ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 2, Spring 1990

DIRECT INTEGRALS OF STANDARD FORMS OF W^*-ALGEBRAS

LOTHAR M. SCHMITT

ABSTRACT. Bös [Invent. Math. **37** (1976), p. 241] proved that standard forms of W^*-algebras behave naturally with respect to direct integrals. We give a new approach to disintegration of standard forms, which uses the characterization of matrix-ordered Hilbert spaces in standard forms of W^*-algebras obtained by Wittstock and the author [Math. Scand. **51** (1982), p. 241].

Introduction. Araki [**1**], Connes [**3**] and Haagerup [**7**] developed standard forms of W^*-algebras. Connes [**3**] characterized the ordered Hilbert spaces arising in these standard forms. Penney [**8**] developed direct integrals of selfdual cones. Based on [**3**], and [**8**], Bös showed in [**2**] that standard forms behave naturally with respect to direct integrals. Wittstock and the author [**11, 12**] characterized the Hilbert spaces arising in standard forms of W^*-algebras among matrix ordered spaces. In this note we give a self contained and simplified approach to disintegration of standard forms. In fact proper use of a result of Elliott [**5**] makes it possible to work with only a few consequences of the measurable choice theorem due to Sainte-Beuve [**9**]. Furthermore disintegration of matrix order allows us to dispense with the rather technical direct integral of orientations [**2**] and is therefore more natural from a categorial point of view.

1. Technical preliminaries.

1.1 *Separability conditions.*

PROPOSITION. *Let $\mathcal{M}$ be a W^*-algebra. Then the following conditions are equivalent*:

a) *$\mathcal{M}$ has a separable predual $\mathcal{M}_*$.*

This research was supported by NSF grant #DMS 8403631 a.4.
Received by the editors on July 17, 1987 and, in revised form, on February 8, 1988.

b) *The Hilbert space* $\mathcal{H}$ *in the standard form* $(\mathcal{M}, \mathcal{H}, \mathcal{J}, \mathcal{H}^+)$ [**7**, p. 241] *of* $\mathcal{M}$ *is separable.*

c) $\mathcal{M}$ *has a faithful* W^**-representation on a separable Hilbert space.*

PROOF. (a) $\Rightarrow$ (b) follows from Bures's inequality, see [**13**, §10.24, Proposition]. (c) $\Rightarrow$ (a) is [**10**, Proposition 2.1.1]. □

Disintegration of W^*-algebras exists in the above case [**4, 14**] and other special situations.

1.2 *Direct integrals of selfdual cones.* Let (Γ, μ) be a σ-finite measure space. Let $\{\mathcal{H}(\gamma), \gamma \in \Gamma\}$ be a measurable family of Hilbert spaces and set

$$\mathcal{H} = \int_\Gamma^\oplus \mathcal{H}(\gamma) d\mu(\gamma).$$

In what follows we shall assume that a disintegration of $\mathcal{H}$ as above is given, but we do not assume $\mathcal{H}$ to be separable. If $\mathcal{H}^+$ is a cone in $\mathcal{H}$, then we shall call $\mathcal{H}^+$ compatible with Γ if the projections in the diagonal algebra $\mathcal{L} = L^\infty(\Gamma, \mu)$ map $\mathcal{H}^+$ into $\mathcal{H}^+$. The following Lemma is essentially due to Penney [**8**] and is used to fix our notation.

1.2.1. LEMMA. *Let* $\mathcal{H}^+$ *be a selfdual cone in* $\mathcal{H}$ *compatible with* Γ.

(a) *The conjugate linear symmetry* $\mathcal{J}$ *associated with* $\mathcal{H}^+$ *by* [**3**, Proposition 4.1] *is a decomposable operator, i.e.,*

$$\mathcal{J} = \int_\Gamma^\oplus \mathcal{J}(\gamma) d\mu(\gamma).$$

(b) *There exists a sequence* $\{\xi_k(\gamma), k \in \mathbf{N}\}$ *in* $\mathcal{H}^{\mathcal{J}} = \{\xi \in \mathcal{H} \mid \xi = \mathcal{J}\xi\}$ *such that* $\{\xi_k(\gamma), k \in \mathbf{N}\}$ *is dense in* $\mathcal{H}(\gamma)^{\mathcal{J}(\gamma)}$ *a.e.*

(c) *With the sequence* $\{\xi_k, k \in \mathbf{N}\}$ *as in (b) set*

$$\mathcal{H}(\gamma)^+ = \overline{\{\xi_k^+(\gamma), k \in \mathbf{N}\}},$$

where $\xi_k = \xi_k^+ - \xi_k^-, \xi_k^+ \perp \xi_k^-, \xi_k^+ \in \mathcal{H}^+$ *is the canonical decomposition of* ξ_k [**3**, Proposition 4.1]. *Then* $\mathcal{H}(\gamma)^+$ *is a selfdual cone in* $\mathcal{H}(\gamma)$ *with associated conjugate linear symmetry* $\mathcal{J}(\gamma)$ *a.e.*

(d) $\xi \in \mathcal{H}^+ \Leftrightarrow \langle \xi(\gamma), \xi_k^+(\gamma)\rangle \geq 0$ a.e. $\forall k \in \mathbf{N}$.

(e) *If* $x = \int_\Gamma^\oplus x(\gamma) d\mu(\gamma) \in \mathcal{B}(\mathcal{H})$ *is a decomposable operator* [**14**, p. 273], *then* x *is positive with respect to* $\mathcal{H}^+$ *if and only if* $x(\gamma)$ *is positive with respect to* $\mathcal{H}^+(\gamma)$ *a.e.*

Proof. (a). $\mathcal{J}$ commutes with $\mathcal{L}$ since $\mathcal{H}^+$ is compatible with Γ. Hence $\mathcal{J}$ is decomposable by a conjugate linear version of [**4**; II.2.5, Theorem 1].

(b). Take $\{(1+\mathcal{J})\xi_k^0, k \in \mathbf{N}\}$ for a fundamental sequence [**14**, p. 270] $\{\xi_k^0, k \in \mathbf{N}\} \subset \mathcal{H}$.

(c). Apply the proof of [**8**, Theorem II.10].

(d). The set $\{\xi | \langle \xi(\gamma), \xi_k^+(\gamma)\rangle \geq 0$ a.e. $\forall k \in \mathbf{N}\}$ is a selfdual cone, which contains $\mathcal{H}^+$.

(e). By (d) x is positive if and only if $\langle x(\gamma)\xi_k^+(\gamma), \xi_j^+(\gamma)\rangle \geq 0$ a.e. $\forall k, j \in \mathbf{N}$. □

Following Penney [**8**, Definition II.6], we shall write

$$\mathcal{H}^+ = \int_\Gamma^\oplus \mathcal{H}(\gamma)^+ d\mu(\gamma)$$

in the situation of Lemma 1.2.1. For the remainder of this section we shall keep the notation of Lemma 1.2.1, and shall moreover assume that the following conditions hold:

C_1: μ is finite and the measurable family of Hilbert spaces

$\{\mathcal{H}(\gamma), \gamma \in \Gamma\}$ is constant, i.e., $\mathcal{H} = L^2(\Gamma, \mathcal{K})$
for some separable Hilbert space $\mathcal{K}$.

C_2: $\mathcal{H}(\gamma)^+ = \{0\}$ for all $\gamma \in \Gamma$ for which $\mathcal{H}(\gamma)^+$ is not selfdual.

For a closed, convex subset A of $\mathcal{K}$ let $d(\eta, A), \eta \in K$, denote the distance between η and A, taken to be ∞ in the case $A = \varnothing$. In addition, let $U(r), r > 0$, be the open ball of radius r in $\mathcal{K}$. The following three lemmata are essentially consequences of the measurable selection theorem due to von Neumann, Aumann and Sainte-Beuve, see [**9**].

1.2.2. LEMMA. *The following functions are measurable*:

(a)
$$h : \Gamma \times \mathcal{K} \to \mathbf{R}^+$$
$$h(\gamma, \eta) = d(\eta, \mathcal{H}(\gamma)^+), \quad \gamma \in \Gamma, \eta \in \mathcal{K}.$$

(b)
$$f : \Gamma \times \mathcal{K}^3 \to \mathbf{R}^+ \cup \{\infty\}$$
$$f(\gamma, \eta) = d(\eta_1, \eta_2 + \mathcal{H}(\gamma)^+ \cap \eta_3 - \mathcal{H}(\gamma)^+),$$
$$\gamma \in \Gamma, \eta = (\eta_1, \eta_2, \eta_3) \in \mathcal{K}^3.$$

PROOF. (a). $h(\gamma, \eta) = \inf ||\eta - \xi_k^+(\gamma)||$ is measurable since $(\gamma, \eta) \to (\xi_k^+(\gamma), \eta) \in \mathcal{K}^2$ and the inner product are measurable.

(b). Let $P_\Gamma(\gamma, \eta) = \gamma$. We conclude from [**9**, Theorem 4] that

$$\Gamma_k^m = \bigcup_{\ell \in \mathbf{N}} P_\Gamma \Big\{ (\gamma, \eta) | \; ||\eta_2 + \xi_k^+(\gamma) - \eta_3 + \xi_\ell^+(\gamma)|| < \frac{1}{m} \Big\}, k, m \in \mathbf{N},$$

is a measurable subset of Γ. Now

$$f_k^m(\gamma, \eta) = \begin{cases} ||\eta_1 - \eta_2 - \xi_k^+(\gamma)||, & \gamma \in \Gamma_k^m \\ \infty, & \gamma \notin \Gamma_k^m \end{cases}$$

is a measurable function on $\Gamma \times \mathcal{K}^3$. We have

$$(1) \qquad d\Big(\eta_1, \eta_2 + \mathcal{H}(\gamma)^+ \bigcap \overline{\eta_3 - \mathcal{H}(\gamma)^+ + U(\frac{1}{m})}\Big) \le \inf_k f_k^m(\gamma, \eta),$$

$$(2) \quad \inf_k f_k^m(\gamma, \eta) \le d\Big(\eta_1, \eta_2 + \mathcal{H}(\gamma)^+ \bigcap \overline{\eta_3 - \mathcal{H}(\gamma)^+ + U\Big(\frac{1}{m+1}\Big)}\Big).$$

An application of the parallelogram law shows that the left hand side of (1) converges to $f(\gamma, \eta)$ as $m \to \infty$. $\square$

1.2.3. LEMMA. *Let* $\rho \in \mathcal{H}, \eta, \eta_1, \eta_2 \in \mathcal{H}^{\mathcal{J}}$ *be such that* $\eta_1 \le \eta \le \eta_2$ *and* $||\rho - \eta|| = d(\rho, [\eta_1, \eta_2])$. *Then*

$$||\rho(\gamma) - \eta(\gamma)|| = d(\rho(\gamma), [\eta_1(\gamma), \eta_2(\gamma)]) \; a.e.$$

PROOF. Let Ω be the set of $(\gamma, \eta_\gamma) \in \Gamma \times \mathcal{K}$ such that

$$(1) \qquad \left.\begin{cases} d(\eta_\gamma, [\eta_1(\gamma), \eta_2(\gamma)]) = 0 \\ d(\rho(\gamma), [\eta_1(\gamma), \eta_2(\gamma)]) = ||\rho(\gamma) - \eta_\gamma|| \end{cases}\right\} \text{ if } \eta_1(\gamma) \le \eta_2(\gamma),$$

$$(2) \qquad \eta_\gamma = 0 \text{ if } \eta_1(\gamma) \not\le \eta_2(\gamma).$$

Ω is measurable by Lemma 1.2.2. Hence $\gamma \to \eta_\gamma$ is measurable by [**9**, Theorem 4] and has range in $[\eta_1, \eta_2]$. Clearly

$$\int ||\rho(\gamma) - \eta_\gamma||^2 d\mu(\gamma) \le \int ||\rho(\gamma) - \eta(\gamma)||^2 d\mu(\gamma)$$

and therefore $\eta_\gamma = \eta(\gamma)$ a.e. □

For $\rho \in \mathcal{H}^+$ let $F(\rho) = \cup_{\ell \in \mathbf{N}} [0, \ell\rho]$ denote the face generated by ρ. For any face $F \subset \mathcal{H}^+$ let P_F denote the projection onto $\overline{\operatorname{span}_{\mathbf{C}} F}$. P_F commutes with $\mathcal{J}$ and $\mathcal{L}$. The following statement is taken from Bös's paper [**2**].

1.2.4. LEMMA. *Suppose F is a closed face in $\mathcal{H}^+$.*

(a) *There exists $\rho \in F$ such that $F = \overline{F(\rho)}$.*

(b) $\xi \in F \Leftrightarrow \xi(\gamma) \in \overline{F(\rho(\gamma))}$ *a.e.*

(c) $(P_F\eta)(\gamma) = P_{F(\rho(\gamma))}\eta(\gamma)$ *a.e.*, $\eta \in \mathcal{H}$.

(d) $\overline{F(\rho(\gamma))}$ *is a face a.e.*

(e) *If $\rho, \rho' \in \mathcal{H}^+$ are such that $F = \overline{F(\rho)}$ and $F^\perp = \overline{F(\rho')}$, then $F(\rho(\gamma))^\perp = \overline{F(\rho'(\gamma))}$ a.e.*

PROOF. Let $\rho_k \in F$ be such that $||\xi_k^+ - \rho_k|| = d(\xi_k^+, F), k \in \mathbf{N}$. We define $\rho = \sum 2^{-k} ||\rho_k||^{-1} \rho_k$, where the summation is taken over all k with $\rho_k \neq 0$. Fix $\xi \in F$. For a fixed $k \in \mathbf{N}$, set

$$N = \{\gamma \mid ||\xi_k^+(\gamma) - \rho(\gamma)|| > ||\xi_k^+(\gamma) - \xi(\gamma)||\}.$$

Then $\rho' = (1 - \chi_N)\rho_k + \chi_N \xi \in F$ and $||\xi_k^+ - \rho_k'|| < ||\xi_k^+ - \rho_k||$ unless $\mu(N) = 0$. Consequently

$$(1) \qquad ||\xi(\gamma) - \rho_k(\gamma)|| \le 2||\xi(\gamma) - \xi_k^+(\gamma)|| \text{ a.e.}$$

Now let $0 \leq \eta_\ell \leq \ell \cdot \rho$ be such $||\xi - \eta_\ell|| = d(\xi, [0, \ell\rho])$. Lemma 1.2.3 and (1) show that $\gamma \mapsto ||\xi(\gamma) - \eta_\ell(\gamma)||$ is a decreasing sequence of L^2-functions which converges pointwise to zero a.e. Lebesgue's Theorem shows that $\xi = \lim \eta_\ell$. These arguments show (a) and (b). In order to show (c) one can assume that $\eta \in \mathcal{H}^{\mathcal{J}}$ and then apply a similar argument using the order intervals $[-\ell\rho, \ell\rho]$. To show (d) consider the set Ω of $(\gamma, \xi, \eta) \in \Gamma \times \mathcal{K}^2$ with $||\xi|| \leq 1, \eta \neq 0; \eta, \xi - \eta \in \mathcal{H}(\gamma)^+; \inf_\ell d(\xi, [0, \ell\rho(\gamma)]) = 0; \inf_\ell d(\eta, [0, \ell\rho(\gamma)]) > 0$. Ω is measurable by Lemma 1.2.2. Its projection on $\Gamma, N = P_\Gamma(\Omega)$, is measurable by [**9**, Theorem 4]. Let $\Omega' = \Gamma \backslash N \times \{(0,0)\} \cup \Omega$. By [**9**, Theorem 3] there exist measurable functions ξ_1, η_1 such that $(\gamma, \xi_1(\gamma), \eta_1(\gamma)) \in \Omega'$ for $\gamma \in \Gamma$. Hence, by (b), $\xi_1 \in F, 0 \leq \eta_1 \leq \xi_1 \Rightarrow \eta_1 \in F \Rightarrow \mu(N) = 0$. Finally, to prove (e) let Ω be the set of $(\gamma, \eta) \in \Gamma \times \mathcal{K}$ such that

$$0 < ||\eta|| \leq 1; \quad \eta \in \mathcal{H}(\gamma)^+; \quad \langle \eta, \rho(\gamma) \rangle = 0; \; d(\eta, F(\rho'(\gamma))) > 0.$$

As in the proof of (d) the projection of Ω on Γ is a μ-null set. □

1.3. *Direct integrals of matrix ordered Hilbert spaces.* If V is a set, then we shall denote the set of $n \times n$ matrices with entries in V by $M_n(V), n \in \mathbf{N}$. If V is a vector space we shall also write V_n for $M_n(V)$. Let $\Gamma, \mu, \mathcal{H}(\gamma)$ and $\mathcal{H}$ be as in 1.2, without the additional assumptions C1 and C2. Then $\{\mathcal{H}(\gamma)_n, \gamma \in \Gamma\}$ is a measurable field of Hilbert spaces and $\mathcal{H}_n$ can be identified with

$$\int_\gamma^\oplus \mathcal{H}(\gamma)_n d\mu(\gamma).$$

Let $\{\mathcal{H}_n^+ \subset \mathcal{H}_n, n \in \mathbf{N}\}$ be a family of selfdual cones such that $(\mathcal{H}, \mathcal{H}_n^+, n \in \mathbf{N})$ is a matrix-ordered space. We shall say that $\{\mathcal{H}_n^+, n \in \mathbf{N}\}$ is compatible with Γ if each projection in $\mathcal{L}$ is completely positive. By [**12**, Theorem 2.2] it is equivalent to say that $\mathcal{L}$ is in the center of the matrix multiplier algebra [**12**, Definition 2.1] of $(\mathcal{H}, \mathcal{H}_n^+)$. [**12**, Lemma 1.3] shows that the antilinear symmetry $\mathcal{J}_n$ associated with $\mathcal{H}_n^+$ equals $\mathcal{J}_1 \otimes \mathrm{st}$, where st is the adjoint operation on $M_n(\mathbf{C})$.

Let $\{\xi_n^k, k \in \mathbf{N}\}$ be an enumeration of the elements in $(1 + \mathcal{J}_n) M_n(\{\xi_k^0, k \in \mathbf{N}\})$ for a fundamental sequence $\{\xi_k^0, k \in \mathbf{N}\}$ in $\mathcal{H}$. We define, for $\gamma \in \Gamma$,

$$\mathcal{H}(\gamma)_n^+ = \{\alpha^* \xi_m^{k+}(\gamma)\alpha | \alpha \in M_{m,n}(\mathbf{Q}); \; m, k \overline{\in \mathbf{N}\}},$$

where $M_{m,n}(\mathbf{Q})$ denotes the $m \times n$ matrix over $\mathbf{Q}$. The construction in the proof of Lemma 1.2.1 shows that $(\mathcal{H}(\gamma), \mathcal{H}(\gamma)_n^+)$ is a matrix ordered Hilbert space a.e. In this situation we shall write

$$(\mathcal{H}, \mathcal{H}_n^+) = \int_\Gamma^\oplus (\mathcal{H}(\gamma), \mathcal{H}(\gamma)_n^+ d\mu(\gamma)$$

and call this a direct integral of matrix-ordered Hilbert spaces with selfdual cones.

2. Direct integrals of standard forms.

2.1. Theorem. *let (Γ, μ) be a σ-finite measure space and let*

$$(\mathcal{H}, \mathcal{H}_n^+) = \int_\Gamma^\oplus (\mathcal{H}(\gamma), \mathcal{H}(\gamma)_n^+) d\mu(\gamma)$$

be a direct integral of matrix-ordered Hilbert spaces with selfdual cones. Let $\mathcal{M}$ respectively $\mathcal{M}_\gamma$, denote the matrix multiplier algebra of $(\mathcal{H}, \mathcal{H}_n^+)$, respectively $(\mathcal{H}(\gamma), \mathcal{H}(\gamma)_n^+)$, for $\gamma \in \Gamma \backslash N$, where N is the μ-null set for which $\mathcal{M}_\gamma$ is not defined. Let $\mathcal{M}_\gamma = \mathbf{C}$ for $\gamma \in N$. Then the following statements are equivalent:

(a) $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ *is a matrix-ordered standard form* [**12**, Definition 1.4]

(b) $(\mathcal{M}_\gamma, \mathcal{H}(\gamma), \mathcal{H}(\gamma)_n^+)$ *is a matrix-ordered standard form a.e. If one of the above conditions is satisfied then*

$$\mathcal{M} = \int_\Gamma^\oplus \mathcal{M}_\gamma d\mu(\gamma).$$

Proof. We may assume without loss of generality that the additional conditions C1 and C2 hold.

(a) $\Rightarrow$ (b). $\mathcal{L}$ is in the center of $\mathcal{M}$. By [**5**, Lemma 4] $\mathcal{M}$ can be disintegrated into a direct integral of W^*-algebras $\mathcal{M}(\gamma), \gamma \in \Gamma$. $\mathcal{M}' = \mathcal{J}\mathcal{M}\mathcal{J}$ implies $\mathcal{M}(\gamma)' = \mathcal{J}(\gamma)\mathcal{M}(\gamma)\mathcal{J}(\gamma)$ a.e. To obtain (b) and the last statement of the theorem it is sufficient to show that

$\mathcal{M}(\gamma) = \mathcal{M}_\gamma$ a.e. Let $\{x_k, k \in \mathbf{N}\}$ be a countable *-subalgebra of $\mathcal{M}$ over $\mathbf{Q}$ such that $\mathcal{M}(\gamma) = \{x_k(\gamma), k \in \mathbf{N}\}''$. Then $x_k(\gamma) \in \mathcal{M}_\gamma$ a.e., since $\mathcal{M}$ is the matrix multiplier algebra of $(\mathcal{H}, \mathcal{H}_n^+)$. Kaplansky's density theorem shows that $\mathcal{M}(\gamma) \subset \mathcal{M}_\gamma$. Now, by [**12**, Theorem 2.2],

$$\mathcal{M}(\gamma)' = \mathcal{J}(\gamma)\mathcal{M}(\gamma)\mathcal{J}(\gamma) \subset \mathcal{J}(\gamma)\mathcal{M}_\gamma\mathcal{J}(\gamma) \subset \mathcal{M}'_\gamma \subset \mathcal{M}(\gamma)' \text{ a.e.}$$

(b) $\Rightarrow$ (a). Suppose that $F = F^{\perp\perp}$ is a face in $\mathcal{H}_n^+$ and $\eta \in P_F\mathcal{H}_n \cap \mathcal{H}_n^{\mathcal{J}}$. We apply Lemma 1.2.4: let $\rho \in F$ be such that $F = \overline{F(\rho)}$; then $\eta(\gamma) \in P_{F(\rho(\gamma))}\mathcal{H}(\gamma)_n \cap \mathcal{H}(\gamma)_n^{\mathcal{J}}$ and $F(\rho(\gamma))^{\perp\perp} = \overline{F(\rho(\gamma))}$ a.e. By [**11**; Theorem 1.3, Lemma 1.5] it follows that $\eta(\gamma)^+ \in \overline{F(\rho(\gamma))}$ a.e. Hence $\eta^+ \in F$. Applying [**11**; Theorem 1.3, Lemma 1.5], again we are done. $\square$

The following two theorems include the main results of Bös [**2**]. Recall that, for a selfdual cone $\mathcal{H}^+$ in a Hilbert space $\mathcal{H}$,

$$\mathcal{D}(\mathcal{H}^+) = \{\delta \in \mathcal{B}(\mathcal{H}) | \exp(t\delta)\mathcal{H}^+ = \mathcal{H}^+ \quad \forall t \in \mathbf{R}\}.$$

2.2. Theorem. *Let (Γ, μ) be a σ-finite measure space and let*

$$(\mathcal{H}, \mathcal{H}^+) = \int_\Gamma^\oplus (\mathcal{H}(\gamma), \mathcal{H}(\gamma)^+) d\mu(\gamma)$$

be a direct integral of ordered Hilbert spaces with selfdual cones. Then

(a) $\mathcal{D}(\mathcal{H}^+) = \{\delta = \int_\Gamma^\oplus \delta(\gamma)d\mu(\gamma) \,|\, \delta(\gamma) \in \mathcal{D}(\mathcal{H}(\gamma)^+)$ *a.e.*$\}$

(b) $\mathcal{H}^+$ *is homogeneous* [**3**, Definition 5.1] *and orientable* [**3**, Definition 4.1.1] *if and only if the following conditions hold*:

(1) $\mathcal{H}(\gamma)^+$ *is homogeneous and there exists an orientation* I_γ *on* $\mathcal{D}(\mathcal{H}(\gamma)^+)/Z(\mathcal{D}(\mathcal{H}(\gamma)^+))$ *a.e.*

(2) *If* $\delta = \int_\Gamma^\oplus \delta(\gamma)d\mu(\gamma) \in \mathcal{D}(\mathcal{H}^+)$, *then there exists a measurable, bounded field* $\delta_i(\gamma) \in \mathcal{D}(\mathcal{H}(\gamma)^+)$ *such that* $\delta_i(\gamma) \in I_\gamma(\delta(\gamma) + Z(\mathcal{D}(\mathcal{H}(\gamma)^+)))$ *a.e.*

Proof. Assume without loss of generality that the additional conditions C1 and C2 hold.

(a). If $\delta \in \mathcal{D}(\mathcal{H}^+), p \in L$ is a projection and $\xi, \eta \in \mathcal{H}^+$, then $\langle \delta p\xi, (1-p)\eta\rangle = 0$ by [**6**, Theorem 3]. Hence $p\delta = p\delta p = \delta p$ and δ is decomposable. Also $p \in Z(\mathcal{D}(\mathcal{H})^+)$. If $\delta = \int_\Gamma^\oplus \delta(\gamma)d\mu(\gamma)$, then by [**6**, Theorem 3] and Lemma 1.2.1 there exists $\lambda_0 > 0$ such that, for all $\lambda \in \mathbf{Q}$ with $|\lambda| > \lambda_0, (\lambda - \delta(\gamma))^{-1}$ is positive a.e. This shows $\subset$ in (a). The converse inclusion follows directly from [**6**, Theorem 1(iii)].

(b). Suppose that $\mathcal{H}$ is homogeneous and orientable. Let $\mathcal{M}$ be the W^*-algebra with standard form $(\mathcal{M}, \mathcal{H}, \mathcal{J}, \mathcal{H}^+)$, which exists by [**3**, Theorem 5.2]. The proof of (a) and [**3**, Proposition 4.10] shows that $\mathcal{L} \subset \mathcal{M}$. Hence $\mathcal{M} = \int_\Gamma^\oplus \mathcal{M}(\gamma)d\mu(\gamma)$ is decomposable by [**5**, Lemma 4]. Now one checks that $(\mathcal{M}(\gamma), \mathcal{H}(\gamma), \mathcal{J}(\gamma), \mathcal{H}(\gamma)^+)$ is a standard form a.e. This shows (1). If $\delta \in \mathcal{D}(\mathcal{H}^+)$ then $\delta = x + \mathcal{J}x\mathcal{J}, x \in \mathcal{M}$, by [**3**, Theorem 3.4]. Let $x = \int_\Gamma^\oplus x(\gamma)d\mu(\gamma)$ and $\delta_i(\gamma) = ix(\gamma) + \mathcal{J}(\gamma)ix(\gamma)\mathcal{J}(\gamma)$. $\delta_i(\gamma)$ satisfies (2). Conversely, $\mathcal{H}^+$ is orientable if (1) and (2) are satisfied. Let F be a closed face in $\mathcal{H}^+$. By Lemma 1.2.4 there exists ρ and ρ' in $\mathcal{H}^+$ such that $F = \overline{F(\rho)}$ and $F^\perp = \overline{F(\rho')}$. Let $\xi, \eta \in \mathcal{H}^+$ with $\xi^\perp\eta$. By Lemma 1.2.4 and [**6**, Theorem 1(iii)] applied to $\mathcal{H}(\gamma)^+$, we obtain

$$\begin{aligned}\langle P_F\xi, \eta\rangle &= \int_\Gamma \langle P_{F(\rho(\gamma))}\xi(\gamma), \eta(\gamma)\rangle d\mu(\gamma)\\ &= \int_\Gamma \langle P_{F(\rho'(\gamma))}\xi(\gamma), \eta(\gamma)\rangle d\mu(\gamma) = \langle P_{F^\perp}\,\xi, \eta\rangle.\end{aligned}$$

Hence $\mathcal{H}^+$ is homogeneous, again by [**6**, Theorem 1(iii)]. □

2.3. Theorem. *Let* (Γ, μ) *be a* σ*-finite measure space and let*

$$(M, H) = \int_\Gamma^\oplus (M(\gamma), H(\gamma))d\mu(\gamma)$$

be a direct integral of W^**-algebras* $\{M(\gamma), \gamma \in \Gamma\}$ *acting on Hilbert spaces* $\{H(\gamma), \gamma \in \Gamma\}$*. Let* $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ *be the matrix-ordered standard form of* M*. Then there exists a direct integral of matrix-ordered standard forms* $(\mathcal{M}(\gamma), \mathcal{H}(\gamma), \mathcal{H}(\gamma)_n^+)$ *of* $M(\gamma)$ *such that*

$$(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+) = \int_\Gamma^\oplus (\mathcal{M}(\gamma), \mathcal{H}(\gamma), \mathcal{H}(\gamma)_n^+)d\mu(\gamma).$$

If ϕ, respectively ϕ_γ, is the W^*-isomorphism between M and $\mathcal{M}$, respectively $M(\gamma)$ and $\mathcal{M}(\gamma)$, then

$$\phi = \int_\Gamma^\oplus \phi_\gamma d\mu(\gamma) \quad [\mathbf{4}; \text{§II.3, Definition 3}].$$

PROOF. As Bös [**2**] points out, [**4**; II.3, Proposition 11] remains valid under the above hypothesis by virtue of [**5**, Lemma 4]. Now the existence of the disintegration

$$(\mathcal{M}, \mathcal{H}) = \int_\Gamma^\oplus (\mathcal{M}(\gamma), \mathcal{H}(\gamma)) d\mu(\gamma)$$

follows, as well as the last statement in the theorem. §1.3 and Theorem 2.1 show that the matrix-order automatically disintegrates as stated. □

REFERENCES

1. H. Araki, *Some properties of modular conjugation operator of von Neumann algebras and a non-commutative Radon-Nikodym theorem with a chain rule,* Pacific J. Math. **50** (1974), 309–354.

2. W. Bös, *Direct integrals of selfdual cones and standard forms of von Neumann algebras,* Invent. Math. **37** (1976), 241–251.

3. A. Connes, *Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann,* Ann. Inst. Fourier (Grenoble) **24** (1974), 121–155.

4. J. Dixmier, *Von Neumann algebras,* North Holland. Amsterdam-New York-Oxford, 1981.

5. G.A. Elliott, *An extension of some results of Takesaki in the reduction theory of von Neumann algebras,* Pacific J. Math. **39** (1971), 145–148.

6. D.E. Evans and H.Hanche-Olsen, *The generators of positive semigroups,* J. Funct. Anal. **32** (1979), 207–212.

7. U. Haagerup, *The standard form of von Neumann algebras,* Math Scand. **37** (1975), 271–283.

8. R.C. Penney, *Selfdual cones in Hilbert space,* J. Funct. Anal. **21** (1976), 305–215.

9. M.F. Sainte-Beuve, *On the extension of von Neumann-Aumann's theorem,* J. Funct. Anal. **17** (1974), 112–129.

10. S. Sakai, *C^*-algebras and W^*-algebras,* Springer Verlag, Berlin-Heidelberg-New York, 1971.

11. L.M. Schmitt, *Characterization of $L^2(\mathcal{M})$ for injective W^*-algebras $\mathcal{M}$.* Math. Scand. **57** (1985), 267–280.

12. ——— and G. Wittstock, *Characterization of matrix-ordered standard form of W^*-algebras,* Math. Scand. **51** (1982), 241–260.

13. S. Strătilă and L. Zsidó, *Lectures on von Neumann algebras,* Abacus Press, Turnbridge Wells, 1979.

14. M. Takesaki, *Theory of operator algebras,* Springer Verlag, Berlin-Heidelberg-New York, 1979.

Department of Mathematics, University of Kansas, Lawrence, KS 66045

Permanent address: Fachbereich Mathematik/Informatik, Universitat Osnabrück, D-4500 Osnabrück, West Germany