ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 4, 2007

A PARTICULAR TEST ELEMENT OF A FREE SOLVABLE LIE ALGEBRA OF RANK TWO

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ABSTRACT. We prove that a free solvable Lie algebra of solvability class 3 generated by two elements has test rank 1 by giving a particular test element.

1. Introduction. Let F be an n-generator Lie algebra. A set of elements $g_1, g_2, \dots, g_r$, $r \leq n$, is a test set if for every endomorphism φ of F the conditions $\varphi(g_i) = g_i$ for $i = 1, 2, \dots, r$ imply that φ is an automorphism. The test rank of F is minimal cardinality of a test set. A test element is a test set consisting of one element. Well known examples of test elements for free groups were described by Nielsen [**8**] and Turner [**11**]. Mikhalev and Yu in [**6**] described an algorithm to determine test elements of free algebras of rank two.

Recently, Chirkov and Shevelin [**1**] and Esmerligil and Ekici [**4**] have independently shown that all nontrivial elements of a commutant of a free metabelian Lie algebra of rank 2 are test elements. They have further proved that the test rank of a free metabelian Lie algebra of rank n is equal to $n-1$. In the case of free solvable Lie algebras the situation is different from the metabelian case. Roman'kov [**9**] showed that a free solvable group of rank 2 and class 3 has test rank 1, and he constructed a test element for such groups.

The purpose of this article is to construct a test element for free solvable Lie algebras of rank 2 and solvability class 3.

2. Preliminaries and notations. Let F be a free Lie algebra over a field K with free generating set $\{x, y\}$. By $\delta^i F$ we denote the ith term of the derived series of F. We fix the notation $L = F \,/\, \delta^3 F$ for the free solvable Lie algebra generated by the set $\{\overline{x}, \overline{y}\}$ of solvability class 3, where $\overline{x} = x + \delta^3 F$, $\overline{y} = y + \delta^3 F$. Let $\widetilde{x}$, $\widetilde{y}$ denote the cosets $\widetilde{x} = x + \delta^2 F$ and $\widetilde{y} = y + \delta^2 F$. We know from [**2, 3**] that the universal

Received by the editors on Nov. 9, 2004, and in revised form on Feb. 17, 2005.

enveloping algebra $U(L)$ of L is embeddable in the skew field $\mathcal{Q}(L)$ of fractions of the algebra $U(L)$.

Let M be the free metabelian Lie algebra $F \diagup \delta^2 F$. The adjoint representation of M induces a representation of $M \diagup \delta^1 M$ on $\delta^1 M$. Thus, $\delta^1 M$ is furnished with the structure of a left $U\left(M \diagup \delta^1 M\right)$-module. We will denote the action by a dot. For $h \in \delta^1 M$ and g_1, $g_2, \dots, g_n \in M \diagup \delta^1 M$, then

$$g_n \cdot \dots \cdot g_2 \cdot g_1 \cdot h = ((\dots((h, g_1), g_2), \dots), g_n).$$

All necessary information on Fox derivatives can be found in [**5**]. We recall some of this.

For any free Lie algebra G over a field K with free generators $x_1, x_2, \dots, x_n$, we define the left Fox derivatives as the mappings $\partial/(\partial x_i) : U(G) \to U(G)$, $1 \le i \le n$, satisfying the following conditions whenever $\alpha, \beta \in K$, $u, v \in U(G)$:

1) $\partial/(\partial x_i)(x_j) = \delta_{ij}$ (Kronecker delta);
2) $\partial/(\partial x_i)(\alpha u + \beta v) = \alpha(\partial u/\partial x_i) + \beta(\partial v/\partial x_i)$;
3) $\partial/(\partial x_i)(uv) = (\partial u/\partial x_i)\varepsilon(v) + u(\partial v/\partial x_i)$;

where $\varepsilon : U(K) \to K$ is the augmentation homomorphism defined by $\varepsilon(x_i) = 0$, $1 \le i \le n$. The kernel Δ of the augmentation homomorphism ε is a free left $U(G)$-module with free basis $\{x_1, x_2, \dots, x_n\}$, and the mappings $\partial/(\partial x_i)$ are projections to the corresponding free cyclic direct summands. Thus, any element $u \in \Delta$ can be uniquely written in the form

$$u = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} x_i.$$

We need the following technical lemmas. The first lemma is an immediate consequence of the definitions and the second one can be found in [**13**].

Lemma 1. *Let J be an arbitrary ideal of $U(G)$, and let $u \in \Delta$. Then $u \in J\Delta$ if and only if $(\partial u/\partial x_i) \in J$ for each i, $1 \le i \le n$.*

Lemma 2 [**13**]. *Let R be an ideal of G, and let $u \in G$. Then $u \in I_R\Delta$ if and only if $u \in \delta^1 R$, where I_R is the ideal of $U(G)$ generated by R.*

A criterion for n elements of a free Lie algebra of rank n to be a generating set has been obtained by Shpilrain in [**10**].

Theorem 3 [**10**]. *Let R be an ideal of G, and let $y_1, y_2, \dots, y_n$ be elements of G. Then the Lie algebra $G \,/\, \delta^1 R$ is generated by the images $\widehat{y_1}, \widehat{y_2}, \dots, \widehat{y_n}$ of $y_1, y_2, \dots, y_n$ if and only if the matrix $(\partial \widehat{y_i} / \partial x_j)_{1 \le i,j \le n}$ has a left inverse over $U(G \,/\, R)$.*

Now we consider the free solvable Lie algebra $L = F \,/\, \delta^3 F$. On the universal enveloping algebra $U(L)$, left Fox derivatives are defined so that their values are in $U(M)$. Hence, for every element $f = f\,(\overline{x}, \overline{y})$ of L, we have

$$f\,(\widetilde{x}, \widetilde{y}) = \frac{\partial f}{\partial x}\,\widetilde{x} + \frac{\partial f}{\partial y}\,\widetilde{y}. \tag{1}$$

For α_1, $\alpha_2 \in U\,(M)$ and $g = g\,(\widetilde{x}, \widetilde{y}) \in M$

$$\alpha_1 \widetilde{x} + \alpha_2 \widetilde{y} = g\,(\widetilde{x}, \widetilde{y}) \tag{2}$$

implies the existence of an element $h \in L$ such that $h\,(\widetilde{x}, \widetilde{y}) = g\,(\widetilde{x}, \widetilde{y})$ and

$$\alpha_1 = \frac{\partial h}{\partial x}, \quad \alpha_2 = \frac{\partial h}{\partial y}.$$

Assume that $v = v\,(x, y) \in U\,(F)$. As in [**7**] we need the formula

$$v(a_1 + h_1,\ a_2 + h_2) = v(a_1, a_2) + \frac{\partial v}{\partial a_1}\,h_1 + \frac{\partial v}{\partial a_2}\,h_2 \tag{3}$$

where a_1, $a_2 \in L$, h_1, $h_2 \in \delta^2 L$.

We use the notation $\operatorname{adv}(w) = (w, v)$, $v, w \in L$. Clearly, if $v \in \delta^2 L$, then $\mathrm{ad}^2 v = 0$ and $e^{\mathrm{adv}}(w) = w + \operatorname{adv}(w)$ is an inner automorphism.

Define the element $u \in L$ as

$$u = \left(\left(\left(\left(\left(\overline{y}, \overline{x}\right), \overline{x}\right), \overline{x}\right), \overline{y}\right), \left(\overline{y}, \overline{x}\right)\right) + \left(\left(\left(\left(\overline{y}, \overline{x}\right), \overline{x}\right), \overline{y}\right), \left(\left(\overline{y}, \overline{x}\right), \overline{y}\right)\right).$$

3. Test elements. In this section we prove that the free solvable Lie algebra L has test elements.

Proposition 4. *Let Ψ be an endomorphism of M such that $\Psi(u) = u$. Then $\Psi(\widetilde{x}) = \widetilde{x}(mod\,\delta^1 M)$ and $\Psi(\widetilde{y}) = \widetilde{y}(mod\,\delta^1 M)$.*

Proof. Let Ψ be defined by

$$\begin{aligned}\Psi :\widetilde{x} &\longrightarrow a\widetilde{x} + b\widetilde{y} + f\\ \widetilde{y} &\longrightarrow c\widetilde{x} + d\widetilde{y} + g\end{aligned}$$

where $f, g \in \delta^1 M$, $a, b, c, d \in K$. Using the Jacobi identity, we compute $\Psi(u)$ as follows:

$$\begin{aligned}\Psi(u) = {} & (((((c\widetilde{x} + d\widetilde{y}, a\widetilde{x} + b\widetilde{y}), a\widetilde{x} + b\widetilde{y}), a\widetilde{x} + b\widetilde{y})\, c\widetilde{x} + d\widetilde{y}),\\ & \qquad (c\widetilde{x} + d\widetilde{y}, a\widetilde{x} + b\widetilde{y}))\\ & + ((((c\widetilde{x} + d\widetilde{y}, a\widetilde{x} + b\widetilde{y}), a\widetilde{x} + b\widetilde{y}), c\widetilde{x} + d\widetilde{y}),\\ & \qquad ((c\widetilde{x} + d\widetilde{y}, a\widetilde{x} + b\widetilde{y}), c\widetilde{x} + d\widetilde{y})) + v(\widetilde{x}, \widetilde{y}, f, g)\\ = {} & (ad - bc)^2 \big[\, a^2 d\, (((((\widetilde{y}, \widetilde{x}), \widetilde{x}), \widetilde{x}), \widetilde{y}), (\widetilde{y}, \widetilde{x}))\\ & + ad^2\, ((((\widetilde{y}, \widetilde{x})\, \widetilde{,x})\, \widetilde{y}), ((\widetilde{y}, \widetilde{x}), \widetilde{y}))\\ & + a^2 c\, (((((\widetilde{y}, \widetilde{x}), \widetilde{x}), \widetilde{x})\, \widetilde{x},), (\widetilde{y}, \widetilde{x}))\\ & + abc\, (((((\widetilde{y}, \widetilde{x}), \widetilde{y}), \widetilde{x}), \widetilde{x}), (\widetilde{y}, \widetilde{x}))\\ & + abc\, (((((\widetilde{y}, \widetilde{x}), \widetilde{x}), \widetilde{y}), \widetilde{x}), (\widetilde{y}, \widetilde{x}))\\ & + b^2 c\, (((((\widetilde{y}, \widetilde{x}), \widetilde{y}), \widetilde{y}), \widetilde{x}), (\widetilde{y}, \widetilde{x}))\\ & + abd\, (((((\widetilde{y}, \widetilde{x}), \widetilde{y}), \widetilde{x}), \widetilde{y}), (\widetilde{y}, \widetilde{x}))\\ & + abd\, (((((\widetilde{y}, \widetilde{x}), \widetilde{x}), \widetilde{y}), \widetilde{y}), (\widetilde{y}, \widetilde{x}))\\ & + b^2 d\, (((((\widetilde{y}, \widetilde{x}), \widetilde{y}), \widetilde{y}), \widetilde{y}), (\widetilde{y}, \widetilde{x}))\\ & + ac^2\, ((((\widetilde{y}, \widetilde{x}), \widetilde{x}), \widetilde{x})\, ((\widetilde{y}, \widetilde{x}), \widetilde{x}))\\ & + bc^2\, ((((\widetilde{y}, \widetilde{x}), \widetilde{y}), \widetilde{x}), ((\widetilde{y}, \widetilde{x}), \widetilde{x}))\\ & + acd\, ((((\widetilde{y}, \widetilde{x}), \widetilde{x}), \widetilde{y})\, ((\widetilde{y}, \widetilde{x}), \widetilde{x}))\\ & + bcd\, ((((\widetilde{y}, \widetilde{x}), \widetilde{y}), \widetilde{y})\, ((\widetilde{y}, \widetilde{x}), \widetilde{x}))\\ & + acd\, ((((\widetilde{y}, \widetilde{x}), \widetilde{x}), \widetilde{x}), ((\widetilde{y}, x), \widetilde{y}))\\ & + bcd\, ((((\widetilde{y}, \widetilde{x}), \widetilde{y}), \widetilde{x}), ((\widetilde{y}, \widetilde{x}), \widetilde{y}))\\ & + bd^2\, ((((\widetilde{y}, \widetilde{x}), \widetilde{y}), \widetilde{y}), ((\widetilde{y}, \widetilde{x}), \widetilde{y}))\, \big] + v(\widetilde{x}, \widetilde{y}, f, g)\\ = {} & (ad - bc)^2 \big[\, \big(2abc + a^2 d\big)\, (((((\widetilde{y}, \widetilde{x}), \widetilde{x}), \widetilde{x}), \widetilde{y}), (\widetilde{y}, \widetilde{x}))\\ & + (bcd + ad^2)\, ((((\widetilde{y}, \widetilde{x}), \widetilde{x})\, \widetilde{y}), ((\widetilde{y}, \widetilde{x}), \widetilde{y}))\end{aligned}$$

$$
\begin{aligned}
&+ a^2c\,(((((\widetilde{y},\widetilde{x})\,,\widetilde{x})\,,\widetilde{x})\,\widetilde{x},)\,,(\widetilde{y},\widetilde{x}))\\
&+ 2abc\,((((\widetilde{y},\widetilde{x})\,,\widetilde{x})\,,(\widetilde{y},\widetilde{x}))\,,(\widetilde{y},\widetilde{x}))\\
&+ b^2c\,((((\widetilde{y},\widetilde{x})\,,\widetilde{y})\,,(\widetilde{y},\widetilde{x}))\,,(\widetilde{y},\widetilde{x}))\\
&+ (b^2c+2abd)\,(((((\widetilde{y},\widetilde{x})\,,\widetilde{x})\,,\widetilde{y})\,,\widetilde{y})\,,(\widetilde{y},\widetilde{x}))\\
&+ b^2d\,(((((\widetilde{y},\widetilde{x})\,,\widetilde{y})\,,\widetilde{y})\,,\widetilde{y})\,,(\widetilde{y},\widetilde{x}))\\
&+ ac^2\,((((\widetilde{y},\widetilde{x})\,,\widetilde{x})\,,\widetilde{x})\,((\widetilde{y},\widetilde{x})\,,\widetilde{x}))\\
&+ (bc^2+acd)\,((((\widetilde{y},\widetilde{x})\,,\widetilde{x})\,,\widetilde{y})\,,((\widetilde{y},\widetilde{x})\,,\widetilde{x}))\\
&+ bcd\,((((\widetilde{y},\widetilde{x})\,,\widetilde{y})\,,\widetilde{y})\,,((\widetilde{y},\widetilde{x})\,,\widetilde{x}))\\
&+ acd\,((((\widetilde{y},\widetilde{x})\,,\widetilde{x})\,,\widetilde{x})\,,((\widetilde{y},\widetilde{x})\,,\widetilde{y}))\\
&+ bd^2\,((((\widetilde{y},\widetilde{x})\,,\widetilde{y})\,,\widetilde{y})\,,((\widetilde{y},\widetilde{x})\,,\widetilde{y}))\,\big] + v\,(\widetilde{x},\widetilde{y},f,g)
\end{aligned}
$$

where $v(\widetilde{x},\widetilde{y},f,g)$ stands for the terms containing $\widetilde{x}$, $\widetilde{y}$, f, g. Comparing the degrees of the basis commutators in the equality $\Psi(u)=u$ we obtain $a=d=1$ and $b=c=0$. Hence, $\Psi(\widetilde{x})=\widetilde{x}(\operatorname{mod}\delta^1 M)$ and $\Psi(\widetilde{y})=\widetilde{y}(\operatorname{mod}\delta^1 M)$. □

Remark 5. Since $\delta^1 M$ is a free left $U\left(M\,/\,\delta^1 M\right)$-module generated by $(\widetilde{y},\widetilde{x})$, every element h of $\delta^1 M$ can be represented as

$$h=\left(\sum_{i,j}\alpha_{ij}\widetilde{x}^i\cdot\widetilde{y}^j\right)\cdot(\widetilde{y},\widetilde{x})$$

where $\alpha_{ij}\in K$, $i,j\geq 0$. If f is any element of $\delta^2 L$, then the values of the left Fox derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are in $U(\delta^1 M)$. The algebra $U(\delta^1 M)$ is a free K-module generated by the monomials of the form

$$x^i\cdot y^j\cdot(y,x)^k,\quad i,j\geq 0,\quad k\geq 1.$$

Proposition 6. *Let Ψ be an endomorphism of M given by*

$$
\begin{aligned}
\Psi:\widetilde{x}&\longrightarrow\widetilde{x}+\left(\sum_{i,j}\alpha_{ij}\widetilde{x}^i\cdot\widetilde{y}^j\right)\cdot(\widetilde{y},\widetilde{x})\\
\widetilde{y}&\longrightarrow\widetilde{y}+\left(\sum_{k,l}\beta_{kl}\widetilde{x}^k\cdot\widetilde{y}^l\right)\cdot(\widetilde{y},\widetilde{x})
\end{aligned}
$$

where α_{ij}, $\beta_{kl}\in K$. If $\Psi(u)=u$, then $i=k+1$ and $l=j+1$.

Proof. By the concrete calculations we see that

$$\begin{aligned}\Psi(u) &= (((((\widetilde{y}+g,\widetilde{x}+f),\widetilde{x}+f),\widetilde{x}+f),\widetilde{y}+g),(\widetilde{y}+g,\widetilde{x}+f))\\ &\quad+((((\widetilde{y}+g,\widetilde{x}+f),\widetilde{x}+f),\widetilde{y}+g),((\widetilde{y}+g,\widetilde{x}+f),\widetilde{y}+g))\\ &= u+(((((\widetilde{y},\widetilde{x}),\widetilde{x}),\widetilde{x}),\widetilde{y}),(g,\widetilde{x}))-(((((\widetilde{y},\widetilde{x}),\widetilde{x}),\widetilde{x}),\widetilde{y}),(f,\widetilde{y}))\\ &\quad+(((((g,\widetilde{x}),\widetilde{x}),\widetilde{x})\,\widetilde{y}),(\widetilde{y},\widetilde{x}))-(((((f,\widetilde{y}),\widetilde{x}),\widetilde{x})\,\widetilde{y}),(\widetilde{y},\widetilde{x}))\\ &\quad+(((((\widetilde{y},\widetilde{x}),f),\widetilde{x})\,\widetilde{y}),(\widetilde{y},\widetilde{x}))+(((((\widetilde{y},\widetilde{x}),\widetilde{x})\,f),\widetilde{y}),(\widetilde{y},\widetilde{x}))\\ &\quad+((((\widetilde{y},\widetilde{x}),\widetilde{x}),\widetilde{x}),g),(\widetilde{y},\widetilde{x})-((((f,\widetilde{y}),\widetilde{x}),\widetilde{y}),((\widetilde{y},\widetilde{x}),\widetilde{y}))\\ &\quad+((((g,\widetilde{x}),\widetilde{x}),\widetilde{y}),((\widetilde{y},\widetilde{x}),\widetilde{y}))+((((\widetilde{y},\widetilde{x}),f),\widetilde{y}),((\widetilde{y},\widetilde{x}),\widetilde{y}))\\ &\quad+((((\widetilde{y},\widetilde{x}),\widetilde{x}),\widetilde{y}),((g,\widetilde{x}),\widetilde{y}))+((((\widetilde{y},\widetilde{x}),\widetilde{x}),g),((\widetilde{y},\widetilde{x}),\widetilde{y}))\\ &\quad-((((\widetilde{y},\widetilde{x}),\widetilde{x}),\widetilde{y}),((f,\widetilde{y}),\widetilde{y}))-((((\widetilde{y},\widetilde{x}),\widetilde{x}),\widetilde{y}),(g,(\widetilde{y},\widetilde{x}))),\end{aligned}$$

where $f = \left(\sum_{i,j}\alpha_{ij}\widetilde{x}^i\cdot\widetilde{y}^j\right)\cdot(\widetilde{y},\widetilde{x})$, $g = \left(\sum_{k,l}\beta_{kl}\widetilde{x}^k\cdot\widetilde{y}^l\right)\cdot(\widetilde{y},\widetilde{x})$. Comparing the degrees in the equality $\Psi(u)=u$, we see that the term

$$(((((\widetilde{y},\widetilde{x}),\widetilde{x}),\widetilde{x}),\widetilde{y}),(g,\widetilde{x}))-(((((\widetilde{y},\widetilde{x}),\widetilde{x}),\widetilde{x}),\widetilde{y}),(f,\widetilde{y}))$$

must be equal to zero. Therefore, $(g,\widetilde{x})=(f,\widetilde{y})$, and this implies that $i=k+1$ and $l=j+1$. □

Lemma 7. *Let φ be the endomorphism of L defined as*

$$\begin{aligned}\varphi : \overline{x} &\longrightarrow \overline{x}+a\\ \overline{y} &\longrightarrow \overline{y}+b\end{aligned}$$

where. a, $b\in\delta^2L$. Assume that φ acts identically on δ^2L. Then φ is an inner automorphism of L induced by some element of δ^2L.

Proof. Consider the element $g=g(\overline{x},\overline{y})$ of δ^2L. Using the formula (3), we compute

$$\begin{aligned}\varphi(g) &= g\left(\varphi(\overline{x}),\varphi(\overline{y})\right)\\ &= g(\overline{x},\overline{y})+\frac{\partial g}{\partial x}\cdot a+\frac{\partial g}{\partial y}\cdot b\\ &= g(\overline{x},\overline{y}),\end{aligned}$$

that is,

$$\frac{\partial g}{\partial x} \cdot a + \frac{\partial g}{\partial y} \cdot b = 0. \tag{4}$$

Computing Fox derivatives of (4) in $U(L)$, we obtain

$$\begin{aligned} \frac{\partial g}{\partial x} \cdot \frac{\partial a}{\partial x} + \frac{\partial g}{\partial y} \cdot \frac{\partial b}{\partial x} &= 0 \\ \frac{\partial g}{\partial x} \cdot \frac{\partial a}{\partial y} + \frac{\partial g}{\partial y} \cdot \frac{\partial b}{\partial y} &= 0. \end{aligned} \tag{5}$$

Since $g \in \delta^2 L$ then

$$\frac{\partial g}{\partial x} \cdot \widetilde{x} + \frac{\partial g}{\partial y} \cdot \widetilde{y} = 0.$$

Passing from $U(F \diagup \delta^2 F)$ to a skew field $\mathbf{Q}(F \diagup \delta^2 F)$, we get

$$\frac{\partial g}{\partial x} = - \frac{\partial g}{\partial y} \cdot \widetilde{y} \cdot \widetilde{x}^{-1}.$$

Substituting this in (5) yields

$$\begin{aligned} - \frac{\partial g}{\partial y} \cdot \widetilde{y} \cdot \widetilde{x}^{-1} \cdot \frac{\partial a}{\partial x} + \frac{\partial g}{\partial y} \cdot \frac{\partial b}{\partial x} &= 0 \\ - \frac{\partial g}{\partial y} \cdot \widetilde{y} \cdot \widetilde{x}^{-1} \cdot \frac{\partial a}{\partial y} + \frac{\partial g}{\partial y} \cdot \frac{\partial b}{\partial y} &= 0. \end{aligned}$$

This implies that

$$\begin{aligned} - \widetilde{y} \cdot \widetilde{x}^{-1} \cdot \frac{\partial a}{\partial x} + \frac{\partial b}{\partial x} &= 0 \\ - \widetilde{y} \cdot \widetilde{x}^{-1} \cdot \frac{\partial a}{\partial y} + \frac{\partial b}{\partial y} &= 0, \end{aligned} \tag{6}$$

that is,

$$\frac{\partial b}{\partial x} = \widetilde{y} \cdot \widetilde{x}^{-1} \cdot \frac{\partial a}{\partial x} \quad \text{and} \quad \frac{\partial b}{\partial y} = \widetilde{y} \cdot \widetilde{x}^{-1} \cdot \frac{\partial a}{\partial y}. \tag{7}$$

Since the derivatives of a and b are contained in $U(\delta^1 M)$, the elements $\widetilde{y} \cdot \widetilde{x}^{-1} \cdot \partial a/\partial x$ and $\widetilde{y} \cdot \widetilde{x}^{-1} \cdot \partial a/\partial y$ can be written in the form

$$\widetilde{y} \cdot \widetilde{x}^{-1} \cdot \frac{\partial a}{\partial x} = \sum \alpha_{ijk} \widetilde{x}^i \cdot \widetilde{y}^j \cdot (\widetilde{y}, \widetilde{x})^k$$
$$\widetilde{y} \cdot \widetilde{x}^{-1} \cdot \frac{\partial a}{\partial y} = \sum \beta_{\ell mn} \widetilde{x}^\ell \cdot \widetilde{y}^m \cdot (\widetilde{y}, \widetilde{x})^n,$$

where $\alpha_{ijk}, \ \beta_{\ell mn} \in K, \ i, \ell \geq 0, \ j, k, m, n \geq 1$. Therefore, for the elements $\alpha_1 = \sum \alpha_{ijk} \widetilde{x}^i \cdot \widetilde{y}^{j-1} \cdot (\widetilde{y}, \widetilde{x})^k$ and $\alpha_2 = \sum \beta_{\ell mn} \widetilde{x}^\ell \cdot \widetilde{y}^{m-1} \cdot (\widetilde{y}, \widetilde{x})^n$ of $U(M)$ the elements $\partial a/\partial x$ and $\partial a/\partial y$ are equal to

$$\frac{\partial a}{\partial x} = \widetilde{x}\alpha_1 \quad \text{and} \quad \frac{\partial a}{\partial y} = \widetilde{x}\alpha_2.$$

Using (7) gives

$$\frac{\partial b}{\partial x} = \widetilde{y}\alpha_1 \quad \text{and} \quad \frac{\partial b}{\partial y} = \widetilde{y}\alpha_2. \tag{8}$$

By formula (1), we have

$$\frac{\partial b}{\partial x} \cdot \widetilde{x} + \frac{\partial b}{\partial y} \cdot \widetilde{y} = 0. \tag{9}$$

Hence, in view of (8),

$$\alpha_1 \widetilde{x} + \alpha_2 \widetilde{y} = 0.$$

Taking into account (2) we conclude that there exists a $v \in \delta^2 L$ for which $\alpha_1 = \partial v/\partial x$, $\alpha_2 = \partial v/\partial y$.

Now, since

$$\frac{\partial b}{\partial x} = \widetilde{y} \cdot \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial b}{\partial y} = \widetilde{y} \cdot \frac{\partial v}{\partial y},$$

$b = (\overline{y}, v)$. Likewise, $a = (\overline{x}, v)$. Consequently,

$$\varphi(\overline{x}) = \overline{x} + (\overline{x}, v) = e^{\mathrm{ad} v}(\overline{x})$$
$$\varphi(\overline{y}) = \overline{y} + (\overline{y}, v) = e^{\mathrm{ad} v}(\overline{y}). \qquad \square$$

Theorem 8. *The free solvable Lie algebra L contains test elements.*

Proof. Let φ be an endomorphism of L defined by

$$\varphi : \overline{x} \longrightarrow a\ \overline{x} + b\ \overline{y} + \left(\sum_{i,j} \alpha_{ij} (\overline{y}, \overline{x}), \underbrace{\overline{x}), \dots, \overline{x})}_{i\text{-times}}, \underbrace{\overline{y}), \dots, \overline{y}}_{j\text{-times}} \right) + h_1$$

$$\overline{y} \longrightarrow c\overline{x} + d\ \overline{y} + \left(\sum_{k,l} \beta_{kl} (\overline{y}, \overline{x}), \underbrace{\overline{x}), \dots, \overline{x})}_{k\text{-times}}, \underbrace{\overline{y}), \cdots, \overline{y}}_{l\text{-times}} \right) + h_2,$$

where $a, b, c, d, \alpha_{ij}, \beta_{kl} \in K,\ i, j, k, \ell \geq 0,\ h_1, h_2 \in \delta^2 L$. Assume that $\varphi(u) = u$. It is clear that φ induces the homomorphism Ψ of M which is given by

$$\Psi : \widetilde{x} \longrightarrow a\widetilde{x} + b\widetilde{y} + \left(\sum_{i,j} \alpha_{ij} \widetilde{x}^i \cdot \widetilde{y}^j \right) \cdot (\widetilde{y}, \widetilde{x})$$

$$\widetilde{y} \longrightarrow c\widetilde{x} + d\widetilde{y} + \left(\sum_{k,l} \beta_{kl} \widetilde{x}^k \cdot \widetilde{y}^l \right) \cdot (\widetilde{y}, \widetilde{x}) .$$

Now still $\Psi(u) = u$. Then by Proposition 4 and Proposition 6, Ψ acts identically modulo $\delta^1 M$ and $i = k + 1$, $l = j + 1$. Let

$$w = \left(\sum_{i,j} \alpha_{ij} \widetilde{x}^{i-1} \cdot \widetilde{y}^j \right) \cdot (\widetilde{y}, \widetilde{x}) .$$

Then Ψ will be

$$\Psi : \widetilde{x} \longrightarrow \widetilde{x} + (\widetilde{x}, w) = (1 + adw)(\widetilde{x})$$

$$\widetilde{y} \longrightarrow \widetilde{y} + (\widetilde{y}, w) = (1 + adw)(\widetilde{y}).$$

Since Ψ is induced by φ, then φ will be

$$\varphi : \overline{x} \longrightarrow (1 + adw)(\overline{x}) + h_1$$

$$\overline{y} \longrightarrow (1 + adw)(\overline{y}) + h_2.$$

We modify φ by multiplying it from left by the mapping 1-adw. This yields

$$\begin{aligned}(1 - adw)\varphi(\overline{x}) &= \varphi(\overline{x}) - (w, \varphi(\overline{x})) \\ &= \overline{x} + h_1 - (w, (w, \overline{x})) - (w, h_1)\end{aligned}$$

$$
\begin{aligned}
(1-adw)\varphi\,(\overline{y}) &= \varphi\,(\overline{y}) - (w,\varphi\,(\overline{y}))\\
&= \overline{y} + h_2 - (w,(w,\overline{y})) - (w,h_2).
\end{aligned}
$$

Therefore $\theta = (1-adw)\varphi$ is an endomorphism defined as

$$
\begin{aligned}
\theta :\overline{x} &\longrightarrow \overline{x} + c\\
\overline{y} &\longrightarrow \overline{y} + e,
\end{aligned}
$$

where $c = h_1 - (w,(w,\overline{x})) - (w,h_1)$, $e = h_2 - (w,(w,\overline{y})) - (w,h_2)$. In this case,

$$
\theta(u) = u - (w,u). \tag{10}
$$

Computing Fox derivatives of the left and right side of (10), we obtain

$$
\begin{aligned}
\frac{\partial u}{\partial\theta(\overline{x})}\cdot\frac{\partial\theta(\overline{x})}{\partial x} + \frac{\partial u}{\partial\theta(\overline{y})}\cdot\frac{\partial\theta(\overline{y})}{\partial x} &= \frac{\partial u}{\partial x} - w\,\frac{\partial u}{\partial x} + u\,\frac{\partial w}{\partial x}\\
\frac{\partial u}{\partial\theta(\overline{x})}\cdot\frac{\partial\theta(\overline{x})}{\partial y} + \frac{\partial u}{\partial\theta(\overline{y})}\cdot\frac{\partial\theta(\overline{y})}{\partial y} &= \frac{\partial u}{\partial y} - w\,\frac{\partial u}{\partial y} + u\,\frac{\partial w}{\partial y}.
\end{aligned}
$$

In view of the definition of θ,

$$
\begin{aligned}
\frac{\partial u}{\partial\theta(\overline{x})}\cdot\left(1+\frac{\partial c}{\partial x}\right) + \frac{\partial u}{\partial\theta(\overline{y})}\cdot\frac{\partial e}{\partial x} - (1-w)\,\frac{\partial u}{\partial x} &= u\,\frac{\partial w}{\partial x}\\
\frac{\partial u}{\partial\theta(\overline{x})}\cdot\frac{\partial c}{\partial y} + \frac{\partial u}{\partial\theta(\overline{y})}\left(1+\frac{\partial e}{\partial y}\right) - (1-w)\,\frac{\partial u}{\partial y} &= u\frac{\partial w}{\partial y}.
\end{aligned}
$$

It is clear that, since $u, e, c \in \delta^2 L$, the terms on the left sides of these equations are contained in the left ideal J of $U(L)$ generated by $\delta^1 L$. Hence $u(\partial w/\partial x)$, $u(\partial w/\partial y) \in J$. This implies $(\partial w/\partial x)$, $(\partial w/\partial y) \in J$. By Lemmas 1 and 2 we obtain $w \in \delta^2 L$. Keeping in mind that

$$
w = \sum \alpha_{ij}(\overline{y},\overline{x}),\underbrace{\overline{x}),\dots),\overline{x})}_{(i-1)\text{-times}},\underbrace{\overline{y}),\dots),\overline{y})}_{j\text{-times}},
$$

$w \in \delta^2 L$ holds only if $w = 0$. Therefore $\theta = \varphi$ and so $\varphi(\overline{x}) = \overline{x} + c$, $\varphi(\overline{y}) = \overline{y} + e$. If $c = e = 0$, then φ is the identity automorphism. Hence, u is a test element of L.

Assume $(c,e) \neq (0,0)$. We now show that φ acts identically on $\delta^2 L$. Let $v = v(\overline{x},\overline{y}) \in \delta^2 L$. Using (3), compute $\varphi(u)$ as follows:

$$\begin{aligned}\varphi(u(\overline{x},\overline{y})) &= u(\varphi(\overline{x}),\varphi(\overline{y}))\\ &= u(\overline{x}+c,\overline{y}+e)\\ &= u(\overline{x},\overline{y}) + \frac{\partial u}{\partial x}\cdot c + \frac{\partial u}{\partial y}\cdot e\\ &= u.\end{aligned}$$

This implies $(\partial u/\partial x)\cdot c + (\partial u/\partial y)\cdot e = 0$.

Computing $\varphi(v)$ yields

$$\begin{aligned}\varphi(v(\overline{x},\overline{y})) &= v(\varphi(\overline{x}),\varphi(\overline{y}))\\ &= v(\overline{x},\overline{y}) + \frac{\partial v}{\partial x}\cdot c + \frac{\partial v}{\partial y}\cdot e.\end{aligned}$$

Let

$$\frac{\partial v}{\partial x}\cdot c + \frac{\partial v}{\partial y}\cdot e = \alpha.$$

Assume that $\alpha \neq 0$. Since the system of equations

$$\begin{aligned}\frac{\partial u}{\partial x}\cdot c + \frac{\partial u}{\partial y}\cdot e &= 0\\ \frac{\partial v}{\partial x}\cdot c + \frac{\partial v}{\partial y}\cdot e &= \alpha\end{aligned}$$

has a nontrivial solution, the coefficient matrix

$$\begin{bmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{bmatrix}$$

must be invertible over $U(F/\delta^2 F)$. By Theorem 3 the elements u and v are free generators of L. In the expression of u there are no linear terms. So, it is clear that u cannot be a free generator. This contradiction implies $\alpha = 0$, that is, φ acts identically on $\delta^2 L$. By Lemma 7, φ is an inner automorphism of L. Hence u is a test element for L. □

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