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ON THE SIZE OF SETS IN WHICH $xy + 4$ IS ALWAYS A SQUARE

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ABSTRACT. In this paper, we prove that there does not exist a set of 7 positive integers such that the product of any two of its distinct elements increased by 4 is a perfect square.

1. Introduction. Let n be an integer. A set of m positive integers is called a Diophantine m-tuple with the property $D(n)$ or simply $D(n)$-m-tuple, if the product of any two of them increased by n is a perfect square.

The problem of finding such sets was first studied by Diophantus in the case $n = 1$. He found a set of four positive rationals with the above property:

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}.$$

However, the first $D(1)$-quadruple, the set $\{1, 3, 8, 120\}$, was found by Fermat. Later Euler was able to add the fifth positive rational, 777480/8288641, to Fermat's set, see [**5**], [**6**, pages 103–104, 232]. Recently, Gibbs [**17**] found examples of sets of six positive rationals with the property of Diophantus. The conjecture is that there does not exist a $D(1)$-quintuple. In 1969, Baker and Davenport [**1**] proved that Fermat's set cannot be extended to a $D(1)$-quintuple. Recently, Dujella, see [**11**], proved that there does not exist a $D(1)$-sextuple and there are only finitely many $D(1)$-quintuples. This implies that there does not exist a $D(4)$-8-tuple and that there are only finitely many $D(4)$-septuples, see [**15**]. In this paper we will improve this result.

In the case $n = 4$ the conjecture is that there does not exist a $D(4)$-quintuple. Actually there is a stronger version of that conjecture, see [**15**, Conjecture 1].

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Conjecture 1. *There does not exist a $D(4)$-quintuple. Moreover, if $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then*

$$d = a + b + c + \frac{1}{2}(abc + rst),$$

where r, s, t are positive integers defined by

$$ab + 4 = r^2, \quad ac + 4 = s^2, \quad bc + 4 = t^2.$$

It is easy to check that if $d = a+b+c+(abc+rst)/2$ then $\{a, b, c, d\}$ is a $D(4)$-quadruple. From now on we will denote this d_+. We will also define $d_- = a + b + c + (abc - rst)/2$. If $d_- \neq 0$, the set $\{a, b, c, d_-\}$ is also $D(4)$-quadruple, but $d_- < c$.

Definition 1. A $D(4)$-quadruple $\{a, b, c, d\}$ such that $d > \max\{a, b, c\}$ is called regular if $d = d_+$.

We have checked, using a computer program, that all $D(4)$-quadruples $\{a, b, c, d\}$ such that $\max\{a, b, c, d\} \leq 4 \cdot 10^7$ are regular, and we will use this result in our paper.

The first result of nonextendibility of $D(4)$-m-tuples was proven by Mohanty and Ramasamy in [**20**]. There they proved that $D(4)$-quadruple $\{1, 5, 12, 96\}$ cannot be extended to a $D(4)$-quintuple. Later Kedlaya, see [**18**], proved that if $\{1, 5, 12, d\}$ is a $D(4)$-quadruple, then $d = 96$.

One generalization of this result was given by Dujella and Ramasamy in [**15**] where they proved Conjecture 1 for a parametric family of $D(4)$-quadruples. Precisely, they proved that if k and d are positive integers and

$$\{F_{2k}, 5F_{2k}, 4F_{2k+2}, d\}$$

is a $D(4)$-quadruple, then $d = 4L_{2k}F_{4k+2}$, where F_k and L_k are the Fibonacci and Lucas numbers. The second generalization was given by Fujita in [**16**]. There he proved that if $k \geq 3$ is an integer and $\{k-2, k+2, 4k, d\}$ is a $D(4)$-quadruple, then $d = 4k^3 - 4k$. Both these results support Conjecture 1.

Our main result is the following theorem.

Theorem 1. *There does not exist a $D(4)$-septuple.*

In the proof of nonexistence of the $D(4)$-septuple we will mostly use the strategy and methods from [**11**]. First, we will transform the problem of extending the $D(4)$-triple $\{a, b, c\}$ to a quadruple, to solving a system of simultaneous Pellian equations. And this reduces to finding the intersection of binary recurrence sequences. By analysis of elements of the sequences with small indices, we will get some useful gap principles. Using congruence relations, we will get a lower bound for the solutions. In obtaining this bound we will assume that our triple satisfies some gap principles, precisely $c > \max\{b^{12}, 10^{29}\}$. Comparing this with the upper bound obtained from Bennett's theorem on simultaneous approximations of algebraic numbers, we will prove our main theorem.

2. System of Pellian equations. Let us fix some notation at the beginning. Let $\{a, b, c\}$ be a $D(4)$-triple such that $a < b < c$, and let r, s, t be positive integers defined by

$$ab + 4 = r^2, \quad ac + 4 = s^2, \quad bc + 4 = t^2.$$

If we want to extend $\{a, b, c\}$ to a $D(4)$-quadruple $\{a, b, c, d\}$, then we have to solve

$$ad + 4 = x^2, \quad bd + 4 = y^2, \quad cd + 4 = z^2,$$

with positive integers x, y, z. Eliminating d we get the following system of simultaneous Pellian equations

$$az^2 - cx^2 = 4(a - c), \tag{1}$$

$$bz^2 - cy^2 = 4(b - c). \tag{2}$$

From the theory of Pellian equations we can describe the sets of solutions of equations (1) and (2) in the following lemma.

Lemma 1. *There exist positive integers i_0, j_0 and $z_0^{(i)}, x_0^{(i)}, z_1^{(j)}, y_1^{(j)}$, $i = 1, \ldots, i_0$, $j = 1, \ldots, j_0$, with the following properties*:

(i) $(z_0^{(i)}, x_0^{(i)})$ *and* $(z_1^{(j)}, y_1^{(j)})$ *are solutions of* (1) *and* (2).

(ii) $z_0^{(i)}, x_0^{(i)}, z_1^{(j)}, y_1^{(j)}$ *satisfy the following inequalities*:

$$1 \le x_0^{(i)} \le \sqrt{\frac{a(c-a)}{s-2}} < \sqrt{s+2} < 1.236\sqrt[4]{ac}, \tag{3}$$

$$|z_0^{(i)}| \le \sqrt{\frac{(s-2)(c-a)}{a}} < \sqrt{\frac{c\sqrt{c}}{\sqrt{a}}} < 0.468c, \tag{4}$$

$$1 \le y_1^{(j)} \le \sqrt{\frac{b(c-b)}{t-2}} < \sqrt{t+2} < 1.122\sqrt[4]{bc}, \tag{5}$$

$$|z_1^{(j)}| \le \sqrt{\frac{(t-2)(c-b)}{b}} < \sqrt{\frac{c\sqrt{c}}{\sqrt{b}}} < 0.360c. \tag{6}$$

(iii) *If* (z, x) *and* (z, y) *are integer solutions of* (1) *and* (2), *then there exist* $i \in \{1, \dots, i_0\}$, $j \in \{1, \dots, j_0\}$, *and integers* $m, n \ge 0$ *such that*

$$z\sqrt{a} + x\sqrt{c} = \left(z_0^{(i)}\sqrt{a} + x_0^{(i)}\sqrt{c}\right)\left(\frac{s+\sqrt{ac}}{2}\right)^m, \tag{7}$$

$$z\sqrt{b} + y\sqrt{c} = \left(z_1^{(j)}\sqrt{b} + y_1^{(j)}\sqrt{c}\right)\left(\frac{t+\sqrt{bc}}{2}\right)^n. \tag{8}$$

In obtaining estimates (3)–(6) we have used the assumptions $ac \ge 21$ and $bc \ge 60$ because we know that the triple $\{1, 5, 12\}$ can be extended to the quadruple in the unique way, so we will not consider that triple. The rest follows from [**15**, Lemma 2].

Let (x, y, z) be a solution of the system of equations (1) and (2). Then from (7) we get $z = v_m^{(i)}$ for some index i and integer $m \ge 0$, where

$$v_0^{(i)} = z_0^{(i)},\ v_1^{(i)} = \frac{1}{2}\left(sz_0^{(i)} + cx_0^{(i)}\right), \quad v_{m+2}^{(i)} = sv_{m+1}^{(i)} - v_m^{(i)}. \tag{9}$$

From (8) we conclude that $z = w_n^{(j)}$ for some index j and integer $n \ge 0$, where

$$w_0^{(j)} = z_1^{(j)},\ w_1^{(j)} = \frac{1}{2}(tz_1^{(i)} + cy_1^{(j)}),\ w_{n+2}^{(j)} = tw_{n+1}^{(j)} - w_n^{(j)}. \tag{10}$$

By induction, using (9) and (10), the following lemma is easy to prove.

Lemma 2. *For the sequences* (v_m) *and* (w_n) *we have*

$$\begin{aligned} v_{2m}^{(i)} &\equiv v_0^{(i)} \pmod{c}, \\ v_{2m+1}^{(i)} &\equiv v_1^{(i)} \pmod{c}, \\ w_{2n}^{(j)} &\equiv w_0^{(j)} \pmod{c}, \\ w_{2n+1}^{(j)} &\equiv w_1^{(j)} \pmod{c}. \end{aligned}$$

For simplicity, from now on, we will omit indices i and j. Because we are looking for the solution of our system of equations (1) and (2) such that $d = (z^2 - 4)/c$ is an integer, from $z = v_m = w_n$, using Lemma 2, we get

$$z_0^2 \equiv z_1^2 \equiv 4 \pmod{c}.$$

Now we need one result that will give us information on the possible cs in the case $c < 4b$.

Lemma 3 *If* $\{a, b, c\}$ *is a* $D(4)$*-triple such that* $a < b < c$*, then* $c \geq 4b$ *or* $c = a + b + 2r$*.*

Proof. By [**10**, Lemma 3], there exist integers e, x', y', z' such that

$$ae + 16 = (x')^2, \quad be + 16 = (y')^2, \quad ce + 16 = (z')^2,$$

and

$$c = a + b + \frac{e}{4} + \frac{1}{8}(abe + rx'y').$$

We can take x' and y' to be positive integers. If $e = 0$, we get $c = a + b + 2r$. If $e > 0$, then $ae \geq 9$ so we can conclude

$$c > b + \frac{1}{8}\left(9b + \sqrt{ab} \cdot 5 \cdot \sqrt{be}\right) \geq b + \frac{1}{8}(9b + 15b) = 4b.$$

If $e < 0$, then $c \leq 16$, but such a triple does not exist, so we proved our lemma. $\square$

From the proof of the last lemma it follows that $c > 4a$.

Lemma 4. (i) *If the equation* $v_{2m} = w_{2n}$ *has a solution, then* $z_0 = z_1$.

(ii) *If the equation* $v_{2m+1} = w_{2n}$ *has a solution, then* $z_0 \cdot z_1 < 0$ *and* $|z_1| = (cx_0 - s|z_0|)/2$.

(iii) *If the equation* $v_{2m} = w_{2n+1}$ *has a solution, then* $z_0 \cdot z_1 < 0$ *and* $|z_0| = (cy_1 - t|z_1|)/2$.

(iv) *If the equation* $v_{2m+1} = w_{2n+1}$ *has a solution, then* $z_0 \cdot z_1 > 0$ *and* $cx_0 - s|z_0| = cy_1 - t|z_1|$.

Proof. Equation $v_{2m} = w_{2n}$, together with Lemmas 1 and 2, implies that $z_0 \equiv z_1 \pmod{c}$ and $|z_0 - z_1| < c$. This implies $z_0 = z_1$ which proves (i).

We now consider equation $v_{2m+1} = w_{2n}$. It is easy to see that

$$|(sz_0 + cx_0)(sz_0 - cx_0)| = 4c^2 - 4ac - 4z_0^2 < 4c^2$$

and

$$4c^2 - 4ac - 4z_0^2 > 4c^2 - c^2 - 4 \cdot 0.219c^2 > 0,$$

which implies that

$$0 < \frac{1}{2}(cx_0 - s|z_0|) < c.$$

Now from $z_1 \equiv (sz_0 + cx_0)/2 \pmod{c}$ we can conclude that if $z_0 > 0$, $z_1 = (sz_0 - cx_0)/2$, and if $z_0 < 0$, then $z_1 = (sz_0 + cx_0)/2$, which proves (ii).

In the case $v_{2m} = w_{2n+1}$, first it is easy to see

$$|(tz_1 + cy_1)(tz_1 - cy_1)| = 4c^2 - 4bc - 4z_1^2 < 4c^2.$$

Now if $c \geq 4b$, as in case (ii) we get $4c^2 - 4bc - 4z_1^2 > 0$. And if $c < 4b$, then $c = a + b + 2\sqrt{ab+4}$ and we get $c > b + 2\sqrt{b} + 1 = (\sqrt{b}+1)^2$, which implies $b < c - 2\sqrt{c} + 1$. Then we have

$$4c^2 - 4bc - 4z_1^2 > 4c^2 - 4(c - 2\sqrt{c} + 1)c - 4 \cdot 2c = 8c\sqrt{c} - 12c > 0.$$

Actually, we have proved

$$0 < \frac{1}{2}(cy_1 - t|z_1|) < c.$$

Now from $z_0 \equiv (tz_1 + cy_1)/2 \pmod{c}$ we conclude that if $z_1 > 0$, $z_0 = (tz_1 - cy_1)/2$, and if $z_1 < 0$, then $z_0 = (tz_1 + cy_1)/2$, which proves (iii).

Case (iv) can be proved on the same way as cases (ii) and (iii). □

3. The relationship between m and n. In this section we will prove an unconditional relationship between m and n. Later we will improve that result slightly by assuming some additional conditions.

Lemma 5. *If $v_m = w_n$, then $n - 1 \le m \le 2n + 1$.*

Proof. It is easy to get the following estimates for v_1:

$$\begin{aligned} v_1 &= \frac{1}{2}(sz_0 + cx_0) \ge \frac{1}{2}(cx_0 - s|z_0|) = \frac{1}{2} \cdot \frac{4c^2 - 4ac - 4z_0^2}{cx_0 + s|z_0|} \\ &> \frac{4c^2 - 4 \cdot (c^2/4) - 4z_0^2}{2 \cdot 2cx_0} > \frac{2c^2}{4cx_0} > \frac{c}{2x_0} > \frac{c}{2.472\sqrt[4]{ac}}, \\ v_1 &= \frac{1}{2}(sz_0 + cx_0) < \frac{1}{2} \cdot 2cx_0 = cx_0 < 1.236c\sqrt[4]{ac}. \end{aligned}$$

Then we conclude

$$\frac{c}{2.472\sqrt[4]{ac}}(s-1)^{m-1} < v_m < 1.236c\sqrt[4]{ac}s^{m-1},$$

for $m \ge 1$.

Now if $c \ge 4b$, similarly as above we get

$$w_1 = \frac{1}{2}(tz_1 + cy_1) > \frac{c}{2.224\sqrt[4]{bc}}.$$

If $c < 4b$, then we conclude from (6) that $|z_1| < \sqrt{(c\sqrt{c})/\sqrt{b}} < \sqrt{2c}$, and $|z_1|^2 < 2c$. Now from $|z_1|^2 \equiv 4 \pmod{c}$ we conclude that the only possibilities for z_1^2 are given by $z_1^2 = 4$ and $z_1^2 = c+4$. If $z_1^2 = c+4$, then from (2) we get $y_1^2 = b+4$. But z_1 and y_1 are integers and it is possible only if the set $\{1, b, c\}$ is a $D(4)$-triple. Then the relation $c < 4b$ and Lemma 3 imply $a = 1$ and $c = 1 + b + 2\sqrt{b+4}$. It is obviously a contradiction, because now condition (5) is not satisfied, actually the inequality $y_1 \le \sqrt{(b(c-b))/(t-2)}$ is not satisfied. So, we proved that, in the case $c < 4b$, we have $z_1 = \pm 2$, $y_1 = 2$. Then we conclude

$$w_1 = \frac{1}{2}(tz_1 + cy_1) \ge \frac{1}{2}(cy_1 - t|z_1|) = c - t > \sqrt{ac} > \frac{c}{2.224\sqrt[4]{bc}}.$$

On the other side, we have

$$w_1 < 1.112c\sqrt[4]{bc};$$

hence,

$$\frac{c}{2.224\sqrt[4]{bc}}(t-1)^{n-1} < w_n < 1.112c\sqrt[4]{bc}t^{n-1},$$

for $n \ge 1$.

Now $v_m = w_n$ for $m, n \ge 1$ implies

$$(s-1)^{m-1} < 2.749\sqrt[4]{abc^2}t^{n-1}.$$

Since $s - 1 = \sqrt{ac+4} - 1 > 0.781\sqrt{ac}$ and $t = \sqrt{bc+4} < 1.033\sqrt{bc}$, we get $(s-1)^2 > 0.61ac > t$, if $a > 1$ or $c \ge 4b$. But it is easy to see that $(s-1)^2 > t$ also holds for $a = 1$ and $c = 1 + b + 2\sqrt{b+4}$. Now we get

$$(s-1)^{m-1} < 2.749t^n < t^{n+0.27} < (s-1)^{2n+0.54},$$

and $m \le 2n+1$. $v_m = w_n$ also implies

$$(t-1)^{n-1} < 2.749\sqrt[4]{abc^2}s^{m-1} < 2.749\sqrt[4]{abc^2}(t-1)^{m-1} < (t-1)^{m+0.27},$$

and $n - 1 \le m$.

To finish the proof of our lemma, we have to check inequalities $w_2 > v_0$ and $v_2 > w_0$. Now we have

$$w_2 = tw_1 - w_0 > \frac{ct}{2.224\sqrt[4]{bc}} - \sqrt{\frac{c\sqrt{c}}{\sqrt{b}}} > c\left(\frac{\sqrt[4]{bc}}{2.224} - \frac{1}{\sqrt[4]{bc}}\right) > 0.892c > v_0,$$

$$v_2 = sv_1 - v_0 > \frac{cs}{2.472\sqrt[4]{ac}} - \sqrt{\frac{c\sqrt{c}}{\sqrt{a}}} > c\left(\frac{\sqrt[4]{ac}}{2.472} - \frac{1}{\sqrt[4]{ac}}\right) > 0.398c > w_0. \quad \square$$

4. Gap principles. In this section we will get gap principles for the elements of $D(4)$-quadruples by considering equation $v_m = w_n$ for small values of m and n.

Lemma 6. *Let* $v_m = w_n$ *and define* $d = (v_m^2 - 4)/c$. *If* $\{0,1,2\} \cap \{m,n\} \neq \varnothing$, *then* $d < c$ *or* $d = d_+$.

Proof. From the proof of Lemma 5 we conclude

$$v_0 < w_2,\ w_0 < v_2,\ v_1 < w_3,\ w_1 < v_4,\ v_2 < w_4,\ w_2 < v_6.$$

Then condition $\{0,1,2\} \cap \{m,n\} \neq \varnothing$, implies $(m,n) \in S$, where

$$S = \{(0,0),\ (0,1),\ (1,0),\ (1,1),\ (1,2),\ (2,1),\ (2,2),\ (3,1),\\ (2,3),\ (3,2),\ (4,2),\ (5,2)\}.$$

If $0 \in \{m,n\}$, then from Lemma 1 we get $d < c$.

Assume that $(m,n) = (1,1)$. If $z_0 < 0$, then

$$z = v_1 = \frac{1}{2}(sz_0 + cx_0) < \frac{1}{2} \cdot 2c = c,$$

and $d < c$. If $z_0 > 0$, then from Lemma 4 we get $sz_0 - cx_0 = tz_1 - cy_1$. But from $z = v_1 = w_1$ we conclude $sz_0 + cx_0 = tz_1 + cy_1$, and $x_0 = y_1$ and $sz_0 = tz_1$. Hence,

$$az_0^2 - cx_0^2 = 4(a-c), \quad bz_1^2 - cy_1^2 = 4(b-c),$$

implies

$$\begin{aligned} 4(b-a)s^2 = (bz_1^2 - az_0^2)s^2 &= \left(bz_1^2 - a \cdot \frac{t^2 z_1^2}{s^2}\right) \cdots^2 \\ &= (b(ac+4) - a(bc+4))z_1^2 = 4(b-a)z_1^2. \end{aligned}$$

Then we conclude $z_1 = s$, $z_0 = t$, $x_0 = y_1 = r$, which implies $z = v_1 = (st + cr)/2$ and therefore $d = d_+$.

Assume now that $(m,n) = (1,2)$. Then

$$v_1 = \frac{1}{2}(sz_0 + cx_0), \quad w_2 = z_1 + \frac{1}{2}c(bz_1 + ty_1).$$

From Lemma 4, if $z_1 > 0$, then $z_0 < 0$ and $z_1 = (cx_0 + sz_0)/2$, $w_2 > v_1 = w_0$. If $z_1 < 0$, then $z_0 > 0$ and

$$-z_1 = \frac{1}{2}(cx_0 - sz_0). \tag{11}$$

If we insert this in equation $v_1 = w_2$, we get

$$x_0 = \frac{1}{2}(bz_1 + ty_1). \tag{12}$$

Now from (11) and (12) and the system of equations (1) and (2) we conclude $4(b-a)t^2 = 4z_0^2(b-a)$. Then $z_0 = t$, $x_0 = r$, $z_1 = (st-cr)/2$, $y_1 = (rt - bs)/2$, which implies that

$$z = v_1 = \frac{1}{2}(st + cr), \quad d = \frac{z^2 - 4}{c} = d_+.$$

The case $(m, n) = (2, 1)$ is completely analogous to the case $(m, n) = (1, 2)$.

Let $(m, n) = (2, 2)$. Then

$$v_2 = z_0 + \frac{1}{2}c(az_0 + sx_0), \quad w_2 = z_1 + \frac{1}{2}c(bz_1 + ty_1).$$

Hence $v_2 = w_2$ implies $z_0 = z_1$ and $az_0 + sx_0 = bz_0 + ty_1$. Moreover, we get

$$(b-a)(cy_1^2 - cx_0^2 + 4(b-a)) = (b-a)^2 z_0^2 = (sx_0 - ty_1)^2,$$

which implies $4(b-a)^2 = (sy_1 - tx_0)^2$. Since $sy_1 < tx_0$, we have $b - a = (tx_0 - sy_1)/2$. Furthermore, we have

$$(ac+4)(bx_0^2 + 4(a-b)) = a(tx_0 - 2(b-a))^2$$

and

$$4(b-a)(x_0^2 + atx_0 + \frac{1}{4}a^2t^2 = (b-a)r^2s^2.$$

Then $x_0 + (at)/2 = (rs)/2$, $x_0 = (rs - at)/2$, $y_1 = (rt - bs)/2$ and $z_0 = (st - cr)/2$. This implies

$$v_2 = \frac{1}{2}(st - cr) + \frac{1}{2}c\left(a(st - cr) + \frac{1}{2}rs^2 - \frac{1}{2}ast\right) = \frac{1}{2}(st + cr).$$

Finally we get

$$z = v_2 = \frac{1}{2}(st + cr), \quad d = \frac{z^2 - 4}{c} = d_+.$$

Assume now $(m, n) = (3, 1)$. In the proof of Lemma 5 we showed

$$(s-1)^2 < 2.749\sqrt[4]{abc^2}. \tag{13}$$

From $s - 1 > 0.781\sqrt{ac}$, using $c > 4a$ we conclude

$$0.609ac < 1.943\sqrt{[4]bc^3} < 1.943c. \tag{14}$$

Now, if $a > 3$ we get a contradiction. If $a = 3$, from (14) we get $c < 15$, but such a $D(4)$-triple does not exist. If $a = 2$, (14) implies $c < 51$, so the only possible triples are $\{2, 6, 16\}$, $\{2, 16, 30\}$ and $\{2, 30, 48\}$. But then from $v_3 > c/(2.472\sqrt[4]{ac})(s-1)^2$ and $w_1 < cy < c\sqrt{(b(c-b))/(t-2)}$, we get in all three cases, $v_3 > w_1$, which is a contradiction. If $a = 1$ and $c \geq 4b$, relation (13) implies $c < 26$, but it is easy to see that such a triple does not exist. If $c < 4b$, Lemma 3 implies $c = 1 + b + 2\sqrt{b+4}$, which together with (13) gives us $b \leq 77$. So we have to check what is happening for the remaining values of b. But again, in all cases we get $v_3 > w_1$.

Let $(m, n) = (3, 2)$. Assume that $z_0 > 0$. Then $z_1 < 0$ and $v_1 > 1/2 \cdot 2sz_0 = sz_0$. From that we conclude

$$v_3 > (s-1)^2 sz_0 > 0.609(ac)^{3/2} z_0.$$

On the other hand,

$$w_1 < \frac{1}{2} \cdot \frac{4c^2}{2t|z_1|} = \frac{c^2}{t|z_1|}$$

and

$$w_2 < tw_1 < \frac{c^2}{|z_1|}.$$

Since $-z_1 = (cx_0 - sz_0)/2$ we conclude $|z_1| > c/(2x_0)$. Now we have

$$w_2 < 2cx_0 < 2 \cdot 1.236c\sqrt[4]{ac} < 1.75c\sqrt{c}. \tag{15}$$

We want to prove $w_2 < v_3$, which will give us the contradiction. It is obviously valid if $1.75c^{3/2} < 0.609(ac)^{3/2}z_0$. Since $z_0^2 \equiv 4 \pmod{c}$ we can conclude $z_0 \geq 2$, and the inequality will always be true if $a > 1$. If $a = 1$, when we insert this in (15), $w_2 < v_3$ will be valid when $2.472c^{5/4} < 1.218c^{3/2}$. But it is true for $c \geq 17$, and c cannot be smaller because such a $D(4)$-triple does not exist. So we proved $w_2 < v_3$ for $z_0 > 0$.

Let $z_0 < 0$. Then $z_1 > 0$ and $z_1 = (sz_0 + cx_0)/2$. Now, $v_3 = w_2$ implies that

$$2x_0 + 2az_1 = bz_1 + ty_1 > 2bz_1,$$

which gives us

$$2x_0 < 4z_1 < 4 \cdot \frac{c}{2x_0},$$

and $x_0^2 > c$, which is a contradiction to (3).

Assume that $(m, n) = (2, 3)$. If $z_0 > 0$, then $z_1 < 0$ and $v_2 = w_3$ imply

$$2y_1 + 2bz_0 = az_0 + sx_0 < 2az_0 + \frac{2c}{z_0}.$$

We conclude that $z_0^2 < c/2$. On the other hand, $z_0^2 \equiv 4 \pmod{c}$, so we get $z_0 = 2$. If $c \geq 4b$ we have $z_0 > c/(2y_1) > 2$, which is a contradiction. If $c < 4b$, we have proven $z_1 = -2$ and $y_1 = 2$. Now, from Lemma 4, part (iii), we conclude that $2 = (2c - 2t)/2$ and $2 = c - \sqrt{bc+4}$. Then we get $c = b + 4$, which contradicts $c = a + b + 2\sqrt{ab+4}$.

If $z_0 < 0$, then $z_1 > 0$. As in case $(m, n) = (3, 2)$, we get $w_1 > (1/2) \cdot 2tz_1 = tz_1$. Then we conclude

$$w_3 > (t-1)^2 tz_1 > 0.86(bc)^{3/2} z_1,$$

if $bc > 60$. On the other hand, we have

$$v_1 < \frac{1}{2} \cdot \frac{4c^2}{2s|z_0|} = \frac{c^2}{s|z_0|}$$

and $v_2 < sv_1 < c^2/|z_0|$. If $c \geq 4b$, since $y_1 < \sqrt{c}$, we get $|z_0| > c/2y_1$. Then

$$v_2 < 2cy_1 < 2.224c\sqrt[4]{bc} < 1.58c^{3/2}.$$

We also know that $w_3 > 1.72c^{3/2}$. So we proved that for $c \geq 4b$ and $bc > 60$ we have $w_3 > v_2$, which is a contradiction. If $bc = 60$ and $t = 8$, it is easy to see that $v_2 < w_3$. If $c < 4b$, then

$$w_3 > 0.215c^3 > c^2 > v_2,$$

which shows that we cannot have $v_2 = w_3$.

If $(m, n) = (4, 2)$, we have

$$v_4 = z_0 + c(2az_0 + sx_0) + \frac{1}{2}ac^2(az_0 + sx_0),$$
$$w_2 = z_1 + \frac{1}{2}c(bz_1 + ty_1).$$

Then $v_4 = w_2$ implies $z_0 = z_1$ and

$$bz_0 + ty_1 = 4az_0 + 2sx_0 + ac(az_0 + sx_0). \tag{16}$$

Furthermore, we have $ty_1 - b|z_0| < (2c)/|z_0|$. Then, if $z_0 > 0$, the lefthand side of (16) is less than or equal to $2bz_0 + (2c)/z_0$, when the righthand side is greater than $2a^2cz_0$. So if $a > 1$ or $c \geq 4b$ we get a contradiction right away. So let $a = 1$ and $c = 1 + b + 2\sqrt{b+4}$. Then

$$bz_0 + ty_1 \leq 2bz_0 + c$$

and

$$4az_0 + 2sx_0 + ac(az_0 + sx_0) \geq 2cz_0 = c + (2c-1)z_0 > c + 2bz_0,$$

so we cannot have equality. If $z_0 < 0$, we have that the lefthand side of (16) is less than or equal to $(2c)/|z_0| \leq c$. To estimate the righthand side, we have to show $sx_0 - a|z_0| \geq 3$. We get that from

$$\begin{aligned} sx_0 - a|z_0| &= \frac{s^2x_0^2 - a^2z_0^2}{sx_0 + az_0} > \frac{(ac+4)x_0^2 - a^2z_0^2}{2sx_0} \\ &= \frac{a \cdot 4(c-a) + 4x_0^2}{2sx_0} > \frac{4s^2 - 4a^2}{2sx_0} > \frac{4s^2 - 4a^2}{2.475s\sqrt[4]{ac}} \\ &> \frac{4s^2 - s^2}{2.475s\sqrt[4]{ac}} > \frac{s}{\sqrt[4]{ac}} > 3. \end{aligned}$$

Now we have the estimate

$$4az_0 + 2sx_0 + ac(az_0 + sx_0) \geq 3ac - 4a|z_0| > ac \geq c,$$

which gets us a contradiction again.

Assume $(m, n) = (5, 2)$. From the proof of Lemma 5, we get

$$0.37a^2c^2 < 2.749\sqrt[4]{abc^2} \cdot t < 2.840bc. \tag{17}$$

Now if $a \geq 3$, we get a contradiction in (17). If $a = 2$ and $c \geq 4b$, we see again that (17) cannot hold. If $a = 2$ and $c = 2+b+2\sqrt{2b+4}$, relation (17) implies $b < c < 2.714b^{3/4}$ and $b \leq 54$. But for all remaining values of b it is easy to see that $c < 2.714b^{3/4}$ is false. In the case $a = 1$ and $c \geq 4b$ from (17) we get $c \leq 58$ and $b \leq 14$, but such a $D(4)$-triple does not exist. Finally, if $a = 1$ and $c = 1 + b + 2\sqrt{b+4}$ from (17) we get $b \leq 3731$. But for all the remaining values of b, it is easy to see that $v_5 > w_2$. □

Lemma 7. *If $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then $d = d_+$ or $d \geq 0.116c^{2.5}b^{1.5}$.*

Proof. From Lemma 6, if $d \neq d_+$, then $z = v_m = w_n$ for some $m, n \geq 3$ and $d = (z^2 - 4)/c$. We have

$$z \geq w_3 > (t-1)^2 \cdot \frac{c}{2.224\sqrt[4]{bc}} > \frac{0.758}{2.224}\sqrt[4]{b^3c^3} \cdot c > 0.341\sqrt[4]{b^3c^3} \cdot c,$$

and

$$d \geq \frac{0.1163b^{1.5}c^{3.5} - 4}{c} > 0.116c^{2.5}b^{1.5}. \qquad \square$$

Using the gap principle from Lemma 7 we can prove the following lemma.

Lemma 8. *In the notation as above, we have $v_3 \neq w_3$.*

Proof. Assume that $v_3 = w_3$. Define

$$z' = \frac{1}{2}(cx_0 - sz_0) = \frac{1}{2}(cy_1 - tz_1),$$

if $z_0,\ z_1 > 0$, and

$$z' = \frac{1}{2}(cx_0 + sz_0) = \frac{1}{2}(cy_1 + tz_1),$$

if $z_0,\ z_1 < 0$. Also define $d_0 = (z'^2 - 4)/c$. Then d_0 is an integer and

$$cd_0 + 4 = z'^2, ad_0 + 4 = \left(\frac{1}{2}(sx_0 \pm az_0)\right)^2, bd_0 + 4 = (ty_1 \pm bz_1))^2.$$

Now from the proof of Lemma 5, we get

$$|z'| = \frac{1}{2} \cdot \frac{4c^2 - 4ac - 4z_0^2}{cx_0 + s|z_0|} > \frac{c}{2.472\sqrt{[4]ac}}$$

and $|z'| < c$. Then we conclude

$$d_0 > \frac{0.163(c\sqrt{c}/\sqrt{a}) - 4}{c} > 0.$$

Hence, the set $\{a, b, c, d_0\}$ is a $D(4)$-quadruple. Since $d_0 < c$, we have two possibilities, depending upon whether the quadruple is regular or not. If $\{a, b, c, d_0\}$ is a regular $D(4)$-quadruple, then $d_0 = d_-$. This implies that $z' = (cr - st)/2$. Now we have $c(x_0 - r) = s(|z_0| - t)$. If c is odd, we have $(c, s) = 1$ so $|z_0| \equiv t \pmod{c}$. From the relations $|z_0| < c$, $t < c$, we conclude $|z_0| = t$, $x_0 = r$. In exactly the same way, we get $|z_1| = s$, $y_1 = r$. Considering the case when c is even, we can get one more possibility: $|z_0| = t - (c/2)$, $x_0 = r - (s/2)$.

The relation $v_3 = w_3$ implies

$$\frac{1}{2}sz_0 + \frac{3}{2}cx_0 + \frac{1}{2}ac(cx_0 + sz_0) = \frac{1}{2}tz_1 + \frac{3}{2}cy_1 + \frac{1}{2}bc(cy_1 + tz_1). \tag{18}$$

Now if $|z_0| = t$, $x_0 = r$, from (18) we get $a = b$, which is a contradiction. If $|z_0| = t - (c/2)$, $x_0 = r - (s/2)$, using (18) we get a contradiction because the lefthand side is always less than the righthand side.

So the $D(4)$-quadruple $\{a, b, c, d_0\}$ is not regular. Then $c > 4 \cdot 10^7$ (see the remark after Definition 1) and Lemma 7 implies

$$c > 0.116 d_0^{2.5} b^{1.5}. \tag{19}$$

We have

$$|z'0| = \frac{1}{2} \cdot \frac{4c^2 - 4ac - 4z_0^2}{cx_0 + s|z_0|} > \frac{2.987c}{4x_0} > 0.746\frac{c}{\sqrt[4]{ac}}$$

and

$$d_0 > \frac{0.5565(c\sqrt{c}/\sqrt{a}) - 4}{c} > 0.556\sqrt{\frac{c}{a}}.$$

If we insert this in (19) we get

$$c > 0.0267c^{1.25}a^{-1.25}b^{1.5} > 0.0267c^{1.25}a^{0.25},$$

and $ac < 1967683$, which contradicts $c > 4 \cdot 10^7$. $\square$

We are now ready to prove the stronger gap principle, that we will use in the proof of Theorem 1.

Proposition 1. *If $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then $d = d_+$ or $d > 0.036c^{3.5}a^{2.5}$.*

Proof. From Lemmas 5 and 7 we conclude that $d = d_+$ or $z = v_m = w_n$, such that $m \geq 4$ or $n \geq 4$. In the case $m \geq 4$, we have that

$$z = v_4 \geq \frac{c}{2.472\sqrt[4]{ac}}(s-1)^3 > 0.192\sqrt[4]{a^5c^5} \cdot c$$

and

$$d \geq \frac{0.0369a^{2.5}c^{4.5} - 4}{c} > 0.036c^{3.5}a^{2.5}.$$

If $n \geq 4$, we have

$$z = w_4 \geq \frac{c}{2.224\sqrt[4]{bc}}(t-1)^3 > 0.297\sqrt[4]{b^5c^5} \cdot c$$

and

$$d \geq \frac{0.0882b^{2.5}c^{4.5} - 4}{c} > 0.087c^{3.5}b^{2.5} > 0.0036c^{3.5}a^{2.5}. \qquad \square$$

Corollary 1. *If* $\{a,b,c,d,e\}$ *is a* $D(4)$*-quintuple such that* $a<b<c<d<e$, *then* $e>0.036d^{3.5}b^{2.5}$.

Proof. If $\{b,c,d,e\}$ is a regular $D(4)$-quadruple, then

$$e \le d(bc+4) < d\left(\frac{1}{4}d(d-1)+4\right) < \frac{1}{4}d^3.$$

Thus, the $D(4)$-quadruple $\{a,c,d,e\}$ is not regular, so Proposition 1 implies

$$e > 0.036d^{3.5}a^{2.5} > \frac{1}{4}d^3,$$

which is a contradiction. Now the statement of the corollary follows from Proposition 1, because the quadruple $\{b,c,d,e\}$ is not regular. □

Now, using gap principles we will refine Lemma 4 and get more detailed information about initial values of (v_m) and (w_n).

Lemma 9. (i) *If the equation* $v_{2m}=w_{2n}$ *has a solution, then* $z_0=z_1$. *Moreover,* $|z_0|=2$ *or* $|z_0|=(cr-st)/2$ *or* $|z_0|<1.608a^{-(5/14)}c^{(9/14)}$.

(ii) *If the equation* $v_{2m+1}=w_{2n}$ *has a solution, then* $|z_0|=t$, $|z_1|=(cr-st)/2$, $z_0z_1<0$.

(iii) *If the equation* $v_{2m}=w_{2n+1}$ *has a solution, then* $|z_1|=s$, $|z_0|=(cr-st)/2$, $z_0z_1<0$.

(iv) *If the equation* $v_{2m+1}=w_{2n+1}$ *has a solution, then* $|z_0|=t$, $|z_1|=s$, $z_0z_1>0$.

Proof. (i) From Lemma 4 we get $z_0=z_1$. Define $d_0=(z_0^2-4)/c$. Then d_0 is an integer and

$$cd_0+4=z_0^2, \quad ad_0+4=x_0^2, \quad bd_0+4=y_1^2.$$

Now we have three possibilities for d_0. First, if $d_0=0$, we conclude $|z_0|=2$. If $d_0\ne 0$, then $\{a,b,c,d_0\}$ is a $D(4)$-quadruple. If it is regular, then $d_0=d_-$ and $|z_0|=(cr-st)/2$. Note, we showed that $d_0<c$.

If $\{a, b, c, d_0\}$ is not a regular $D(4)$-quadruple, then we conclude from Proposition 1 that

$$c \geq 0.036 d_0^{3.5} a^{2.5}. \tag{20}$$

Since $|z_0| \neq 2$, we have $z_0^2 \geq c+4$ and from $c > 4 \cdot 10^7$, we get

$$d_0 = \frac{z_0^2 - 4}{c} \geq \frac{z_0^2}{c}\left(1 - \frac{4}{c+4}\right) > 0.999 \frac{z_0^2}{c}.$$

If we insert this in (20), we have $c^{4.5} > 0.035 |z_0|^7 a^{2.5}$ and $|z_0| < 1.608 a^{-(5/14)} c^{(9/14)}$.

(ii) Define $z' = z_1 = (cx_0 + sz_0)/2$ if $z_1 > 0$, and $z' = -z_1 = (cx_0 - sz_0)/2$ if $z_1 < 0$. Moreover, define $d_0 = (z'^2 - 4)/c$. Then d_0 is an integer and

$$cd_0 + 4 = z'^2, \quad ad_0 + 4 = \left(\frac{1}{2}(sx_0 \pm az_0)\right)^2, \quad bd_0 + 4 = y_1^2.$$

In the proof of Lemma 8 we showed $0 < d_0 < c$ and $D(4)$-quadruple $\{a, b, c, d_0\}$ is regular. Then $d_0 = d_-$ and $|z_1| = (cr - st)/2$. This implies $|z_0| = t$, $|z_1| = (cr - st)/2$. It is easy to check that if c is even, relation $|z_0| = t - c/2$, $x_0 = r - s/2$ cannot be valid, because if we insert this in (1), we get

$$a\left(t - \frac{c}{2}\right)^2 - c\left(r - \frac{s}{2}\right)^2 = 4(a - c),$$

and

$$1 = rs - at,$$

which is impossible because s and t are even.

The last two statements can be proved in a similar manner. $\square$

We will now get the inequality with linear form in logarithms of algebraic numbers, which will be used to refine Lemma 5 in the special case.

Lemma 10. *If* $v_m = w_n$, $m, n \neq 0$, *then*

$$0 < m \log\left(\frac{s+\sqrt{ac}}{2}\right) - n \log\left(\frac{t+\sqrt{bc}}{2}\right) + \log \frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})}$$
$$< 2ac\left(\frac{s+\sqrt{ac}}{2}\right)^{-2m}.$$

Proof. From recurrence relations we have

$$v_m = \frac{1}{2\sqrt{a}}\Bigg[\left(z_0\sqrt{a}+x_0\sqrt{c}\right)\left(\frac{s+\sqrt{ac}}{2}\right)^m + \left(z_0\sqrt{a}-x_0\sqrt{c}\right)\left(\frac{s-\sqrt{ac}}{2}\right)^m\Bigg]$$

and

$$w_n = \frac{1}{2\sqrt{b}}\left[\left(z_1\sqrt{b}+y_1\sqrt{c}\right)\left(\frac{t+\sqrt{bc}}{2}\right)^n + \left(z_1\sqrt{b}-y_1\sqrt{c}\right)\left(\frac{t-\sqrt{bc}}{2}\right)^n\right].$$

If we define

$$P = \frac{1}{\sqrt{a}}\left(z_0\sqrt{a}+x_0\sqrt{c}\right)\left(\frac{s+\sqrt{ac}}{2}\right)^m,$$
$$Q = \frac{1}{\sqrt{b}}\left(z_1\sqrt{b}+y_1\sqrt{c}\right)\left(\frac{t+\sqrt{bc}}{2}\right)^n,$$

then $v_m = w_n$ implies

$$P - \frac{4(c-a)}{a}P^{-1} = Q - \frac{4(c-b)}{b}Q^{-1}.$$

Furthermore, it is easy to see that $P > 1$, $Q > 1$ and $P > Q$. Moreover, $(P-Q)/P < (4(c-a)/a)P^{-2} < 1/2$, since $P > 3\sqrt{c/a}$. Now the inequality from [**22**, Lemma B.2] implies

$$\begin{aligned}
0 < \log\frac{P}{Q} &= -\log(1-\frac{P-Q}{P}) \leq \frac{-\log(1-(1/2))}{1/2}\cdot\frac{4(c-a)}{a}P^{-2}\\
&< \frac{8(c-a)}{a}P^{-2} = \frac{8(c-a)}{a}\cdot\frac{a}{(z_0\sqrt{a}+x_0\sqrt{c})^2}\left(\frac{s+\sqrt{ac}}{2}\right)^{-2m}\\
&= \frac{(z_0\sqrt{a}-x_0\sqrt{c})^2}{2(c-a)}\left(\frac{s+\sqrt{ac}}{2}\right)^{-2m}\\
&\leq \frac{(|z_0|\sqrt{a}+x_0\sqrt{c})^2}{c-a}\left(\frac{s+\sqrt{ac}}{2}\right)^{-2m}\\
&\leq \frac{4x_0^2c}{3/4c}\left(\frac{s+\sqrt{ac}}{2}\right)^{-2m} < 2ac\left(\frac{s+\sqrt{ac}}{2}\right)^{-2m}. \qquad \square
\end{aligned}$$

Lemma 11. *If* $c > \max\{b^{12}, 10^{29}\}$, *then* $v_m = w_n$, $n > 2$, *implies* $m \leq 3/2n$.

Proof. From Lemma 10 we get

$$\frac{m}{n} < \frac{\log(t+\sqrt{bc}/2)}{\log(s+\sqrt{ac}/2)} - \frac{\log\gamma}{n\log(s+\sqrt{ac}/2)} + \frac{2ac(s+\sqrt{ac}/2)^{-2m}}{n\log(s+\sqrt{ac}/2)},$$

where $\gamma = (\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a}))/(\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b}))$. We will now estimate all three summands. We have

$$\begin{aligned}
\frac{\log((t+\sqrt{bc})/2)}{\log((s+\sqrt{ac})/2)} &= 1 + \frac{\log((t+\sqrt{bc})/(s+\sqrt{ac}))}{\log((s+\sqrt{ac})/2)}\\
&< 1 + \frac{\log\sqrt{b/a}}{\log((s+\sqrt{ac})/2)} < 1 + \frac{1}{12},
\end{aligned}$$

$$\begin{aligned}
\gamma &\geq \frac{\sqrt{b}(x_0\sqrt{c}-|z_0|\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+|z_1|\sqrt{b})} = \frac{\sqrt{b}\cdot 4(c-b)}{\sqrt{a}(x_0\sqrt{c}+|z_0|\sqrt{a})(y_1\sqrt{c}+|z_1|)}\\
&\geq \frac{\sqrt{b}\cdot 4(c-a)}{\sqrt{a}\cdot 2x_0\sqrt{c}\cdot 2y_1\sqrt{c}} > c^{-0.5437},
\end{aligned}$$

where we used $c > b^{12}$ and estimates for x_0 and y_1. From that we conclude

$$\frac{\log \gamma}{n \log((s+\sqrt{ac})/2)} > \frac{-0.5437}{0.5n} > -0.3625,$$

since we know that $n \geq 3$. Finally,

$$\frac{2ac((s+\sqrt{ac})/2)^{-2m}}{n \log((s+\sqrt{ac})/2)} < \frac{2ac}{3(ac)^2 \log(\sqrt{ac})} < 0.001.$$

When we sum all estimates we get the statement. $\square$

5. There does not exist a $D(4)$-septuple. In this section we will finish the proof of Theorem 1. The following lemma can be easily proven by induction.

Lemma 12. *If c is odd, then*

$$v_{2m} \equiv z_0 + \frac{1}{2}c(az_0m^2 + sx_0m) \pmod{c^2},$$

$$v_{2m+1} \equiv \frac{1}{2}sz_0 + \frac{1}{2}c\Big(\frac{1}{2}asz_0m(m+1) + x_0(2m+1)\Big) \pmod{c^2},$$

$$w_{2n} \equiv z_1 + \frac{1}{2}c(bz_1n^2 + ty_1n) \pmod{c^2},$$

$$w_{2n+1} \equiv \frac{1}{2}tz_1 + \frac{1}{2}c\Big(\frac{1}{2}btz_1n(n+1) + y_1(2n+1)\Big) \pmod{c^2}.$$

If c is even, then

$$v_{2m} \equiv z_0 + \frac{1}{2}c(az_0m^2 + sx_0m)\Big(\bmod \frac{1}{2}c^2\Big),$$

$$v_{2m+1} \equiv \frac{1}{2}sz_0 + \frac{1}{2}c\Big(\frac{1}{2}asz_0m(m+1) + x_0(2m+1)\Big)\Big(\bmod \frac{1}{2}c^2\Big),$$

$$w_{2n} \equiv z_1 + \frac{1}{2}c(bz_1n^2 + ty_1n)\Big(\bmod \frac{1}{2}c^2\Big),$$

$$w_{2n+1} \equiv \frac{1}{2}tz_1 + \frac{1}{2}c(\frac{1}{2}btz_1n(n+1) + y_1(2n+1))\Big(\bmod \frac{1}{2}c^2\Big).$$

Now we will get the lower bound of n, depending on c. We will prove the cases from Lemma 9.

Lemma 13. *Let $\{a,b,c\}$ be a $D(4)$-triple such that $c>\max\{b^{12},10^{29}\}$. Then $v_m=w_n$, $n>2$, implies $n>c^{0.06}$.*

Proof. Assume the opposite, that $v_m=w_n$, $n>2$ and $n\le c^{0.06}$.

Case 1.1. $v_{2m}=w_{2n}$, $|z_0|=2$. Lemma 12 implies

$$\pm am^2+sm\equiv \pm bn^2+tn \pmod c.$$

We also have the following estimates

$$am^2<c^{(1/12)+0.12}<\frac{1}{2}c,\quad sm<c^{(7/12)+0.06}<\frac{1}{2}c.$$

In the same manner, we get bn^2, $tn<1/2c$. Then our congruence becomes the equality

$$\pm am^2+sm=\pm bn^2+tn.$$

If we square it twice we get

$$((am^2-bn^2)^2-4m^2-4n^2)^2\equiv 64m^2n^2 \pmod c.$$

But again the absolute value of both sides is less than c. We have

$$64m^2n^2<c^{0.1+0.12}<c$$

and

$$((am^2-bn^2)^2-4m^2-4n^2)^2<c^{0.34+0.48}<c.$$

Then we have equality again so we get $m(s\pm 2)=n(t\pm 2)$ and

$$n=\frac{(s\pm 2)\,[t(s\pm 2)-(t\pm 2)s]}{\pm\,[a(t\pm 2)^2-b(s\pm 2)^2]}=\frac{(s\pm 2)(\pm t\mp s)}{\pm(\pm 4at+8a\pm 4bs-8b)}.$$

Finally, from the estimates

$$\begin{aligned}|(s\pm 2)(\pm t\mp s)|\ge (s-2)(t-s)&=\frac{(s-2)c(b-a)}{t+s}\\ &>\frac{2c(s-2)}{2\sqrt{bc}}>c\cdot\frac{\sqrt{a}}{2\sqrt{b}}\end{aligned}$$

and
$$| \pm 4at + 8a \pm 4bs - 8b| \le 8bs + 16b < 16b\sqrt{ac},$$
we get
$$n > \frac{\sqrt{c}}{32\sqrt{b}} > c^{0.279},$$
a contradiction.

Case 1.2. $v_{2m} = w_{2n}$, $|z_0| = (cr - st)/2$. We have
$$|z_0| = |z_1| = \frac{4c^2 - 4ac - 4bc - 16}{cr + st} > \frac{3c^2}{2rc} > \frac{3c}{2.309\sqrt{ab}}.$$
Then $|z_1| < \sqrt{(c\sqrt{c})/\sqrt{b}}$ implies
$$c < 0.352a^2b < 0.352b^2 < b^{12},$$
a contradiction.

Case 1.3. $v_{2m} = w_{2n}$, $|z_0| \neq 2$, $(cr - st)/2$. Then from Lemma 12 we get
$$az_0m^2 + sx_0m \equiv bz_0n^2 + ty_1n \pmod{c}.$$
We have the following estimates
$$|az_0m^2| < a \cdot \sqrt{\frac{c\sqrt{c}}{\sqrt{a}}} \cdot c^{0.12} = a^{0.75} \cdot c^{0.75+0.12} < \frac{1}{2}c,$$
$$|sx_0m| < c^{0.55+0.28+0.06} = c^{0.89} < \frac{1}{2}c.$$
In exactly the same way we conclude that the absolute value of both summands on the righthand side is less $1/2c$. Then we have equality again,
$$az_0m^2 + sx_0m = bz_0n^2 + ty_1n.$$
If $b > 4a$, then $z_0^2 > \max\{c + 4, (5c/a)\}$. Now we have
$$0 \le \frac{sx_0}{a|z_0|} - 1 = \frac{4x_0^2 + 4ac - 4a^2}{a|z_0|(sx_0 + a|z_0|)} \le \frac{4 \cdot 1.001ac}{2a^2z_0^2} < 0.401$$

and

$$0 \le \frac{ty_1}{b|z_0|} - 1 = \frac{4y_1^2 + 4bc - 4b^2}{b|z_0|(ty_1 + b|z_0|)} < 0.101.$$

If $z_0 > 2$, we conclude

$$az_0m(m + 1.401) \ge bz_0n(n + 1),$$

and $m,\ n \ge 2$, $a \cdot 1.701m^2 > bn^2$ which implies $(m/n) \ge 1.533$, which contradicts Lemma 11. If $z_0 < -2$, we have

$$a|z_0|m(m - 1) \ge b|z_0|n(n - 1.101),$$

and $4n(n - 1.101) < (3/2)n((3/2)n - 1)$, which implies $n \le 1$, a contradiction. If $b > 4a$, similar to Case 1.1, we get the congruence relation

$$((am^2 - bn^2)^2 - 4x_0^2m^2 - 4y_1^2n^2)^2 \equiv 64x_0^2y_1^2m^2n^2 \pmod{c}.$$

Now from Lemma 9 we have

$$y_1^2 < \frac{b}{c}z_0^2 + 4 < \frac{4a}{c} \cdot a^{-5/7}c^{9/7} \cdot 2.586 + 4 < 10.343(ac)^{2/7} + 4 < c^{0.35}.$$

Then the lefthand side of the congruence is less than or equal to

$$\max\left\{c^{1/3+0.48},\ c^{1/3+0.48},\ c^{0.022+0.7+0.24}\right\} < c,$$

and for the righthand side we have

$$64x_0^2y_1^2m^2n^2 < c^{0.065+0.694+0.24} < c;$$

therefore, again, equality holds. We get $z_0(am^2-bn^2) = \pm 2x_0m \pm 2y_1n$, and therefore together with $az_0m^2 + sx_0m = bz_0n^2 + ty_1n$ we conclude $x_0m(s \pm 2) = y_1n(t \pm 2)$. Now $n = A/B$, where A and B are defined by

$$A = x_0^2y_1(s \pm 2)(\pm t \mp s),$$
$$B = z_0\left(abc(y_1^2 - x_0^2) + 16(a - b) \pm 4aty_1^2 \mp 4bsx_0^2\right).$$

Furthermore, we have the following estimates

$$|A| \le x_0^2y_1(s + 2)(t + s) < 2.01x_0^2y_1c\sqrt{ab}$$

and

$$|B| > |z_0|\Big(abc(2y_1-1)+16\Big(\frac{b}{4}-b\Big)-4aty_1^2-16s(b-a)\Big)$$
$$> |z_0|y_1abc\Big(2-\frac{1}{y_1}-\frac{12}{acy_1}-\frac{4ty_1}{bc}-\frac{12s}{acy_1}\Big) > 1.49|z_0|y_1abc.$$

Then

$$n < \frac{2.01x_0^2y_1c\sqrt{ab}}{1.49|z_0|y_1abc} < 1.349\frac{x_0^2}{|z_0|\sqrt{ab}} < 1.349\frac{az_0^2}{0.9|z_0|c\sqrt{ab}}$$
$$< 1.5\frac{|z_0|}{c} < c^{0.007+0.78-1} = c^{-0.213} < 1,$$

which is impossible.

Case 2. $v_{2m+1} = w_{2n}$. It can be proven in the same way as Case 1.2.

Case 3. $v_{2m} = w_{2n+1}$. It can also be proven in the same way as Case 1.2.

Case 4. $v_{2m+1} = w_{2n+1}$. Lemma 12 with the relations $|z_0| = t$, $|z_1| = s$ and $z_0z_1 > 0$ implies

$$\pm\frac{1}{2}astm(m+1)+x_0(2m+1) \equiv \pm\frac{1}{2}bstn(n+1)+y_1(2n+1) \pmod{c},$$

and since $x_0 \equiv y_1 \equiv r \pmod{c}$,

$$\pm\frac{1}{2}astm(m+1)+2rm \equiv \pm\frac{1}{2}bstn(n+1)+2rn \pmod{c}.$$

If we multiply this congruence by s, respectively t, we get

$$\pm 2atm(m+1)+2rsm \equiv \pm 2btn(n+1)+2rsn \pmod{c}$$

and

$$\pm 2asm(m+1)+2rtm \equiv \pm 2bsn(n+1)+2rtn \pmod{c}.$$

From the estimates

$$|2btn(n+1)| < c^{0.011+(1/12)+0.55+0.12} < c^{0.765} < \frac{1}{2}c$$

and

$$2rtnc^{0.011+0.7+0.06} = c^{0.771} < \frac{1}{2}c,$$

we see that we have equalities again, which implies

$$rm = rn, \quad am(m+1) = bn(n+1),$$

and $m = n = 0$, a contradiction. $\square$

Let us define $\theta_1 = s/a\sqrt{a/c}$, $\theta_2 = t/b\sqrt{b/c}$.

Lemma 14. *If x, y, z are positive solutions of the system of equations* (1) *and* (2), *then*

$$\max\left\{\left|\theta_1 - \frac{sbx}{abz}\right|, \ \left|\theta_2 - \frac{tay}{abz}\right|\right\} < \frac{2c}{a}z^{-2}.$$

Proof. The statement can be shown in exactly the same way as [**15**, Lemma 6]. $\square$

Now we will use Bennett's theorem about simultaneous rational approximations of the square roots of the numbers that are close to 1.

Theorem 2 [**3**, Theorem 3.2]. *Let a_i, p_i, q and N be integers for $0 \le i \le 2$ such that $a_0 < a_1 < a_2$, $a_j = 0$, for some $0 \le j \le 2$, $q \ne 0$ and $N > M^9$ where $M = \max\{|a_i| : 0 \le i \le 2\}$. Then*

$$\max\left\{\left|\sqrt{1+\frac{a_i}{N}} - \frac{p_i}{q}\right| : 0 \le i \le 2\right\} > (130N\gamma)^{-1}\, q^{-\lambda},$$

where

$$\lambda = 1 + \frac{\log(33N\gamma)}{\log\left(1.7N^2 \prod_{0\le i<j\le 2}(a_i - a_j)^{-2}\right)}$$

and

$$\gamma = \begin{cases} ((a_2-a_0)^2(a_2-a_0)^2/2a_2-a_0-a_1) & \text{if } a_2-a_1 \geq a_1-a_0, \\ ((a_2-a_0)^2(a_1-a_0)^2/a_1+a_2-2a_0) & \text{if } a_2-a_1 < a_1-a_0. \end{cases}$$

Proposition 2. *Let $\{a, b, c, d\}$ be a $D(4)$-quadruple such that $a<b<c<d$ and $c > \max\{b^{12}, 10^{29}\}$. Then $d < c^{53.9}$.*

Proof. Let

$$ad+4=x^2, \quad db+4=y^2, \quad cd+4=z^2.$$

Now we will apply Bennett's theorem (Theorem 2) to the following numbers:

$$\begin{array}{lll} a_0=0, \quad a_1=4a, & a_2=4b,\ N=abc, & M=4b, \\ q=abz, & p_1=sbx, & p_2=tay. \end{array}$$

It is easy to check that $N>M^9=4^9b^9$. If $b>4^3$ we have $M^9=4^9b^9<b^{12}<c<N$. And if $b\leq 4^3$, we get $m^9<10^{28}<c<N$. Moreover, $\gamma=(16b^2(b-a)^2/2b-a)$, if $b\geq 2a$, and $\gamma=(16a^2b^2)/(a+b)$, if $a<b<2a$. We conclude that $(8b^3/3)\leq\gamma<8b^3$. Furthermore,

$$\lambda = 1+\frac{\log(33abc\gamma)}{\log(1.7a^2b^2c^2\cdot(1/16^3a^2b^2(b-a)^2))} = 2-\lambda_1,$$

where

$$\lambda_1 = \frac{\log\left(1.7c/33\cdot 16^3(b-a)^2ab\gamma\right)}{\log(1.7c^2/16^3(b-a)^2)}.$$

Now Lemma 14 implies

$$\frac{2c}{az^2} > (130abc\gamma)^{-1}(abz)^{\lambda_1-2},$$

$z^{\lambda_1} < 260a^2b^3c^2\gamma$, and finally

$$\log z < \frac{\log(260a^2b^3c^2\gamma)\log(1.7c^2/16^3(b-a)^2)}{\log\left(1.7c/33\cdot 16^3(b-a)^2ab\gamma\right)}.$$

We also have the following estimates

$$260a^2b^3c^2\gamma < 2080a^2b^6c^2 < 2080b^8c^2 < c^{2.768},$$

$$\frac{1.7c^2}{16^3(b-a)^2} < c^2,$$

$$\frac{1.7c}{33\cdot 16^3(b-a)^2ab\gamma} > \frac{1.7c}{33\cdot 16^3b^2\cdot ab\cdot 8b^3} > 1.5\cdot 10^{-6}cb^{-7} > c^{0.203}.$$

Then

$$\log z < \frac{2\cdot 2.786\log^2 c}{0.203\log c} < 27.45\log c,$$

and $z < c^{27.45}$, which implies $d < (z^2-4)/c < c^{53.9}$. $\square$

Lemma 15. *Let $\{a,\, b,\, c,\, d\}$ be a $D(4)$-quadruple such that $a < b < c < d$ and $c > \max\{b^{12},\, 10^{29}\}$. Then $d = d_+$.*

Proof. If we define

$$ad + 4 = x^2,\; bd + 4 = y^2,\; cd + 4 = z^2,$$

then there exist $m,\, n \geq 0$ such that $z = v_m = w_n$. But, if $n > 2$, we get

$$z = w_n > \frac{c}{2.224\sqrt{[4]bc}}(t-1)^{n-1} > \frac{c}{2.224\sqrt[4]{bc}}(0.999\sqrt{bc})^{n-1} > c^{n/2}.$$

Then Proposition 2 implies $n \leq 54$. If we apply Lemma 13, we get $c \leq 6.47\cdot 10^{28}$, a contradiction. Therefore, $n \leq 2$, and by Lemma 6 it follows that $d = d_+$. $\square$

Now we are ready to prove our main result.

Proof of Theorem 1. Assume that $\{a,b,c,d,e,f,g\}$ is a $D(4)$-septuple such that $a < b < c < d < e < f < g$. Then Proposition 1 implies

$$f > 0.036e^{3.5}c^{2.5} > 0.036^{4.5}(d^{3.5}b^{2.2})^{3.5}b^{2.5} > 1.34\cdot 10^{-14}b^{37.75} > b^{12}.$$

Also, since $e > 4\cdot 10^7$, (see the remark after Definition 1) we get $f > 10^{29}$. So the quadruple $\{a,b,f,g\}$ satisfies the condition of Lemma 15. Then $g = g_+ < f(ab+4) < f^3$, while on the other hand we have $g > 0.036f^{3.5}d^{2.5} > f^3$, which gives us the contradiction. $\square$

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