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SOME SERIES OF HONEY-COMB SPACES

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ABSTRACT. We study the topology and geometry of some series of closed connected orientable 3-manifolds constructed as honey-comb spaces. These manifolds are quotients of certain polyhedral 3-cells by pairwise identification of their boundary faces. We determine geometric presentations of the fundamental group and study the split extension of it. Then we describe geometric structures, homeomorphism type and covering properties of our manifolds which are shown to be cyclic coverings of the 3-sphere branched over known links with two components. Finally, we answer open questions on certain manifolds, defined by Kim and Kostrikin, and give a complete classification of them.

1. The Seifert-Weber dodecahedron space. Following [**16, 22**], we start with a detailed discussion on a classical example of hyperbolic closed 3-manifold, i.e., the *Seifert-Weber dodecahedron space.* Recall from [**5**] that the Coxeter group $[p,q,r]$ is generated by four elements α, β, γ and δ subject to the following relations:

$$\begin{gathered}\alpha^2=\beta^2=\gamma^2=\delta^2=1\\(\alpha\beta)^p=(\beta\gamma)^q=(\gamma\delta)^r=1\\(\alpha\gamma)^2=(\beta\delta)^2=(\alpha\delta)^2=1.\end{gathered}$$

The group $[3,3,5]$ has order 14400, and is also generated by the elements α_1, β, γ and δ, where $\alpha_1=(\alpha(\beta\gamma\delta)^5)^2\alpha$. Since these elements satisfy the relations of $[5,3,5]$, there is a unique epimorphism $\varphi\colon[5,3,5]\to[3,3,5]$ such that $\varphi(\alpha)=\alpha_1$, $\varphi(\beta)=\beta$, $\varphi(\gamma)=\gamma$ and

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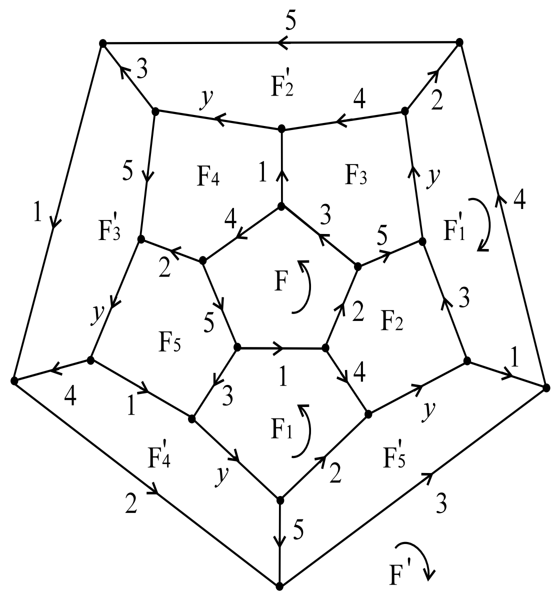

FIGURE 1. Polyhedral representation of the Seifert-Weber hyperbolic manifold.

$\varphi(\delta) = \delta$. Let Γ denote the subgroup of $[3,3,5]$ generated by the elements $\alpha\delta\gamma\beta\delta\gamma\delta\alpha\beta\gamma\alpha\beta$, $\alpha\gamma\beta\delta\gamma\delta\beta\gamma\delta\gamma\alpha\beta$ and $\alpha\beta(\gamma\delta)^2\beta\gamma\delta\beta\gamma\beta$. It has index 120 and intersects $\langle\beta,\gamma,\delta\rangle$ trivially. As is well known, the hyperbolic 3-space $\mathbf{H}^3$ can be tessellated by regular dodecahedrons with 12 meeting at each vertex and 5 meeting at each edge. The symmetry group of this tessellation can be described in terms of the generators α, β, γ and δ, and it turns out to be isomorphic to $[5,3,5]$. Furthermore, $[5,3,5]$ acts sharply transitively on the dodecahedral cells of the tessellation. Let us consider the orbit space obtained from $\mathbf{H}^3$ under the action of $\varphi^{-1}(\Gamma)$. The interior of a dodecahedral cell Δ is a fundamental region for the orbit space. This means that each point in the interior of Δ lies in a different orbit, and each point of $\mathbf{H}^3$ is in the same orbit as at least one point of Δ. Then the orbit space can be obtained from Δ with certain identifications among vertices, edges and faces in $\partial\Delta$. The boundary identifications, induced by the action of $\varphi^{-1}(\Gamma)$, are illustrated in Figure 1.

The resulting 3-manifold, which we denote by $M_{n,k} = M_{5,2}$, is hyperbolic. The fundamental group $G_{5,2}(\cong \varphi^{-1}(\Gamma))$ of $M_{5,2}$ has the finite presentation

$$G_{5,2} \cong \langle x_1, x_2, x_3, x_4, x_5, y : x_1x_2x_3x_4x_5 = 1, \\ x_i^{-1}x_{i+3}x_{i+1}x_{i+4}^{-1} = y \text{ (indices mod 5)}\rangle,$$

which corresponds to a spine of the manifold. The first, respectively second, relation arises by walking around the boundary of the face $F \equiv F'$, respectively $F_{i+3} \equiv F'_{i+3}$. It is a routine matter to compute the integral homology of $M_{5,2}$ from the above presentation, i.e., $H_1(M_{5,2}) \cong \mathbf{Z}_5^3$. The dodecahedral representation in Figure 1 can be depicted in a $\mathbf{Z}_5$-symmetric form as shown in Figure 2, where the shift $k = 2$ is pointed out.

Let a_i, $i = 1, 2, 3, 4, 5$, and b denote the isometries which identify the pairs of faces (F_i, F_i') and (F, F'), respectively. Then $G_{5,2}$ admits the further presentation

$$G_{5,2} \cong \langle a_1, a_2, a_3, a_4, a_5, b : a_1a_3a_5a_2a_4 = 1, \\ a_ia_{i-1}^{-1}a_{i-2}^{-1}a_{i-3} = b \text{ (indices mod 5)}\rangle,$$

which corresponds to a spine of the manifold (dual to the previous one). Let $H_{n,k} = H_{5,2}$ be the split extension group of $G_{5,2}$ by $\mathbf{Z}_5 = \langle \rho : \rho^5 = 1\rangle$, where ρ is the cyclic automorphism of $G_{5,2}$ given by $\rho(a_i) = a_{i+1}$ (indices mod 5), and $\rho(b) = b$. Then $H_{5,2}$ has the finite presentation

$$\begin{aligned} H_{5,2} &\cong \langle a, b, \rho : \rho^5 = 1, \quad \rho^{-1}b\rho = b, \quad (a\rho^{-2})^5 = 1, \quad b = \rho^2a\rho(a^{-1}\rho)^2a\rangle \\ &\cong \langle a, \rho : \rho^5 = 1, \quad (a\rho^{-2})^5 = 1, \quad a\rho(a^{-1}\rho)^2a\rho = \rho a\rho(a^{-1}\rho)^2a\rangle, \end{aligned}$$

where $a_1 = a$ and $a_{i+1} = \rho^{-i}a\rho^i$. Setting $\tau = a\rho^{-2}$ and eliminating $a = \tau\rho^2$ yields the presentation

$$H_{5,2} = \langle \rho, \tau : \rho^5 = \tau^5 = 1, \quad w\rho = \rho w\rangle,$$

where $w = \tau\rho\tau^{-1}\rho^{-1}\tau^{-1}\rho\tau$. Since $w = \tau^{\varepsilon_1}\rho^{\varepsilon_2}\tau^{\varepsilon_3}\rho^{\varepsilon_4}\tau^{\varepsilon_5}\rho^{\varepsilon_6}\tau^{\varepsilon_7}$, where $\varepsilon_1 = \varepsilon_2 = \varepsilon_6 = \varepsilon_7 = 1$ and $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = -1$, the exponent ε_i is the sign (± 1) of $3i$ reduced mod16 in the interval $(-8, 8)$. Then the word w corresponds to the 2-bridge link $8/3$, i.e., the *Whitehead link*

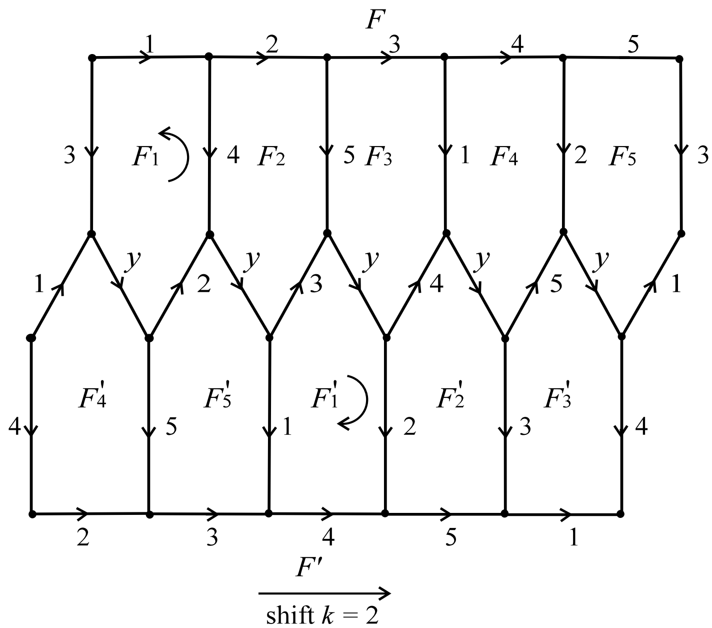

FIGURE 2. The $\mathbf{Z}_5$-symmetric form of the Seifert-Weber manifold $M_{5,2}$ (with shift $k=2$).

$\mathcal{W}$, see [**9, 20**]. In particular, the finite presentation $\langle \rho, \tau : w\rho = \rho w \rangle$ defines the link group of $\mathcal{W}$, where ρ and τ are meridians around the components of $\mathcal{W}$. Therefore, $H_{5,2}$ is the fundamental group of the hyperbolic orbifold $\mathcal{O}_{5,5}(\mathcal{W})$, whose underlying space is the 3-sphere and whose singular set is the Whitehead link $\mathcal{W} = 8/3$ with branching index 5 on its components. This agrees with the well-known fact that the Seifert-Weber 3-manifold $M_{5,2}$ is the strongly cyclic 5-fold covering of $\mathbf{S}^3$ branched over the Whitehead link (for further information on strongly cyclic branched coverings of 2-bridge knots and links see also [**4, 18, 19**]. As general references on branched coverings of links we refer, for example, to [**2, 10, 20**]).

2. Branched coverings of generalized Whitehead links. The $\mathbf{Z}_5$-symmetric complex with boundary identifications shown in Figure 2 was first generalized by Helling, Kim and Mennicke [**7**] to construct

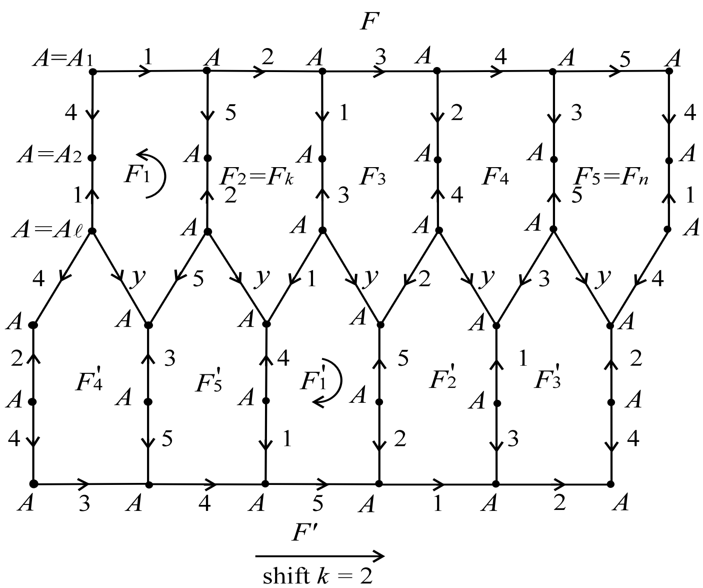

FIGURE 3. The polyhedral scheme $P_{n,k,l} = P_{5,2,3}$ with identifications.

a nice family of polyhedral schemata representing closed connected orientable 3-manifolds $M_{n,k}$ for n and k coprime, $n \geq 2$ and $1 \leq k \leq n-1$. Then they proved that these manifolds are n-fold strongly cyclic coverings of the 3-sphere branched over the Whitehead link. Subsequently, the polyhedral description of the whole family of cyclic branched coverings of the Whitehead link was given by Cavicchioli and Paoluzzi in [**3**]. More recently, it was constructed in [**11**] (for n and k coprime) and in [**14**] (in the general case) a family of closed connected orientable 3-manifolds $M_{n,k,l}$ which contains the Seifert-Weber dodecahedral space $M_{5,2}$ ($= M_{5,2,2}$) and the manifolds $M_{n,k}$ ($= M_{n,k,l}$ for $l = 2$). This section is devoted to describe the polyhedral construction of the manifolds $M_{n,k,l}$ from [**11, 14**], where they were denoted by $M(2m+1, n, k)$ and $\mathcal{M}(2m+1, n, k)$, respectively, for $m = l$ and $(n, k) = 1$ in [11], and $d = (n, k) \geq 1$ in [**14**]. Then we recall the results on these manifolds obtained in the quoted papers. As remarked in [**14**, page 804], the technique for such results is the same as the one developed by Cavicchioli and Paoluzzi in [**3**]: it basically depends on the cancelation of handles on Heegaard diagrams representing threedimensional orbifolds. This generalizes to the orbifold case a method described in [**6**] for link complements. Finally, we complete the known results with more information on certain geometric presentations of the fundamental group and study the split extension of it in the general case.

For every $n \geq 2$, $l \geq 2$ and $1 \leq k \leq n-1$, let us consider the combinatorial 3-cell $P_{n,k,l}$ whose 2-sphere boundary consists of two n-gons F and F' in the northern and southern hemispheres, respectively, and $2n$ $(2l+1)$-gons, labeled by F_i and F_i', $i = 1, 2, \ldots, n$, in the equatorial zone. Then $\partial P_{n,k,l}$ has exactly $2n+2$ faces, $2n(l+1)$ edges and $2nl$ vertices. The side pairing of index k is determined by identifying the pairs of faces (F_i, F_i') and (F, F'), and also the corresponding oriented edges on the boundary with the same labels are identified (see Figure 3 for $P_{n,k,l} = P_{5,2,3}$, where $d = (n,k) = (5,2) = 1$, and Figure 4 for $P_{n,k,l} = P_{8,4,4}$, where $d = (n,k) = (8,4) = 4$). The integer k is again the number of $(2l+1)$-gons which we have to shift before gluing the face F_i with F_i'. The resulting identification space $M_{n,k,l}$ has d vertices, $n+d$ 1-cells, $n+1$ 2-cells, and one 3-cell. Since the Euler characteristic vanishes, the quotient complex $M_{n,k,l}$ is a closed connected orientable 3-manifold. Let us denote, for simplicity, by $G_{n,k,l}$ the fundamental group of $M_{n,k,l}$. We can obtain a finite presentation of $G_{n,k,l}$ by considering the isometries a_i, $i = 1, 2, \ldots, n$, and b which identify the pairs of faces (F_i, F_i') and (F, F'), respectively. Following the cycles of equivalent edges we get for the label x_i ($\equiv i$ in the figures), $i = 1, \ldots, n$, the relation

$$(l \text{ even}) \qquad (a_i a_{i-1}^{-1})^{l/2} (a_{i-k}^{-1} a_{i-k-1})^{l/2} b^{-1} = 1$$
$$(l \text{ odd}) \qquad (a_i a_{i-1}^{-1})^{(l-1)/2} a_i (a_{i+k-1} a_{i+k}^{-1})^{(l-1)/2} a_{i+k-1} b^{-1} = 1,$$

and, for the label y_i, $i = 1, \ldots, d$, $d = (n,k)$, the relation

$$a_i a_{i+k} a_{i+2k} \cdots a_{i+k(n-1)} = 1.$$

Then we have the following result, compare with [**14**, Theorem 1, page 807].

Theorem 2.1. *The polyhedral 3-cell $P_{n,k,l}$ with identifications, $n \geq 2$, $l \geq 2$, $1 \leq k \leq n-1$, constructed above, defines a closed*

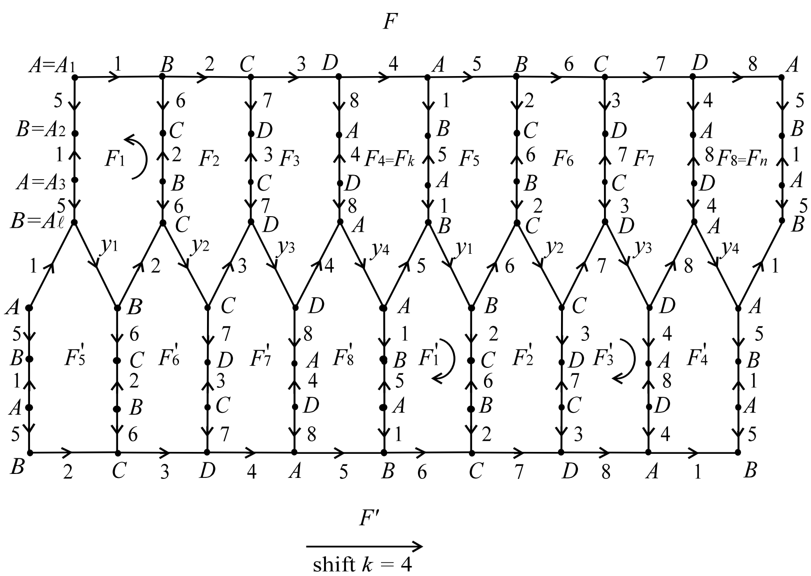

FIGURE 4. The polyhedral scheme $P_{n,k,l} = P_{8,4,4}$ with identifications.

connected orientable 3-manifold $M_{n,k,l}$ *which has a spine modeled on the finite presentation*

$$G_{n,k,l} \cong \Big\langle a_1, \dots, a_n, b\colon$$

$$b = \begin{cases} (a_i a_{i-1}^{-1})^{l/2}(a_{i-k}^{-1} a_{i-k-1})^{l/2} & \text{if } l \text{ even} \\ (a_i a_{i-1}^{-1})^{(l-1)/2} a_i (a_{i+k-1} a_{i+k}^{-1})^{(l-1)/2} a_{i+k-1} & \text{if } l \text{ odd} \end{cases}$$

$$i = 1, \dots, n$$

$$a_i a_{i+k} a_{i+2k} \cdots a_{i+k(n-1)} = 1, \quad i = 1, \dots, d \Big\rangle,$$

where the indices are taken mod$n$, *and* $d = (n, k)$.

If $l = 2$, then the manifolds $M_{n,k,l}$ are exactly the manifolds $M_{n,k}$ considered in [**7**] for $(n, k) = 1$ and in [**3**] for the general case. Suppose now that n and k are coprime, i.e., $d = (n, k) = 1$. Then the quotient complex $M_{n,k,l}$ has exactly one vertex, so we can obtain a further presentation for the fundamental group $G_{n,k,l} = \pi_1(M_{n,k,l})$

with generators $x_1, \dots, x_n, y$ and relators arising from the boundaries of the 2-cells $F_1, \dots, F_n, F$ of the polyhedral scheme. For each $i = 1, \dots, n$, the boundary cycle of the polygons F_i and F_i' is

$$(l \text{ even}) \qquad (x_i^{-1}x_{i-k})^{l/2}(x_{i+1}x_{i-k+1})^{l/2} = y$$
$$(l \text{ odd}) \qquad (x_i x_{i-k}^{-1})^{(l-1)/2} x_i (x_{i-k+1}x_{i+1}^{-1})^{(l-1)/2} x_{i-k+1} = y,$$

where the indices are taken $\mod n$, and the boundary cycle of the polygons F and F' is

$$x_1 x_2 \cdots x_n = 1.$$

Then we have the following result which corrects a misprint in the exponents of the relations in the presentation obtained in [**11**, page 68], at the end of Section 2.

Theorem 2.2. *If n and k are coprime, then the fundamental group $G_{n,k,l}$ of the manifold $M_{n,k,l}$, $n \geq 2$, $l \geq 2$, $1 \leq k \leq n-1$, admits the finite presentation*

$$G_{n,k,l} \cong \Big\langle x_1, \dots, x_n, y \colon x_1 x_2 \cdots x_n = 1,$$
$$y = \begin{cases} (x_i^{-1}x_{i-k})^{l/2}(x_{i+1}x_{i-k+1}^{-1})^{l/2} & \textit{if } l \textit{ even} \\ (x_i x_{i-k}^{-1})^{(l-1)/2} x_i (x_{i-k+1}x_{i+1}^{-1})^{(l-1)/2} x_{i-k+1} & \textit{if } l \textit{ odd} \end{cases}$$
$$i = 1, \dots, n; \; \textit{indices} \mod n \Big\rangle,$$

which corresponds to a spine of the manifold (*hence it is geometric*).

To see the equivalence between the presentations obtained for $G_{n,k,l}$, we use some Tietze transformations, and set $x_j := a_{kj}$, $j = 1, \dots, n$.

Case l even. Taking an integer k' such that $kk' \equiv -1 \pmod n$, we get $a_i = x_{-k'i}$, $a_{i-1} = x_{-k'i+k'}$, $a_{i-k} = x_{-k'i-1}$ and $a_{i-k-1} = x_{-k'i+k'-1}$. Substituting these formulae in the relation

$$(a_i a_{i-1}^{-1})^{l/2}(a_{i-k}^{-1}a_{i-k-1})^{l/2} b^{-1} = 1$$

and setting $y = b^{-1}$, we get

$$(x_{-k'i}x_{-k'i+k'}^{-1})^{l/2}(x_{-k'i-1}^{-1}x_{-k'i+k'-1})^{l/2} y = 1,$$

whose inverse relation is

$$y^{-1}(x^{-1}_{-k'i+k'-1}x_{-k'i-1})^{l/2}(x_{-k'i+k'}x^{-1}_{-k'i})^{l/2}=1.$$

Setting $j=-k'i+k'-1$, we obtain

$$y^{-1}(x_j^{-1}x_{j-k'})^{l/2}(x_{j+1}x^{-1}_{j-k'+1})^{l/2}=1.$$

Now we can interchange the role of k and k' to get the relation of the second presentation for $G_{n,k,l}$. Substituting the formulae $a_i=x_{-k'i}$, $a_{i+k}=x_{-k'i+1},\ldots,a_{i+k(n-1)}=x_{-k'i+n-1}$ in the relation

$$a_ia_{i+k}\cdots a_{i+k(n-1)}=1$$

yields

$$x_{-k'i}x_{-k'i+1}\cdots x_{-k'i+n-1}=1.$$

Setting $i=k$, hence $-k'i\equiv 1 \pmod n$, we get

$$x_1x_2\cdots x_n=1.$$

Case l **odd.** Taking an integer k' such that $kk'\equiv 1 \pmod n$, we get $a_i=x_{k'i}$, $a_{i-1}=x_{k'i-k'}$, $a_{i+k-1}=x_{k'i-k'+1}$ and $a_{i+k}=x_{k'i+1}$. Substituting these formulae in the relation

$$(a_ia^{-1}_{i-1})^{(l-1)/2}a_i(a_{i+k-1}a^{-1}_{i+k})^{(l-1)/2}a_{i+k-1}b^{-1}=1,$$

and setting $y=b$, we get

$$(x_{k'i}x^{-1}_{k'i-k'})^{(l-1)/2}x_{k'i}(x_{k'i-k'+1}x^{-1}_{k'i+1})^{(l-1)/2}x_{k'i-k'+1}y^{-1}=1.$$

Setting $j=k'i$, we obtain

$$(x_jx^{-1}_{j-k'})^{(l-1)/2}x_j(x_{j-k'+1}x^{-1}_{j+1})^{(l-1)/2}x_{j-k'+1}=y.$$

Now we can interchange the role of k and k' to get the relation of the second presentation for $G_{n,k,l}$. The relation

$$a_ia_{i+k}\cdots a_{i+k(n-1)}=1$$

becomes

$$x_{k'i}x_{k'i+1}\cdots x_{k'i+n-1}=1$$

and setting $i = k$, hence $k'i \equiv 1 \pmod n$, we get the relation $x_1x_2\cdots x_n = 1$.

Now we study the split extension group of $G_{n,k,l}$ by the cyclic automorphism corresponding to the presentation in Theorem 2.1. More precisely, we have the following new result which generalizes Theorem 1 of [**11**], case $d=(n,k)=1$.

Theorem 2.3. *Let $H_{n,k,l}$ be the split extension group of $G_{n,k,l}$, where $n,l \geq 2$ and $1 \leq k \leq n-1$. Then $H_{n,k,l}$ is isomorphic to the fundamental group of the orbifold $\mathcal{O}_{n,(n/d)}(\mathcal{W}_l)$ whose underlying space is the 3-sphere and whose singular set is the 2-bridge link $\mathcal{W}_l = (4l)/(2l-1)$ (if $l=2$, then $\mathcal{W}_l = \mathcal{W} = (8/3)$ is the Whitehead link) with branching indices n and n/d on its components, where $d=(n,k)$, see Figure 5.*

Proof. Let us consider the finite presentation of $G_{n,k,l}$ given in the statement of Theorem 2.1. Let ρ be the automorphism of $G_{n,k,l}$ defined by $\rho(a_i)=a_{i+1}$ (indices mod$n$) and $\rho(b)=b$.

(l *even*). The split extension group $H_{n,k,l}$ of $G_{n,k,l}$ by $\mathbf{Z}_n = \langle \rho \colon \rho^n = 1\rangle$ has the finite presentation

$$\begin{aligned} H_{n,k,l} = \langle a,b,\rho \colon & \rho^n = 1,\ \rho b = b\rho, \\ & a\rho^{-k}a\rho^k\rho^{-2k}a\rho^{2k}\ldots\rho^{-k(n-1)}a\rho^{k(n-1)} = 1, \\ & b = (\rho^{-1}a\rho a^{-1})^{l/2}(\rho^{k-1}a^{-1}\rho^{-k+1}\rho^k a\rho^{-k})^{l/2}\rangle \\ \cong \langle a,b,\rho \colon & \rho^n = 1,\ \rho b = b\rho,\ (a\rho^{-k})^{n/d} = 1, \\ & b = (\rho^{-1}a\rho a^{-1})^{l/2}(\rho^{k-1}a^{-1}\rho a\rho^{-k})^{l/2}\rangle \end{aligned}$$

where $a_1 = a$, $a_{i+1} = \rho^{-i}a\rho^i$ and $b = \rho^{-1}b\rho$. Setting $\tau = a\rho^{-k}$ and eliminating $a = \tau\rho^k$ and

$$b = (\rho^{-1}\tau\rho\tau^{-1})^{l/2}(\rho^{-1}\tau^{-1}\rho\tau)^{l/2}$$

yields the finite presentation

$$H_{n,k,l} = \langle \tau,\rho \colon \rho^n = 1,\ \tau^{n/d} = 1,\ w_l\rho = \rho w_l\rangle,$$

where

$$w_l = \tau\rho\tau^{-1}(\rho^{-1}\tau\rho\tau^{-1})^{(l/2)-1}(\rho^{-1}\tau^{-1}\rho\tau)^{l/2}$$

(note that if $l = 2$, then w_l is exactly the word w considered in Section 1). Because

$$w_l = \tau^{\varepsilon_1}\rho^{\varepsilon_2}\tau^{\varepsilon_3}\cdots\tau^{\varepsilon_{4l-3}}\rho^{\varepsilon_{4l-2}}\tau^{\varepsilon_{4l-1}}$$

where ε_i is the sign (± 1) of $(2l-1)i$ reduced $\bmod 8l$ in the interval $(-4l, 4l)$, the word w_l corresponds to the 2-bridge link $(4l)/(2l-1)$, i.e., the link $\mathcal{W}_l$. In particular, the finite presentation $\langle \rho,\, \tau \colon w_l\rho = \rho w_l\rangle$ defines the link group of $\mathcal{W}_l$, where ρ and τ are meridians around its components. Therefore, $H_{n,k,l}$ is the fundamental group of the orbifold $\mathcal{O}_{n,(n/d)}(\mathcal{W}_l)$ whose underlying space is the 3-sphere and whose singular set is the link $\mathcal{W}_l$ with branching indices n and n/d on its components.

(l *odd*). The split extension group $H_{n,k,l}$ has the finite presentation

$$\begin{aligned} H_{n,k,l} = \langle a, b, \rho \colon & \rho^n = 1,\ \rho b = b\rho,\ (a\rho^{-k})^{n/d} = 1,\\ & b = (\rho^{-1}a\rho a^{-1})^{(l-1)/2}\rho^{-1}a\rho\\ & \cdot(\rho^{-k}a\rho^{k}\rho^{-k-1}a^{-1}\rho^{k+1})^{(l-1)/2}\rho^{-k}a\rho^{k}\rangle\\ \cong \langle a, b, \rho \colon & \rho^n = 1,\ \rho b = b\rho,\ (a\rho^{-k})^{(n/d)} = 1,\\ & b = (\rho^{-1}a\rho a^{-1})^{(l-1)/2}\rho^{-1}a\rho\\ & \cdot(\rho^{-k}a\rho^{-1}a^{-1}\rho^{k+1})^{(l-1)/2}\rho^{-k}a\rho^{k}\rangle. \end{aligned}$$

Setting $\tau = a\rho^{-k}$ and eliminating $a = \tau\rho^k$ and

$$b = (\rho^{-1}\tau\rho\tau^{-1})^{(l-1)/2}\rho^{-1}\tau\rho(\tau\rho^{-1}\tau^{-1}\rho)^{(l-1)/2}\tau\rho^{2k}$$

yields the finite presentation

$$H_{n,k,l} = \langle \tau, \rho \colon \rho^n = 1,\ \tau^{n/d} = 1,\ w_l\rho = \rho w_l\rangle,$$

where

$$w_l = \tau\rho\tau^{-1}(\rho^{-1}\tau\rho\tau^{-1})^{(l-3)/2}\rho^{-1}\tau\rho(\tau\rho^{-1}\tau^{-1}\rho)^{(l-1)/2}\tau.$$

Because

$$w_l = \tau^{\varepsilon_1}\rho^{\varepsilon_2}\tau^{\varepsilon_3}\cdots\tau^{\varepsilon_{4l-3}}\rho^{\varepsilon_{4l-2}}\tau^{\varepsilon_{4l-1}},$$

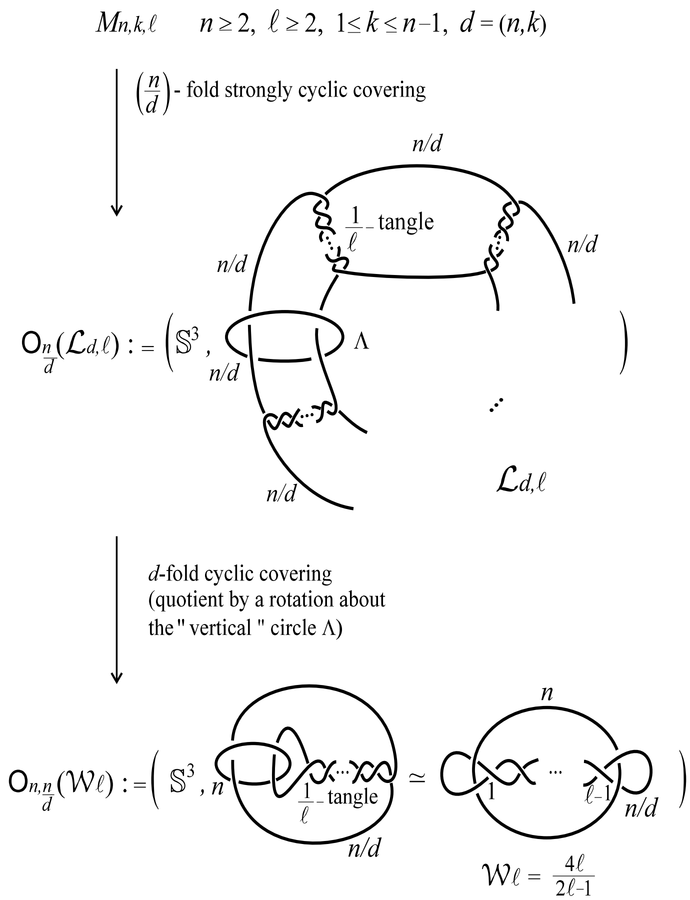

FIGURE 5. Representing the manifolds $M_{n,k,l}$ as branched coverings.

where ε_i is the sign (± 1) of $(2l-1)i$ reduced mod $8l$ in the interval $(-4l, 4l)$, the word w_l corresponds to the 2-bridge link $(4l)/(2l-1) = \mathcal{W}_l$. This completes the proof. $\square$

Theorem 2.4 (**Branched covering representation**). *The closed connected orientable* 3*-manifolds* $M_{n,k,l}$, $n \geq 2$, $l \geq 2$, $1 \leq k \leq n-1$, *are strongly cyclic* (n/d)*-fold coverings of the* 3*-sphere* $\mathbf{S}^3$ *branched over* *the link $\mathcal{L}_{d,l}$ pictured in Figure 5, where $d = (n,k)$ (if $l = 2$, then $\mathcal{L}_{d,l}$ is exactly the link $\mathcal{L}_d$ considered in* [**3**]*). Furthermore, $M_{n,k,l}$ are cyclic branched n-fold coverings of the 2-bridge link $\mathcal{W}_l = (4l)/(2l-1)$ in the 3-sphere, where the branching indices of its components are n and n/d, respectively.*

Theorem 2.4 was first proved in [**3**, Theorem 3.1, page 464], for $l = 2$ and recently generalized in [**14**, Theorem 2, page 807, and Theorem 3, page 809]. As remarked in [**14**, page 804], the authors apply methods from [**3**] to modify Heegaard diagrams of closed 3-orbifolds by simplifications along closed curves and cancelations of handles. These extend to the orbifold case the techniques described in [**6**] for link complements. A proof of the second part of the statement in Theorem 2.4 can also be found in [**11**, Theorem 2, page 70].

To end the section, we describe geometric structures and the homeomorphism type of the manifolds $M_{n,k,l}$. The next two theorems complete Corollary 1 and Theorem 3 of [**11**, pages 70–71], where $(n,k) = 1$, and Theorems 4 and 5 of [**14**, pages 811–812].

By [**17**, Remark 5.2], the orbifold $\mathcal{O}_{n,m}((4l)/(2l-1))$ is hyperbolic for every $l \geq 2$ and $n, m \geq 3$, see also [**23**]. So we get the following result

Theorem 2.5 (**Geometric structures**). *The manifolds $M_{n,k,l}$, $n \geq 2$, $l \geq 2$, $1 \leq k \leq n-1$, are hyperbolic for all $n \geq 3$ and $d < n/2$, where $d = (n,k)$. In these cases, $G_{n,k,l} = \pi_1(M_{n,k,l})$ are hyperbolic groups (hence infinite and torsion free). If $d = n/2$, then the manifolds are Seifert fibered spaces.*

By Theorem 4.1 of [**21**] the symmetry group of the 2-bridge link $\mathcal{W}_l = (4l)/(2l-1)$ is isomorphic to either $(\mathbf{Z}_2)^3$ or the dihedral group D_4, see also [**8**]. Therefore, it has order 8. So we can apply Theorem 1 of [**24**] for the case $(n,k) = 1$ and Theorem 2.2 of [**3**] for $d = (n,k) \neq 1$ to get the following classification of the manifolds $M_{n,k,l}$.

Theorem 2.6 (**Homeomorphism type**). *If $n \geq 3$ and $d = (n,k) = (n,k') = 1$, then $M_{n,k,l}$ is isometric (homeomorphic) to $M_{n,k',l'}$ if and only if $l = l'$ and $k' \equiv (\pm 1)^{l+1} k^{\pm 1} \pmod n$. If*

$d = (n,k) = (n,k') \neq 1$, then the manifolds $M_{n,k,l}$ and $M_{n,k',l'}$ are homeomorphic if and only if $l = l'$ and $k' \equiv (\pm 1)^{l+1}k \pmod{n}$.

To illustrate the arithmetic conditions in Theorem 2.6, we recall that two oriented 2-bridge links α/β and α'/β' are *equivalent* if and only if $\alpha = \alpha'$ and $\beta' \equiv \beta^{\pm 1} \pmod{2\alpha}$, as shown, for example, in [**2**, page 184]. In this case, the n-fold strongly cyclic branched coverings $M_{n,k}(\alpha/\beta)$ and $M_{n,k}(\alpha'/\beta')$ are homeomorphic ($\cong$). By [**19**, Proposition 2.1], we have $M_{n,k}(\alpha/\beta) \cong M_{n,k}(-\alpha/\beta) \cong M_{n,-k}(\alpha/(\beta-\alpha))$, and $M_{n,k}(\alpha/\beta) \cong M_{n,k'}(\alpha/\beta)$ if $kk' \equiv 1 \pmod{n}$. In our case, we have $\alpha = 4l$ and $\beta = 2l-1$, hence $M_{n,k,l} = M_{n,k}(4l/(2l-1))$ is homeomorphic to $M_{n,-k}(4l/-(2l+1))$, where $d = (n,k) = 1$. Now the oriented links $4l/(2l-1)$ and $4l/-(2l+1)$ are equivalent if and only if $-(2l-1)(2l+1) = -4l^2+1 \equiv 1 \pmod{8l}$, i.e., l must be even. This is coherent with the statement of Theorem 2.4 in [**19**].

3. A class of Kim-Kostrikin manifolds. In this section we give a complete classification of a class of closed connected orientable 3-manifolds $M_1(n)$, constructed by Kim and Kostrikin in [**12, 13**] as honey-comb spaces. In fact, let us consider the polyhedral 3-cell $P_1(n)$ whose 2-sphere boundary consists of $8n$ faces, $20n$ edges and $12n+2$ vertices (see Figure 6 for $n=2$ and $n=3$). All the edges are numbered and oriented so that to each face there corresponds precisely one distinct face with the same induced orientation. Identifying the pairs of pentagons (F_i, F_i'), $i = 1, \dots, 3n$, and (E_i, E_i'), $i = 1, \dots, n$, as well as the corresponding edges and vertices, yields a closed connected orientable 3-manifold $M_1(n)$. Computing the homology of $M_1(n)$ [**13**] implies that $M_1(n)$ is homeomorphic to $M_1(n')$ if and only if $n = n'$.

The following result answers open questions on the manifolds $M_1(n)$ stated in [**12, 13**] (the case $n=1$ was solved in [**15**]).

Theorem 3.1. *For any $n \geq 1$, the Kim-Kostrikin manifolds $M_1(n)$ are $(3n)$-fold cyclic covering of $\mathbf{S}^3$ branched over the Whitehead link $\mathcal{W}$, where the branching indices of its components are 3 and $3n$, respectively. In particular, $M_1(n)$ is hyperbolic, and its fundamental group $G_1(n) = \pi_1(M_1(n))$ is infinite and torsion-free.*

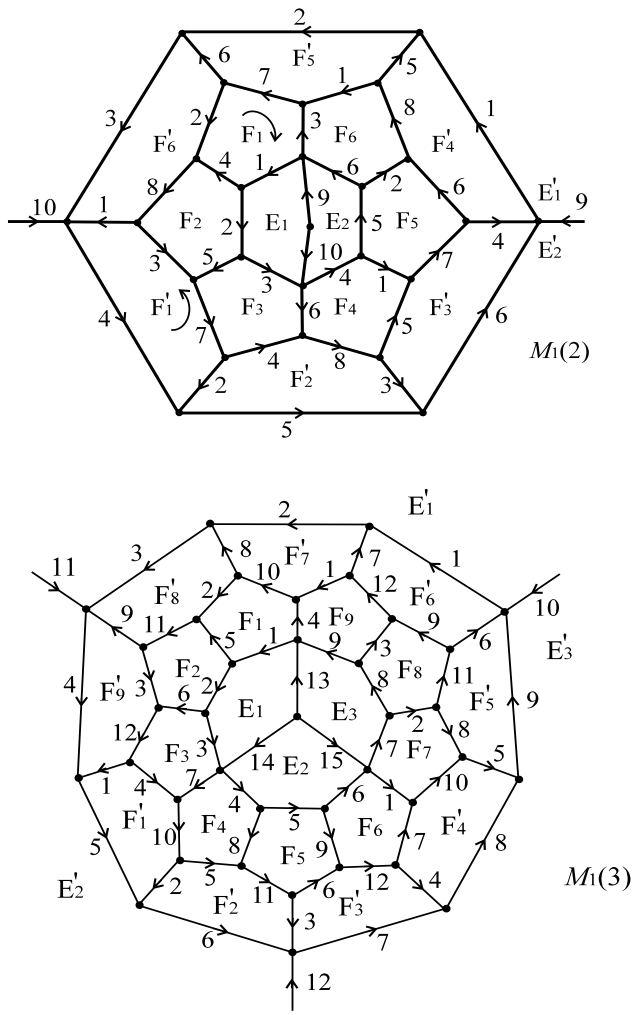

FIGURE 6. The Kim-Kostrikin manifolds $M_1(n)$ for $n = 2, 3$.

Proof. Let ρ denote the n-rotational symmetry of the polyhedron $P_1(n)$ around the north-south axis. Let us consider the orbifold $M_1(n)/\langle\rho\rangle$ obtained from $M_1(n)$ under the action of ρ. If n is coprime with 3, then the singular set of $M_1(n)/\langle\rho\rangle$ is a knot which is the image of the symmetry axis with branching index n. If n is divided by 3, then the singular set is formed by the image of the symmetry axis with branching index n plus the images of three singular edges with branching index 3. The fundamental group of the underlying space of $M_1(n)/\langle\rho\rangle$ can be determined from the cellular structure of the quotient.

If $n \equiv 2 \pmod 3$, this space (topologically) has a fundamental group generated by four elements x, y, z, u subject to relations $zuy = x^2$, $yux = z^2$, $xuz = y^2$ and $xyz = 1$. The polyhedron $P_1(n)$ with identifications induces a cellular decomposition of the underlying space of $M_1(n)/\langle\rho\rangle$. If $n \equiv 2 \pmod 3$, then the induced triangulation is exactly that of the manifold $M_{3,2} = M_{3,2,2}$ considered in Section 2.

If $n \equiv 1 \pmod 3$, the underlying space (topologically) has a fundamental group generated by four elements x, y, z, u subject to relations $zuz = yx$, $xux = zy$, $yuy = xz$ and $xyz = 1$. This presentation arises from the honey-comb description of the manifold $M_{3,1} = M_{3,1,2}$ considered in Section 2. A further rotation of order 3 about the symmetry axis of the orbifold $M_1(n)/\langle\rho\rangle$ gives the representation of the manifold $M_1(n)$, $n \equiv 1, 2 \pmod 3$ as a $(3n)$-fold covering of the orbifold $\mathcal{O}_{3n,3}(\mathcal{W})$, where $\mathcal{W}$ is the Whitehead link. The singular set $\mathcal{W}$ consists of the image of the symmetry axis with branching index $3n$ and the image of the edge u with branching index 3.

If $n \equiv 0 \pmod 3$, then the underlying space (topologically) is the 3-sphere, and $M_1(n)$ is the n-fold covering of $\mathbf{S}^3$ branched over a 4-component link. One component is the image of the north-south axis of the polyhedron $P_1(n)$ under the n-rotational action, and it has branching index n. The other three components have branching index 3. A further rotation of order 3 about the symmetry axis of the orbifold $M_1(n)/\langle\rho\rangle$ gives the representation of the manifold $M_1(n)$, $n \equiv 0 \pmod 3$, as a $(3n)$-fold covering of the orbifold $\mathcal{O}_{3n,3}(\mathcal{W})$, where $\mathcal{W}$ is the Whitehead link. $\square$

Kim and Kostrikin defined in [**12, 13**] other infinite series of groups $G_i(n)$ and manifolds $M_i(n)$ arising from polyhedral schemata, $i =$

$2, 3, 4, 5$. The manifolds $M_5(n)$ were completely classified in [**1**], and they are Seifert fibered spaces. The topological classification of manifolds $M_i(n)$, $2 \leq i \leq 4$, will be given in a forthcoming paper.

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