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ON THE EVALUATION OF A DEFINITE INTEGRAL INVOLVING NESTED SQUARE ROOT FUNCTIONS

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1. Introduction. There are many and varied methods one can employ when seeking the evaluation of definite integrals in closed form. In many circumstances the method employed reflects some intrinsic property of the integrand in question. For example, certain symmetry considerations may be exploited as in the case of determining the convergent improper integral of e^{-x^2} over $\mathbf{R}$. In like manner, we shall in this paper take advantage of a structural feature of a class of nested square root functions, in order that we may evaluate their corresponding definite integrals which on first acquaintance appear rather intractable. In particular the functions in question will be composed of a finite product of reciprocals of the form

$$R_N(x) = \sqrt{2+\sqrt{2+\cdots+\sqrt{2+x}}}, \tag{1}$$

in which $R_N(x)$ consists of N nested square roots. By restricting the intervals of integration to finite subintervals of $[2,\infty)$, we shall see that a simple application of a hyperbolic function substitution together with some standard identities will result in closed-form expressions for these definite integrals. We begin with a technical lemma in which, for all values of x in $[2,\infty)$ it is shown that the expression in (1) can be rewritten in terms of a hyperbolic cosine function of a suitable variable.

Lemma 1. *If $x \geq 2$ and N is a positive integer, then there exists a unique $\theta \geq 0$, such that*

$$R_N(x) = \sqrt{2+\sqrt{2+\cdots+\sqrt{2+x}}} = 2\cosh\left(\frac{\theta}{2^N}\right).$$

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Proof. As the mapping $2\cosh(\cdot) : [0,\infty) \to [2,\infty)$ is a bijection, a unique θ in $[0,\infty)$ must exist such that $x = 2\cosh\theta$. Using this hyperbolic function substitution, we show by induction that

$$R_k(x) = 2\cosh\left(\frac{\theta}{2^k}\right) \tag{2}$$

for $k = 1, 2, \ldots$. For $k = 1$, this follows directly from the identity $2\cosh(\phi/2) = \sqrt{2 + 2\cosh\phi}$ upon setting $\phi = \theta$. Suppose that $R_{\kappa-1}(x) = 2\cosh(\theta/2^{\kappa-1})$ for a fixed $\kappa \geq 2$. By definition, $R_\kappa(x) = \sqrt{2 + R_{\kappa-1}(x)}$. From the inductive assumption together with the foregoing hyperbolic function identity we deduce, upon setting $\phi = \theta/2^{\kappa-1}$, that (2) holds when $k = \kappa$. Hence, by induction (2) holds for all k. $\square$

Applying Lemma 1 we can now easily evaluate the definite integral for the function constructed from a product of reciprocals of nested square roots $R_i(x)$ for $i = 1, 2, \ldots, N$. Central to our method will be the use of a hyperbolic function substitution from which the product $\prod_{i=1}^{N} R_i(x)$ can be shown to be telescoping.

Theorem 1. *If $\xi \geq 2$ and N is a positive integer, then the definite integral of the function $\prod_{i=1}^{N} R_i(x)^{-1}$ over the interval $[2, \xi]$ is given by*

$$\begin{aligned} \frac{1}{2^N}\int_2^\xi & \frac{dx}{\sqrt{2+x}\sqrt{2+\sqrt{2+x}}\cdots\sqrt{2+\sqrt{2+\cdots+\sqrt{2+x}}}} \\ &= \left(\frac{\xi+\sqrt{\xi^2-4}}{2}\right)^{1/2^N} + \left(\frac{\xi+\sqrt{\xi^2-4}}{2}\right)^{-1/2^N} - 2. \end{aligned} \tag{3}$$

Proof. Note the integrand in (3) is well defined over the interval $[2, \xi]$. As $x \geq 2$ we may make the variable substitution $x = 2\cosh(\theta)$ from which we see the left-hand side of (3) transforms via Lemma 1 to

$$\frac{1}{2^N}\int_0^{\cosh^{-1}(\xi/2)} \frac{2\sinh(\theta)}{\prod_{i=1}^{N} 2\cosh(\theta/2^i)}\, d\theta. \tag{4}$$

Recalling the identity $\sinh(2\phi) = 2\cosh(\phi)\sinh(\phi)$, observe upon setting $\phi = \xi/2^i$ the following telescoping product

$$\prod_{i=1}^{N} 2\cosh\left(\frac{\theta}{2^i}\right) = \prod_{i=1}^{N} \frac{\sinh(\theta/(2^{i-1}))}{\sinh(\theta/(2^i))} = \frac{\sinh(\theta)}{\sinh(\theta/2^N)}. \tag{5}$$

Consequently, the definite integral in (4) reduces to

$$\begin{aligned} \frac{1}{2^N}\int_0^{\cosh^{-1}(\xi/2)} 2\sinh\left(\frac{\theta}{2^N}\right) d\theta &= \frac{1}{2^N}\left[2^{N+1}\cosh\left(\frac{\theta}{2^N}\right)\right]_0^{\cosh^{-1}(\xi/2)} \\ &= 2\cosh\left(\frac{1}{2^N}\cosh^{-1}\left(\frac{\xi}{2}\right)\right) - 2. \end{aligned} \tag{6}$$

By an application of the logarithmic form of the inverse hyperbolic cosine function, see [**1**], we see that

$$\cosh^{-1}\left(\frac{\xi}{2}\right) = \ln\left(\frac{\xi + \sqrt{\xi^2 - 4}}{2}\right),$$

from which (2) now immediately follows after substituting the previous expression into (6) and recalling that $\cosh(\phi) = (e^{\phi} + e^{-\phi})/2$. $\square$

To illustrate Theorem 1 we give an example of an evaluation of (3) in terms of the golden ratio $\Phi = (1+\sqrt{5})/2$.

Example 1. Setting $\xi = \sqrt{5}$ in (3), we see that

$$\begin{aligned} \frac{1}{2^N}\int_2^{\sqrt{5}} &\frac{dx}{\sqrt{2+x}\sqrt{2+\sqrt{2+x}}\cdots\sqrt{2+\sqrt{2+\cdots+\sqrt{2+x}}}} \\ &= \Phi^{1/2^N} + \Phi^{-1/2^N} - 2. \end{aligned}$$

To conclude we include an evaluation of a definite integral involving just the function $R_N(x)$ over the interval $[2, \xi]$.

Theorem 2. *If $\xi \geq 2$ and N is a positive integer, then the definite integral of $R_N(x)$ over the interval $[2, \xi]$ is given by*

$$\frac{1}{2^N}\int_2^{\xi}\sqrt{2+\sqrt{2+\cdots+\sqrt{2+x}}}\,dx$$

$$= \frac{1}{2^N+1}\left(\left(\frac{\xi+\sqrt{\xi^2-4}}{2}\right)^{(2^N+1/2^N)} + \left(\frac{\xi+\sqrt{\xi^2-4}}{2}\right)^{-(2^N+1/2^N)}\right)$$

$$+ \frac{1}{2^N-1}\left(\left(\frac{\xi+\sqrt{\xi^2-4}}{2}\right)^{(2^N-1/2^N)} + \left(\frac{\xi+\sqrt{\xi^2-4}}{2}\right)^{-(2^N-1/2^N)}\right) - \left(\frac{2^{N+2}}{2^{2N}-1}\right).$$

Proof. Applying the variable substitution as in the proof of Theorem 1, we again deduce from Lemma 1 that the definite integral in the above transforms to

$$I = \frac{1}{2^N}\int_0^{\cosh^{-1}(\xi/2)} 4\sinh(\theta)\cosh\left(\frac{\theta}{2^N}\right)d\theta. \tag{7}$$

Expanding the integrand in (7) in terms of exponential functions yields

$$I = \frac{1}{2^N}\int_0^{\cosh^{-1}(\xi/2)} e^{\theta(2^N+1/2^N)} + e^{\theta(2^N-1/2^N)} - e^{-\theta(2^N-1/2^N)} - e^{-\theta(2^N+1/2^N)}\,d\theta$$

$$= \left[\frac{1}{2^N+1}\,e^{\theta(2^N+1/2^N)} + \frac{1}{2^N-1}\,e^{\theta(2^N-1/2^N)} + \frac{1}{2^N-1}\,e^{-\theta(2^N-1/2^N)} + \frac{1}{2^N+1}\,e^{-\theta(2^N+1/2^N)}\right]_0^{\cosh^{-1}(\xi/2)}.$$

Finally, applying again the logarithmic form of the inverse hyperbolic cosine function yields the required result. □

REFERENCES

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