ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 4, 2007

RECURSION FORMULA OF SECOND-ORDER RECURRENT SEQUENCES

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ABSTRACT. Let $\{w_n\}$ be a second order recurrence sequence. A recursion formula is proved for certain reciprocal sums whose denominators are products of consecutive elements of $\{w_n\}$.

1. Introduction. Let $\mathbf{Z}$ and $\mathbf{R}$ denote the ring of the integers and the field of real numbers, respectively. For a field $\mathbf{F}$, we put $\mathbf{F}^* = \mathbf{F} \setminus \{0\}$. Fix $A \in \mathbf{R}$ and $B \in \mathbf{R}^*$, and let $\mathcal{L}(A,B)$ consist of all those second-order recurrent sequences $\{w_n\}_{n\in\mathbf{Z}}$ of complex numbers satisfying the recursion:

$$w_{n+2} = Aw_{n+1} - Bw_n \quad (\text{i.e., } Bw_n = Aw_{n+1} - w_{n+2}) \quad \text{for} \quad n = 0, \pm 1, \pm 2, \dots . \tag{1}$$

For sequences in $\mathcal{L}(A,B)$, the corresponding characteristic equation is $x^2 - Ax + B = 0$, whose roots $(A \pm \sqrt{A^2-4B})/2$ are denoted by α and β. If $A \in \mathbf{R}$ and $\Delta = A^2 - 4B \geq 0$, then we have

$$\alpha = \frac{A - \operatorname{sg}(A)\sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A + \operatorname{sg}(A)\sqrt{\Delta}}{2},$$

where $\operatorname{sg}(A) = 1$ if $A > 0$, and $\operatorname{sg}(A) = -1$ if $A < 0$.

The Lucas sequences $\{u_n\}_{n\in\mathbf{Z}}$ and $\{v_n\}_{n\in\mathbf{Z}}$ in $\mathcal{L}(A,B)$ take special values at $n = 0, 1$, namely,

$$u_0 = 0, \quad u_1 = 1, \quad v_0 = 2, \quad v_1 = A. \tag{2}$$

If $A = 1$ and $B = -1$, then those $F_n = u_n$ and $L_n = v_n$ are called Fibonacci numbers and Lucas numbers, respectively.

2000 AMS *Mathematics Subject Classification.* Primary 11B39, 11B37.
Key words and phrases. Second-order recurrent sequences, Lucas numbers, recursion formula.
Received by the editors on Jan. 22, 2005, and in revised form on March 25, 2005.

Let m, n and k be integers. If $w_n \neq 0$ for all $n = 1, 2, \ldots$, the sums are defined as follows:

$$S_{m,k} = \sum_{n=1}^{\infty} \frac{B^{k(n-1)}}{w_n w_{n+1} \cdots w_{n+m}}. \tag{3}$$

In [**1**] Brousseau proved $S_{2,-1} = (5/12) - (3/2)S_{4,0}$, $S_{4,0} = (97/2640) - (40/11)S_{6,1}$ and $S_{6,-1} = (589/1900080) - (273/29)S_{8,0}$ when $\{w_n\} = \{F_n\}$. In [**5**], under the same condition, Melham showed $S_{m,-1} = r_1 + r_2 S_{m+2,0}$ and $S_{m,0} = r_3 + r_4 S_{m+2,1}$, where the r_i are rational numbers that depend on m. In this paper we obtain the following theorem.

Theorem. *Let k be an integer, and let m and n be positive integers. If $w_n \neq 0$ for all $n = 1, 2, \ldots$,*

$$\begin{aligned} S_{m+2,k+1} = {} & \frac{B^{m-k+1} w_{m+2} - w_{2m+3}}{eB^{k+1} w_1 w_2 \cdots w_{m+2} u_{m+1} u_{m+2}} \\ & - \frac{B^k + B^{m-k+1} - v_{m+1}}{eB^{k+1} u_{m+1} u_{m+2}} S_{m,k} \end{aligned} \tag{4}$$

where $e = w_0 w_2 - w_1^2$.

Remark 1. The theorem of Melham [**5**] is essentially our (4) in the special case $A = 1$, $B = -1$, $k = 0$, $k = 1$ and $\{w_n\} = \{F_n\}$.

2. Some lemmas. To complete the proof of the theorem, we need the following two lemmas:

Lemma 1. *Let m and n be nonnegative integers; then we have*

$$\begin{aligned} w_{n+m}\, w_{n+m+2} - B^k w_n w_{n+m+1} & \\ = B^{k-m-1} u_{m+1} w_{n+m+1} w_{n+m+2} & \\ + (1 - B^{k-m-1} u_{m+2}) w_{n+m} w_{n+m+2} & \\ + eB^{n+k-1} u_{m+2}. & \end{aligned} \tag{5}$$

Proof. The following identity is well known, see [**4, 7**], that

$$B^{m+1} w_n = w_{n+m+1} u_{m+2} - w_{n+m+2} u_{m+1}, \tag{6}$$

and

$$w_{n+m+1}^2 = w_{n+m}w_{n+m+2} - eB^{n+m}. \tag{7}$$

Thus, we find that

$$\begin{aligned}
w_{n+m}\, w_{n+m+2} - B^k w_n w_{n+m+1} \\
&= w_{n+m}w_{n+m+2} \\
&\quad - B^k w_{n+m+1} B^{-m-1}(u_{m+2}w_{n+m+1} - u_{m+1}w_{n+m+2}) \\
&= w_{n+m}w_{n+m+2} \\
&\quad - B^{k-m-1}(w_{n+m+1}^2 u_{m+2} - u_{m+1}w_{n+m+1}w_{n+m+2}) \\
&= B^{k-m-1}u_{m+1}w_{n+m+1}w_{n+m+2} \\
&\quad + (1 - B^{k-m-1}u_{m+2})w_{n+m}w_{n+m+2} \\
&\quad + eB^{n+k-1}u_{m+2}.
\end{aligned}$$

This proves Lemma 1. □

Lemma 2. *Let k be an integer, and let m and n be positive integers. If $w_n \neq 0$ for all $n = 1, 2, \ldots,$*

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{B^{k(n-1)}}{w_n w_{n+1} \cdots w_{n+m-1} w_{n+m+1}} &= \frac{-B^{m-k}}{w_1 w_2 \cdots w_{m+1} u_{m+1}} \\
&\quad + \frac{B^{m-k} + u_m}{u_{m+1}}\, S_{m,k}.
\end{aligned} \tag{8}$$

Proof. For k an integer, and m and n positive integers, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{B^{k(n-1)}}{w_n w_{n+1} \cdots w_{n+m-1} w_{n+m+1}} - \frac{B^{m-k} + u_m}{u_{m+1}}\, S_{m,k} \\
&= \sum_{n=1}^{\infty} \frac{B^{k(n-1)}[u_{m+1}w_{n+m} - u_m w_{n+m+1} - B^{m-k}w_{n+m+1}]}{w_n w_{n+1} \cdots w_{n+m+1} u_{m+1}} \\
&= \sum_{n=1}^{\infty} \frac{B^{k(n-1)}[B^m w_n - B^{m-k}w_{n+m+1}]}{w_n w_{n+1} \cdots w_{n+m+1} u_{m+1}} \\
&= \frac{-B^{m-k}}{w_1 w_2 \cdots w_{m+1} u_{m+1}}.
\end{aligned}$$

This completes the proof of Lemma 2. □

3. Proof of Theorem. Let k be an integer, and let m be a positive integer. We define

$$\sum = \sum_{n=1}^{\infty} \frac{B^{k(n-1)}(w_{n+m}w_{n+m+2} - B^k w_n w_{n+m+1})}{w_n w_{n+1} \cdots w_{n+m+2}}. \tag{9}$$

Then, we get

$$\begin{aligned}\sum &= \sum_{n=1}^{\infty} \frac{B^{k(n-1)}}{w_n w_{n+1} \cdots w_{n+m-1} w_{n+m+1}} \\ &\quad - \sum_{n=1}^{\infty} \frac{B^{kn}}{w_{n+1} w_{n+2} \cdots w_{n+m} w_{n+m+2}} \\ &= \frac{1}{w_1 w_2 \cdots w_m w_{m+2}}.\end{aligned}$$

By Lemmas 1 and 2, we obtain

$$\begin{aligned}\sum &= \sum_{n=1}^{\infty} B^{k(n-1)} \Bigg[\frac{B^{k-m-1} u_{m+1} w_{n+m+1} w_{n+m+2}}{w_n w_{n+1} \cdots w_{n+m+2}} \\ &\qquad + \frac{(1 - B^{k-m-1} u_{m+2}) w_{n+m} w_{n+m+2} + eB^{n+k-1} u_{m+2}}{w_n w_{n+1} \cdots w_{n+m+2}} \Bigg] \\ &= B^{k-m-1} u_{m+1} S_{m,k} + (1 - B^{k-m-1} u_{m+2}) \\ &\quad \times \sum_{n=1}^{\infty} \frac{B^{k(n-1)}}{w_n w_{n+1} \cdots w_{n+m-1} w_{n+m+1}} \\ &\quad + eB^k u_{m+2} S_{m+2,k+1} \\ &= B^{k-m-1} u_{m+1} S_{m,k} + (1 - B^{k-m-1} u_{m+2}) \\ &\quad \times \left(\frac{-B^{m-k}}{w_1 w_2 \cdots w_{m+1} u_{m+1}} + \frac{B^{m-k} + u_m}{u_{m+1}} \, S_{m,k} \right) \\ &\quad + eB^k u_{m+2} S_{m+2,k+1}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{w_1w_2\cdots w_mw_{m+2}}&\\ &= B^{k-m-1}u_{m+1}S_{m,k}+\frac{-B^{m-k}+B^{-1}u_{m+2}}{w_1w_2\cdots w_{m+1}u_{m+1}}\\ &\quad+\frac{B^{m-k}+u_m-B^{-1}u_{m+2}-B^{k-m-1}u_mu_{m+2}}{u_{m+1}}\,S_{m,k}\\ &\quad+eB^ku_{m+2}S_{m+2,k+1}.\end{aligned}$$

Now, using the well known identities

$$v_{m+1}=u_{m+2}-Bu_m,\qquad u_{m+1}^2-u_mu_{m+2}=B^m$$

and

$$w_{2m+3}=u_{m+2}w_{m+2}-Bu_{m+1}w_{m+1},$$

we obtain

$$\begin{aligned}S_{m+2,k+1}=&\frac{B^{m-k+1}w_{m+2}-w_{2m+3}}{eB^{k+1}w_1w_2\cdots w_{m+2}u_{m+1}u_{m+2}}\\ &-\frac{B^k+B^{m-k+1}-v_{m+1}}{eB^{k+1}u_{m+1}u_{m+2}}\,S_{m,k}.\end{aligned}$$

The proof is now complete. □

4. Corollaries of the Theorem. If $A,B\in R^*$, $A^2\geq 4B$, $w_1\neq\alpha w_0$, and $w_n\neq 0$ for all $n\geq 1$, then letting $f(n)=n+1$ in [**4**, Theorem 2], we obtain

$$S_{1,1}=\sum_{n=1}^{\infty}\frac{B^{(n-1)}}{w_nw_{n+1}}=\frac{1}{\beta w_1(w_1-\alpha w_0)}.$$

Corollary 1. *If $A,B\in R^*$, $A^2\geq 4B$, $w_1\neq\alpha w_0$, and $w_n\neq 0$ for all $n=1,2,\dots$, in the case $k=1$ and $m=1$,* (4) *becomes*

(10)

$$\sum_{n=1}^{\infty}\frac{B^{2(n-1)}}{w_nw_{n+1}w_{n+2}w_{n+3}}=\frac{Bw_3-w_5}{eB^2w_1w_2w_3u_2u_3}-\frac{2B-v_2}{eB^2u_2u_3\beta w_1(w_1-\alpha w_0)}.$$

Remark 2. Equation (3.10) of Melham [**6**] is essentially our (10) in the special case $w_0 = 0$, $w_1 = 1$ and $w_n = 3w_{n-1} - w_{n-2} = F_{2n}$.

Corollary 2. *In the case* $\{w_n\} = \{F_n\}$ *and* $\{w_n\} = \{L_n\}$, (10) *turns out to be*

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+2} F_{n+3}} = \frac{12 - 5\sqrt{5}}{4}, \tag{11}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1} L_{n+2} L_{n+3}} = \frac{5 - 2\sqrt{5}}{40}. \tag{12}$$

Corollary 3. *If* $A, B \in R^*$, $A^2 \geq 4B$, $w_1 \neq \alpha w_0$, *and* $w_n \neq 0$ *for all* $n = 1, 2, \dots,$ *in the case* $k = 2$ *and* $m = 3$, (4) *says that*

$$\begin{aligned}\sum_{n=1}^{\infty} & \frac{B^{3(n-1)}}{w_n w_{n+1} w_{n+2} w_{n+3} w_{n+4} w_{n+5}} \\ &= \frac{B^2 w_5 - w_9}{eB^3 w_1 w_2 w_3 w_4 w_5 u_4 u_5} - \frac{2B^2 - v_4}{eB^3 u_4 u_5} \\ &\quad \times \left(\frac{B w_3 - w_5}{eB^2 w_1 w_2 w_3 u_2 u_3} - \frac{2B - v_2}{eB^2 u_2 u_3 \beta w_1 (w_1 - \alpha w_0)} \right).\end{aligned} \tag{13}$$

Corollary 4. *In the case* $\{w_n\} = \{F_n\}$ *and* $\{w_n\} = \{L_n\}$, (13) *becomes*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4} F_{n+5}} = \frac{421}{450} - \frac{5\sqrt{5}}{12}, \tag{14}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4} L_{n+5}} = \frac{\sqrt{5}}{300} - \frac{41}{5544}. \tag{15}$$

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