$\begin{aligned} C \cdot \langle D_s w_s, D_s w_s \rangle = C \cdot \langle D_s^2 w_s, w_s \rangle &\geq \left\langle C \cdot \frac{\kappa_s}{4} w_s, w_s \right\rangle \\ &= \left\langle \eta \cdot \frac{\kappa_s}{4} w_s, w_s \right\rangle + \left\langle (C - \eta) \cdot \frac{\kappa_s}{4} w_s, w_s \right\rangle. \end{aligned}$$

It follows that, for $L$ at least the greater of zero and the negative of the minimum value of $\kappa_s/4$,

$$C \cdot \langle D_s w_s, D_s w_s \rangle + CL \cdot \langle w_s, w_s \rangle \geq \left\langle \eta \cdot \frac{\kappa_s}{4} w_s, w_s \right\rangle.$$

By our assumptions we may choose such an $L$ that is independent of $s$. By our assumption of properly positive scalar curvature, for a given $C$, the above inequality implies that for any choices of (large) $\lambda > 0$ and (small) $\varepsilon > 0$, we may choose a positive $t_0$ such that if the support of $\eta$ is outside $B_{t_0}$, and if $\langle D_s^2 w_s, w_s \rangle \leq \lambda \cdot \|w_s\|_s^2$, then $\langle \eta w_s, w_s \rangle \leq \varepsilon \cdot \|w_s\|_s^2$. We will call this result the *curvature estimate*.

Let $t$ be an arbitrary element of $(0,1]$. Again we will use the notation for a function to denote also the operator defined by pointwise multiplication by that function. $(1+D^2)^{-1} = (1+D^2)^{-1} \circ (\alpha_t + \beta_t)$, which equals

$$(1+D^2)^{-1}(1+D^2) \circ \alpha_t \circ (1+D^2)^{-1}$$

plus

$$(1+D^2)^{-1} \circ \big(([[D,\alpha_t],D] + 2[\alpha_t, D]D) \circ (1+D^2)^{-1} + \beta_t\big).$$

For $s < t$, $D_s^2 = D_0^2$ in $C_s$, which contains the support of $\alpha_t$. It follows that

$$\begin{aligned} &(1+D^2)^{-1}(1+D^2) \circ \alpha_t \circ (1+D^2)^{-1} \\ &\qquad = (1+D_0^2)^{-1}(1+D^2) \circ \alpha_t \circ (1+D^2)^{-1}. \end{aligned}$$

Let $\widetilde{\alpha}_t$ be a function with values in $[0,1]$ that is identically one on the support of $\alpha_t$ and that has compact support in $C_t$.

$$(1+D_0^2)^{-1}(1+D^2)\circ\alpha_t\circ(1+D^2)^{-1}$$
$$=\widetilde{\alpha}_t\circ(1+D_0^2)^{-1}\circ\widetilde{\alpha}_t\circ(1+D^2)\circ\alpha_t\circ(1+D^2)^{-1}$$

By the nature of the spectrum of $D_0^2$, over $[0,t]$ we may regard $\widetilde{\alpha}_t\circ(1+D_0^2)^{-1}\circ\widetilde{\alpha}_t$ as a constant compact-operator-valued function acting on functions with values in $L_0^2(S)$. Over $[0,t]$ this compact operator has its composition with $(1+D^2)\circ\alpha_t\circ(1+D^2)^{-1}$, which is bounded over $[0,t]$, a compact operator.

To complete our proof, we need to show that, for any $\delta>0$, if $t\in(0,1]$ is small enough, the norm of

$$(1+D^2)^{-1}\circ\big(([[D,\alpha_t],D]+2[\alpha_t,D]D)\circ(1+D^2)^{-1}+\beta_t\big)$$

is no greater than $\delta$. This estimate holds if we can show that for each $w\in\mathcal{E}_0$ of norm one, the norm of the image of $w$ under this operator has norm no greater than $\delta$. We can reach this conclusion by two applications of the curvature estimate.

First note that $[[D_s,\alpha_t],D_s]+2[\alpha_t,D_s]D_s$ is a first-order differential operator with smooth coefficients compactly supported in the intersection of the supports of $\alpha_t$ and $\beta_t$, where $D_s=D_0$ for all $s\le t$. It follows that, for any $s\le t$, $([[D_s,\alpha_t],D_s]+2[\alpha_t,D_s]D_s)$ equals $([[D_0,\alpha_t],D_0]+2[\alpha_t,D_0]D_0)$, and so, for all $s\le t$, $([[D_s,\alpha_t],D_s]+2[\alpha_t,D_s]D_s)\circ(1+D_s^2)^{-1/2}$ equals $([[D_0,\alpha_t],D_0]+2[\alpha_t,D_0]D_0)\circ\gamma_t\circ(1+D_s^2)^{-1/2}$. Our assumptions on $\gamma_t$ imply that multiplication by $\gamma_t$ satisfies a bound, uniform in $t\in(0,1]$ and in $s\le t$, as a map from the domain of $D_s$ to the domain of $D_0$. Hence, using our assumptions on $\alpha_t$, we may choose a $b>1$ that, for all $t$ and for all $s\le t$, bounds $([[D_s,\alpha_t],D_s]+2[\alpha_t,D_s]D_s)\circ(1+D_s^2)^{-1/2}$.

For $w\in\mathcal{E}_0$ of norm one, and for any $s\le t$, decompose $w_s=u_s+v_s$, where $u_s$ lies in the direct sum of the eigenspaces of $1+D_s^2$ for which the eigenvalues are no greater than $(5b/\delta)^2$ and $v_s$ is similarly associated with the eigenvalues greater than $(5b/\delta)^2$. Due to the presence of $(1+D_s^2)^{-1}(v_s)$,

$$(1+D_s^2)^{-1}\circ([[D_s,\alpha_t],D_s]+2[\alpha_t,D_s]D_s)\circ(1+D_s^2)^{-1}(v_s)$$

has norm less than $\delta/5$. Because of the supports of coefficients of the differential operator in the composition,

$$(1+D_s^2)^{-1}\circ([[D_s,\alpha_t],D_s]+2[\alpha_t,D_s]D_s)\circ(1+D_s^2)^{-1}$$

equals

$$(1+D_s^2)^{-1}\circ([[D_s,\alpha_t],D_s]+2[\alpha_t,D_s]D_s)\circ\gamma_t\circ\Gamma_t\circ(1+D_s^2)^{-1}.$$

By the curvature estimate, for small enough $t$ and for all $s\le t$,

$$(1+D_s^2)^{-1}\circ([[D_s,\alpha_t],D_s]+2[\alpha_t,D_s]D_s)\circ\gamma_t\circ\Gamma_t\circ(1+D_s^2)^{-1}(u_s)$$

has norm less than $\delta/5$.

To get analogous norm bounds on $(1+D_s^2)^{-1}\circ\beta_t(w_s)$, again consider separately $(1+D_s^2)^{-1}\circ\beta_t(u_s)$ and $(1+D_s^2)^{-1}\circ\beta_t(v_s)$. $(1+D_s^2)^{-1}\circ\beta_t(v_s)$ equals

$$\beta_t\circ(1+D_s^2)^{-1}(v_s)$$

plus

$$(1+D_s^2)^{-1}\circ([[D_s,\beta_t],D_s]+2[\beta_t,D_s]D_s)\circ(1+D_s^2)^{-1}(v_s).$$

Each of the terms has norm less than $\delta/5$. The curvature estimate applies immediately to $(1+D_s^2)^{-1}\circ\beta_t(u_s)$ to guarantee that, for small enough $t$ and for all $s\le t$, the norm of this term is no greater than $\delta/5$. □

**Lemma 3.5.** $(1+D^2)^{-1/2}$ *is compact.*

*Proof.* As in [**2**], this result follows from norm approximation by Riemann sums of proper integrals approximating the integral in

$$\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(1+D^2+\lambda)^{-1}\,d\lambda=(1+D^2)^{-1/2}. \qquad \square$$

**Theorem 3.6.** *Suppose that* $(L_0^2(S),D_0)$ *and* $(L_1^2(S),D_1)$ *are cycles representing classes in the* $K$*-homology of* $X$ *by virtue of their satisfying* *the hypotheses of Theorem* 2.8*. Suppose also that the metrics used in the construction of these cycles permit the construction of an* $(H, D)$ *as in the discussion at the beginning of this section. Then these cycles represent the same class in the* $K$*-homology of* $X$*.*

*Proof.* The cycles are homotopic because the preceding lemmas show that $(H, D)$ satisfies the hypotheses of the $KK_*(C(X), C([0, 1]))$ analogue of Theorem 2.3. □

We now turn our attention to the case when $(L_0^2(S), D_0)$ and $(L_1^2(S), D_1)$ satisfy the hypotheses of Theorem 2.9 but at least one of them does not satisfy the hypotheses of Theorem 2.8. Again we assume that the metrics used in the construction of these cycles permit the construction of an $(H, D)$ as in the discussion at the beginning of this section.

**Lemma 3.7.** *For nonnegative* $\lambda$*, and for* $f$ *as in Theorem* 2.9*,* $[D, m_f](1+D^2+\lambda)^{-1/2}$ *is a bounded operator on the Hilbert* $C^*$*-module* $H$*. As a function of* $\lambda$*, the norm of this operator is uniformly bounded relative to some negative power of* $1 + \lambda$*.*

*Proof.* $[D, m_f]$ is defined on $\mathcal{E}_0$, with $\|[D, m_f](w)\|^2$ no greater than some constant multiple (independent of $w$) of the square root of $\|w\|^2 + \|Dw\|^2$. As a vector-bundle map, $[D, m_f]$ has a densely defined adjoint, and so $[D, m_f](1 + D^2 + \lambda)^{-1/2}$ has an adjoint. The same argument shows that $[D, m_f]^*[D, m_f](1+D^2+\lambda)^{-1/2}$ is a bounded operator. The assertion about the norm bound of $[D, m_f](1 + D^2 + \lambda)^{-1/2}$ relative to $1 + \lambda$ follows from the observations that

$$\Big([D, m_f](1 + D^2 + \lambda)^{-1/2}\Big)^* \Big([D, m_f](1 + D^2 + \lambda)^{-1/2}\Big) = (1 + D^2 + \lambda)^{-1/2}[D, m_f]^*[D, m_f](1 + D^2 + \lambda)^{-1/2}$$

and that $[D, m_f]^*[D, m_f](1 + D^2 + \lambda)^{-1/2}$ is bounded. □

**Lemma 3.8.** *For nonnegative* $\lambda$*, and for* $f$ *as in Theorem* 2.9*,* $[D, m_f](1+D^2+\lambda)^{-1/2}$ *is a compact operator on the Hilbert* $C^*$*-module* $H$*.*

*Proof.*

$$
\begin{aligned}
&[D, m_f](1 + D^2 + \lambda)^{-1/2} \\
&= [D, m_f] \circ \frac{1}{\pi}\int_0^\infty \xi^{-1/2}(1 + D^2 + \lambda + \xi)^{-1}\, d\xi \\
&= \frac{1}{\pi}\int_0^\infty \xi^{-1/2}[D, m_f](1 + D^2 + \lambda + \xi)^{-1/2}(1 + D^2 + \lambda + \xi)^{-1/2}\, d\xi.
\end{aligned}
$$

The Riemann sums of approximating proper integrals provide a norm approximation by compact operators. □

**Theorem 3.9.** *Suppose that $(L^2_0(S), D_0)$ and $(L^2_1(S), D_1)$ are cycles representing classes in the $K$-homology of $X$ by virtue of their satisfying the hypotheses of Theorem* 2.9. *Suppose also that the metrics used in the construction of these cycles permit the construction of an $(H, D)$ as in the discussion at the beginning of this section. Then these cycles represent the same class in the $K$-homology of $X$.*

*Proof.* In a proof that $(H, D)$ defines a homotopy, the only issue that we have not addressed is the compactness of $[D(1+D^2)^{-1/2}, m_f]$. This operator equals

$$D[(1 + D^2)^{-1/2}, m_f] + [D, m_f](1 + D^2)^{-1/2}.$$

By Lemma 3.8 the second term is compact. The first term can be written

$$D \circ \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}[(1 + D^2 + \lambda)^{-1}, m_f]\, d\lambda.$$

This expression equals

$$\frac{1}{\pi}\int_0^\infty \lambda^{-1/2} D(1 + D^2 + \lambda)^{-1}[m_f, D] (1 + D^2 + \lambda)^{-1/2} D(1 + D^2 + \lambda)^{-1/2}\, d\lambda$$

plus

$$\frac{1}{\pi}\int_0^\infty \lambda^{-1/2} D^2(1 + D^2 + \lambda)^{-1}[m_f, D] (1 + D^2 + \lambda)^{-1/2}(1 + D^2 + \lambda)^{-1/2}\, d\lambda.$$

Again the Riemann sum argument shows that the operator is compact. □

*Remark* 3.10. It is not the purpose of this paper to address the issues associated with incomplete metrics on $M$. However, one case, that of isolated conical singularities, fits easily within our framework. Let $X$ be a compact singular space with isolated singular point $x_0$, a deleted neighborhood of which is homeomorphic to a cylinder with cross-section $P$. Assume that $M = X \setminus \{x_0\}$ has a spin structure. Assume that $P$, with dimension $m \geq 2$, can be given a metric $g_P$ with scalar curvature everywhere greater than $m(m-1)$. (We use this assumption because of its simplicity, but our argument requires only that the scalar curvature of $P$ be everywhere positive and that every eigenvalue of the spin Dirac operator on $P$ have absolute value at least one-half.) We may give $M$ a metric in which a deleted neighborhood of $x_0$ has the incomplete conical metric $dr \otimes dr + r^2 g_P$ on $(0,1] \times P$ (with the singular point at $r = 0$) or a metric in which the deleted neighborhood has the complete warped-product metric $dr \otimes dr + r^{-2} g_P$ on $[1, \infty) \times P$ (with the singular point at $r = \infty$).

The spin Dirac operators constructed using these metrics on $M$ represent the same class in the $K$-homology of $X$. Because the singularity is isolated, the $K$-homology exact sequence reduces the proof of this assertion to an index calculation, which can be done using the relative index theorem of [**14**]. If $M$ is odd-dimensional, each Dirac operator represents a class in $K_1(X)$. Because $K_1(\{x_0\}) = 0$, classes in $K_1(X)$ are determined by their images in $KK_1(C_0(M), \mathbf{C})$. As noted in Section 2, the images of the classes represented by our two Dirac operators are equal. If $M$ is even-dimensional, the Dirac operators' classes in $K_0(X)$ have equal images in $KK_0(C_0(M), \mathbf{C})$ but may differ by elements in the image of $K_0(\{x_0\})$. The maps $\{x_0\} \hookrightarrow X \to \{x_0\}$ define maps $K_0(\{x_0\}) \to K_0(X) \to K_0(\{x_0\})$ whose composition is the identity and whose second factor is the index map. Hence, to show that the Dirac operators represent the same class in $K_0(X)$, it suffices to show that their indices are equal. Introduce a third metric on $M$, a metric in which the deleted neighborhood of the singular point is a complete cylinder $[1, \infty) \times P$ with metric $dr \otimes dr + g_P$. The index formulas of [**1, 10**] (with sign correction required by the different orientation conventions) show that the Dirac operator constructed using the conical end has index equal to the index of the Dirac operator constructed using the cylindrical end. The relative index theorem of [**14**] shows that the index of the Dirac operator constructed using the cylindrical end equals the index of the Dirac operator constructed using the warped-product end, an observation that finishes our proof. In more detail, in the relative index argument one starts with the Dirac operator on one version of $M$ and the Dirac operator on the other version of $M$ with reversed orientation. The sum of the indices of these operators equals the difference of the indices of the operators constructed without an orientation reversal. The relative index theorem says that this index sum remains unchanged if we cut off the ends of the copies of $M$, glue the remaining interiors together along their boundaries, and glue the ends together along their boundaries. The gluing leaves us with a Dirac operator on a compact double and a Dirac operator on a manifold with uniformly positive scalar curvature. Both of these Dirac operators have indices equal to zero.

## REFERENCES

**1.** M.F. Atiyah, V.K. Patodi and I.M. Singer, *Spectral asymmetry and Riemannian geometry* I, Math. Proc. Camb. Phil. Soc. **77** (1975), 43–69.

**2.** S. Baaj and P. Julg, *Théorie bivariante de Kasparov et opérateurs non bornés dans les $C^*$-modules Hilbertiens*, C.R. Acad. Sci. Paris **296** (1983), 875–878.

**3.** P. Baum, R. Douglas and M. Taylor, *Cycles and relative cycles in analytic $K$-homology*, J. Differential Geom. **30** (1989), 761–804.

**4.** B. Blackadar, *$K$-theory for operator algebras*, MSRI Publications **5**, Springer-Verlag, New York, 1986.

**5.** J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Proc. Symp. Pure Math. **36**, American Math. Soc., Providence, RI, 1980.

**6.** ———, *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. **18** (1983), 575–657.

**7.** ———, *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), 2103–2106.

**8.** J. Cheeger, M. Goresky and R. MacPherson, *$L^2$-cohomology and intersection homology of singular algebraic varieties*, in *Seminar on differential geometry*, S.-T. Yau, ed., Princeton University Press, Princeton, NJ, 1982.

**9.** P. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations*, J. Functional Anal. **12** (1973), 401–414.

**10.** A. Chou, *The Dirac operator on spaces with conical singularities and positive scalar curvatures*, Trans. Amer. Math. Soc. **289** (1985), 1–40.

**11.** ———, *Criteria for selfadjointness of the Dirac operator on pseudomanifolds*, Proc. Amer. Math. Soc. **106** (1989), 1107–1116.

**12.** J. Fox, C. Gajdzinski and P. Haskell, *Homology Chern characters of perturbed Dirac operators*, Houston J. Math. **27** (2001), 97–121.

**13.** J. Fox, P. Haskell and W. Pardon, *Two themes in index theory on singular varieties*, Proc. Symp. Pure Math. **51**, Part 2, Amer. Math. Soc., Providence, RI, 1990.

**14.** M. Gromov and H.B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publ. Math. IHES **58** (1983), 83–196.

**15.** N. Higson and J. Roe, *Analytic $K$-homology*, Oxford University Press, Oxford, 2000.

**16.** G.G. Kasparov, *The operator $K$-functor and extensions of $C^*$-algebras*, Math. USSR Izv. **16** (1981), 513–572.

**17.** H.B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, 1989.

**18.** M. Lesch, *Deficiency indices for symmetric Dirac operators on manifolds with conic singularities*, Topology **32** (1993), 611–623.

**19.** W. Pardon and M. Stern, *$L^2$-$\overline{\partial}$-cohomology of complex projective varieties*, J. Amer. Math. Soc. **4** (1991), 603–621.

**20.** N. Wright, *$C_0$ coarse geometry and scalar curvature*, J. Functional Anal. **197** (2003), 469–488.

Mathematics Department, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0123
**Email address: haskell@math.vt.edu**

Gottfried Wilhelm Leibniz Bibliothek, Niedersächsische Landesbibliothek, Waterloostr. 8, 30169 Hannover, Germany
**Email address: ac.wahl@web.de**