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ON EXTENSION AND REFINEMENT OF WILKER'S INEQUALITY

SHANHE WU

ABSTRACT. In this note, an extension of Wilker's inequality involving exponential parameters is given and a refinement of Wilker's inequality is established.

1. Introduction.

In [**5**], Wilker proposed an interesting inequality:

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \quad 0 < x < \frac{\pi}{2}, \tag{1}$$

different proofs of which were given by Guo et al. [**1**] and Zhu [**6**]. Wilker also asked about a certain refinement of inequality (1); such a refinement was proved by Sumner et al. [**4**] and Pinelis [**3**] using different methods. The purpose of this note is to establish an extension of Wilker's inequality involving exponential parameters and to give a refinement of Wilker's inequality.

We first introduce two lemmas:

Lemma 1 [**2**, page 238]. *If* $0 < x < \pi/2$, *then*

$$\cos x < \left(\frac{\sin x}{x}\right)^3 < 1. \tag{2}$$

Lemma 2. *If* $0 < x < \pi/2$, *then*

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2. \tag{3}$$

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Proof. Let

$$f(x) = \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}, \quad 0 < x < \frac{\pi}{2}.$$

Differentiating $f(x)$ with respect to x gives

$$f'(x) = \frac{1}{\sin^3 x}\left(\sin^2 x \cos x - 2x^2 \cos x + x \sin x\right).$$

Using Lemma 1 together with a simple calculation yields

$$\begin{aligned}
f'(x) &= \frac{x^2}{\sin^3 x}\left[\cos x\left(\left(\frac{\sin x}{x}\right)^2 - 2\right) + \frac{\sin x}{x}\right] \\
&= \frac{x^2}{\sin^3 x}\left[\left(\cos x - \left(\frac{\sin x}{x}\right)^3\right)\left(\left(\frac{\sin x}{x}\right)^2 - 2\right)\right. \\
&\qquad\qquad \left. + \left(\frac{\sin x}{x}\right)^3\left(\left(\frac{\sin x}{x}\right)^2 - 2\right) + \frac{\sin x}{x}\right] \\
&= \frac{x^2}{\sin^3 x}\left[\left(\cos x - \left(\frac{\sin x}{x}\right)^3\right)\left(\left(\frac{\sin x}{x}\right)^2 - 2\right)\right. \\
&\qquad\qquad \left. + \left(\frac{\sin x}{x}\right)\left(\left(\frac{\sin x}{x}\right)^2 - 1\right)^2\right] \\
&> 0.
\end{aligned}$$

This means that $f(x)$ is strictly increasing on $(0, (\pi/2))$; consequently, we can deduce from $\lim_{x\to 0^+} f(x) = 2$ that $f(x) > 2$ for $0 < x < \pi/2$, which leads to the desired inequality (3). Lemma 2 is proved. □

2. Main results.

Theorem 1. *If* $0 < x < \pi/2$, $p \le 2q$, $q > 0$ *(or* $q \le -1$*), then*

$$\left(\frac{\sin x}{x}\right)^p + \left(\frac{\tan x}{x}\right)^q > 2. \tag{4}$$

Proof. Case (I). When $p \leq 2q$, $q > 0$.

Using the arithmetic-geometric means inequality and Lemma 1 gives

$$\begin{aligned}\left(\frac{\sin x}{x}\right)^p + \left(\frac{\tan x}{x}\right)^q &\geq 2\left(\frac{\sin x}{x}\right)^{p/2}\left(\frac{\tan x}{x}\right)^{q/2}\\ &= 2\left(\frac{\sin x}{x}\right)^{p/2}\left(\frac{\sin x}{x}\right)^{q/2}\left(\frac{1}{\cos x}\right)^{q/2}\\ &> 2\left(\frac{\sin x}{x}\right)^{p/2}\left(\frac{\sin x}{x}\right)^{q/2}\left(\frac{\sin x}{x}\right)^{-3q/2}\\ &= 2\left(\frac{\sin x}{x}\right)^{(p-2q)/2}\\ &\geq 2,\end{aligned}$$

which is the desired inequality (4).

Case (II). When $p \leq 2q$, $q \leq -1$.

It follows from $p \leq 2q$ and $x/(\sin x) > 1$ that

$$\begin{aligned}\left(\frac{\sin x}{x}\right)^p + \left(\frac{\tan x}{x}\right)^q &= \left(\frac{x}{\sin x}\right)^{-p} + \left(\frac{x}{\tan x}\right)^{-q}\\ &= \left(\frac{x}{\sin x}\right)^{-2q}\left(\frac{x}{\sin x}\right)^{2q-p} + \left(\frac{x}{\tan x}\right)^{-q} \qquad (5)\\ &\geq \left(\frac{x}{\sin x}\right)^{-2q} + \left(\frac{x}{\tan x}\right)^{-q}.\end{aligned}$$

Noting that $-q \geq 1$ and applying the power means of inequality and Lemma 2, we have

$$(6) \quad \left(\frac{x}{\sin x}\right)^{-2q} + \left(\frac{x}{\tan x}\right)^{-q} \geq 2^{1+q}\left[\left(\frac{x}{\sin x}\right)^2 + \left(\frac{x}{\tan x}\right)\right]^{-q} > 2.$$

Combining inequalities (5) and (6) yields inequality (4). The proof of Theorem 1 is complete. □

In Theorem 1, a special case $p = 2$, $q = 1$ yields the Wilker's inequality.

Finally, we show a refinement of Wilker's inequality:

Theorem 2. *If* $0 < x < \pi/2$, *then*

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2. \tag{7}$$

Proof. From Lemma 1, we find

$$\frac{x}{\tan x}\left(\frac{x}{\sin x}\right)^2 = \cos x\left(\frac{x}{\sin x}\right)^3 < 1,$$

so that

$$\begin{aligned}\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} &> \left(\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x}\right)\left(\left(\frac{x}{\sin x}\right)^2 \frac{x}{\tan x}\right) \\ &= \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}.\end{aligned} \tag{8}$$

Combining inequalities (3) and (8) leads to inequality (7) immediately. The proof of Theorem 2 is complete. □

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Department of Mathematics, Longyan College, Longyan Fujian 364012, P.R. China
Email address: wushanhe@yahoo.com.cn