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EXISTENCE AND MULTIPLICITY RESULTS FOR A CLASS OF ELLIPTIC PROBLEMS WITH CRITICAL SOBOLEV EXPONENTS

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0. Introduction. In this paper we consider the boundary value problem

$$\begin{cases} -\Delta u = \lambda u + K(x)|u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{0.1}$$

where Ω is a bounded smooth domain in $\mathbf{R}^n$ ($n \geq 3$) or a compact manifold with boundary, $2^* = 2n/(n-2)$ is the critical exponent for the Sobolev embedding $H_0^1(\Omega) \subset L^p(\Omega)$ and K is a smooth function on Ω.

When $K(x) = 1$ and Ω is a domain, some remarkable results have been obtained: Brézis and Nirenberg proved in [**5**] existence of a positive solution of (0.1), with $n \geq 4$, for all $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue for the negative Laplacian in Ω under Dirichlet boundary conditions; in [**6**] it was proved that (0.1), with $n \geq 4$, has a solution for any $\lambda > 0$; later, in [**7**], the existence and multiplicity problem for (0.1) with λ near an eigenvalue λ_j was studied; their main result was that (0.1) has at least m_j pairs of solutions for $\lambda \in (\bar{\lambda}_j, \lambda_j)$, where m_j is the multiplicity of λ_j and the constant $\bar{\lambda}_j$ can be estimated.

Problem (0.1) has a deep root in Riemannian geometry and physics. If one deforms a metric conformally in a closed manifold $(\mathcal{M}^n, g)$ of dimension $n \geq 3$ by a positive function $u : \mathcal{M} \to \mathbf{R}$, then u satisfies the equation

$$\begin{cases} \frac{4(n-1)}{n-2}\Delta u + Ru + Ku^{(n+2)/(n-2)} = 0 & \text{on } \mathcal{M} \\ u > 0 & \text{on } \mathcal{M}, \end{cases} \tag{0.2}$$

where Δ and R are, respectively, the Laplacian and the scalar curvature with respect to the metric g. The function K represents the scalar curvature of the new metric $u^{4/(n-2)}g$. An outstanding geometric

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problem is whether a given compact Riemannian manifold is necessarily conformally equivalent to one of constant scalar curvature. This problem was formulated by Yamabe [**13**] in 1960. In the case that the scalar curvature R is nonpositive, the problem was solved by Trudinger in 1968 [**12**]. In the case $R > 0$, T. Aubin [**1**] gave a positive answer in many special cases in 1976. In 1984, R. Schoen introduced a new global idea and was able to solve the problem in all remaining cases [**11**]. Using the same idea, J. Escobar and R. Schoen studied the problem of conformally deforming metrics with prescribed scalar curvature, i.e,. solving (0.2) with K a smooth function on M [**9**]. An extensive study of (0.1) with $u > 0$ in Ω, with boundary, was done in [**8**].

The works mentioned above are all based on the following observation, that the corresponding functionals

$$\phi_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)\, dV - \frac{1}{2^*}\int_\Omega K|u|^{2^*}\, dV$$

and

$$\tilde{\phi}_\lambda(u) = \frac{\int_\Omega (|\nabla u|^2 - \lambda u^2)\, dV}{\left(\int_\Omega K|u|^{2^*}\, dV\right)^{2/2^*}} \qquad \left(\text{with } \int_\Omega K|u|^{2^*}\, dV > 0\right)$$

do satisfy some kind of compactness condition despite the fact that the Sobolev embedding $H_0^1 \to L^{2^*}(\Omega)$ is not compact. It was first observed in [**5**] that a Palais-Smale condition $(\mathrm{PS})_c$ was satisfied for c in a certain range. Later, a detailed proof for the functional ϕ_λ was given in [**7**]. For the functional $\tilde{\phi}_\lambda$, an argument originated in [**12**] has been used to show the existence of a solution u which realizes the infimum of the constrained functional $\tilde{\phi}_\lambda$.

The purpose of this paper is two-fold. Firstly, in an attempt to understand the nature of compactness of the constrained functional in terms of the condition $(\mathrm{PS})_c$, we present here a "natural constraint" approach to the variational problem of minimizing a "naturally constrained" functional. More specifically, we consider the constraint $\psi(u) = 0$, where

$$\psi(u) = \int_\Omega (|\nabla u|^2 - \lambda u^2 - K|u|2^*)\, dV.$$

Defining $M = \{u \in H_0^1(\Omega)\backslash\{0\} \mid \psi(u) = 0\}$, we minimize ϕ on M (from now on we write $\phi = \phi_\lambda$). It can be shown that M is a natural constraint in the sense that $0 \neq u \in H_0^1(\Omega)$ is a critical point of ϕ if and only if $u \in M$ and u is a critical point of $\phi|M$ (cf. [**3**, Section 6.3 B], [**10**], where similar arguments have been used). Furthermore, we show that $\phi|M$ satisfies the condition $(\mathrm{PS})_c$ for $c \in (0, (1/n)S_K^{n/2})$, where $S_K = S_{0,K}$ and $S_{\lambda,K}$ is defined by

$$S_{\lambda,K} = \inf \frac{\int_\Omega (|\nabla u|^2 - \lambda u^2)\, dV}{\int_\Omega K|u|^{2^*}\, dV} \qquad \left(\text{for } \int_\Omega K|u|^{2^*} > 0\right).$$

We then show that, if $0 < \lambda < \lambda_1$, we have $\inf_M \phi = (1/n)S_{\lambda,K}^{n/2}$, and that the intermediate results in [**8**] imply that $S_{\lambda,K} \in (0, S_K)$. Hence, $I = \inf_M \phi$ falls in the $(\mathrm{PS})_c$ range and, by a basic result in the calculus of variations, it follows that the infimum I is attained.

Secondly, we study the bifurcation and multiplicity problem for (0.1) removing the assumption $K(x) \equiv 1$. Our main result is the following.

Theorem. *For a nonnegative function $K(x)$ such that $K(x) > 0$ almost everywhere in Ω, a bounded smooth domain in $\mathbf{R}^n$ $(n \geq 3)$, problem* (0.1) *has at least m_j pairs of solutions if $\lambda \in (\bar{\lambda}_j, \lambda_j)$, where $\bar{\lambda}_j$ can be estimated* (*cf. Theorem* 2.1).

The proof given here is a modification of that in [**7**].

1. The natural constraint approach. In this section we consider the boundary value problem

$$\text{(1.1)} \qquad \begin{cases} -\Delta u = \lambda u + K(x)|u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

where $\Omega \subset \mathbf{R}^n$ $(n \geq 3)$ is a bounded smooth domain, $2^* = 2n/(n-2)$, $\lambda \in \mathbf{R}$ and $K \in C^\alpha(\overline{\Omega})$. As is well known, the solutions of (1.1) are precisely the critical points of the C^1 functional $\phi : H_0^1(\Omega) \to \mathbf{R}$ defined by

$$\text{(1.2)} \qquad \phi(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx - \frac{1}{2^*}\int_\Omega K(x)|u|^{2^*}\, dx.$$

On the other hand, if u is a (classical) solution of (1.1), then multiplying the given equation by u and integrating by parts shows that u

satisfies the constraint

$$\psi(u) = \int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx - \int_\Omega K(x)|u|^{2^*}\, dx = 0,$$

where we note that

$$\psi \in C^1(H_0^1(\Omega), \mathbf{R})$$

and, as we shall see, $\psi'(u) \neq 0 \in H^{-1}(\Omega)$ whenever $\psi(u) = 0$, $u \not\equiv 0$ and $0 < \lambda < \lambda_1$. Therefore, it is natural to consider the submanifold of $H_0^1(\Omega) = X$ given by

$$M = \{u \in X \backslash \{0\} \mid \psi(u) = 0\} \subset X$$

and look for the critical points of $\phi|M$. In fact, it turns out that M is a *natural constraint* for ϕ in the sense that $0 \neq u \in X$ is a critical point of ϕ if and only if $u \in M$ and u is a critical point of $\phi|M$.

In what follows we will always assume that $0 < \lambda < \lambda_1$, K is positive somewhere in Ω, and denote the norms in $X = H_0^1$, L^p by

$$||u|| = \left(\int_\Omega |\nabla u|^2\, dx\right)^{\frac{1}{2}}, \qquad |u|_p = \left(\int_\Omega |u|^p\, dx\right)^{\frac{1}{p}},$$

respectively. We will also denote

$$\rho_K^p(u) = \int_\Omega K(x)|u|^p\, dx$$

whenever $u \in L^p$ and define S_K by

$$S_K = \inf\{||u||^2/(\rho_K^{2^*}(u))^{2/2^*} \mid \rho_K^{2^*}(u) > 0\}.$$

Note that if K changes sign, then $|\rho_K^p|^{1/p}$ is not a norm since we have many nonzero $u \in L^p$ such that $|\rho_K^p(u)| = 0$. However, since $0 < \lambda < \lambda_1$, it is immediate that $\rho_K^{2^*}(u) > 0$ for every $u \in M$. Also, arguing by contradiction, it is not hard to show that $S_K > 0$.

Lemma 1.1. *$M \subset X$ is a (nonempty) C^1-submanifold of codimension 1 and is such that $0 \notin \overline{M}$.*

Proof. Let $u_0 \in X$ be such that $||u_0||^2 - \lambda|u_0|_2^2 = A_0 > 0$, $\rho_K^{2^*}(u_0) = B_0 > 0$. Then, since $2^* > 2$, we have $\psi(ru_0) = A_0 r^2 - B_0 r^{2^*} > 0$ for $r > 0$ small and $\psi(ru_0) < 0$ for $r > 0$ big, so that $\psi(r_0 u_0) = 0$ for some $r_0 > 0$ and $M \neq \phi$.

Now, let $u \in M$ and assume, by contradiction, that

$$\psi'(u) \cdot h = \int_\Omega 2(\nabla u \cdot \nabla h - \lambda uh)\, dx - \int_\Omega 2^* K(x)|u|^{2^*-2} uh\, dx = 0$$
$$\text{for every } h \in X.$$

Then, letting $h = u$ gives

$$2(||u||^2 - \lambda|u|_2^2) - 2^* \rho_K^{2^*}(u) = 0$$

or

$$2\rho_K^{2^*}(u) - 2^* \rho_K^{2^*}(u) = 0$$

since $u \in M$. Hence, we obtain $||u||^2 - \lambda|u|_2^2 = \rho_K^{2^*}(u) = 0$ which implies $u = 0$ since $\lambda < \lambda_1$. This contradicts the fact that $u \in M$ and, therefore, $M \subset X$ is a C^1-submanifold of codimension 1.

Finally, from the definition of S_K we obtain for $u \in M$ that

$$0 = \psi(u) = ||u||^2 - \lambda|u|_2^2 - \rho_K^{2^*}(u)| \geq ||u||^2 - \lambda|u|_2^2 - S^{-2^*/2}||u||^{2^*},$$

hence

$$0 \geq C_\lambda ||u||^2 - S_K^{-2^*/2}||u||^{2^*},$$

where $C_\lambda = 1 - \lambda/\lambda_1 > 0$ since $0 < \lambda < \lambda_1$. The above inequality implies

$$||u||^{2^*-2} \geq C_\lambda S_K^{2^*/2} > 0$$

for every $u \in M$, so that $\text{dist}(0, M) > 0$, that is, $0 \notin \overline{M}$. □

Lemma 1.2. *M is a natural constraint for ϕ, that is, $u \in X\backslash\{0\}$ is a critical point of $\phi \Leftrightarrow u \in M$ and u is a critical point of $\phi|M$.*

Proof. If $0 \neq u \in X$ is a critical point of ϕ, then

$$\phi'(u) \cdot h = \int_\Omega (\nabla u \cdot \nabla h - \lambda uh)\, dx - \int_\Omega K(x)|u|^{2^*-2} uh\, dx = 0$$

for every $h \in X$ and, letting $h = u$, we obtain $\psi(u) = 0$, so that $u \in M$ (and clearly u is a critical point of $\phi|M$).

Conversely, if $u \in M$ is such that $(\phi|M)'(u) = 0$, then there is a Lagrange multiplier $\mu \in \mathbf{R}$ such that $\phi'(u) = \mu\psi'(u)$, i.e.,

$$\int_\Omega (\nabla u \cdot \nabla h - \lambda uh) - \int_\Omega K|u|^{2^*-2}uh \\ = 2\mu \int_\Omega (\nabla u \cdot \nabla h - \lambda uh) - 2^*\mu \int_\Omega K|u|^{2^*-2}uh$$

for every $h \in X$. Letting $h = u$ in the above gives

$$(1-2\mu)(||u||^2 - \lambda|u|_2^2) = (1-2^*\mu)\rho_K^{2^*}(u),$$

or, since $u \in M$,

$$(1-2\mu)\rho_K^{2^*}(u) = (1-2^*\mu)\rho_K^{2^*}(u).$$

Therefore, since $\rho_K^{2^*} > 0$ for $u \in M$, we obtain $\mu = 0$, so that $\phi'(u) \cdot h = 0$ for all $h \in X$, i.e., u is a critical point of ϕ. □

Lemma 1.3. *ϕ is bounded from below on M.*

Proof. For $u \in M$ we have $\psi(u) = 0$, so that

$$\phi(u) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\rho_K^{2^*} = \frac{1}{n}(||u||^2 - \lambda|u|_2^2) \geq \frac{1}{n}C_\lambda ||u||^2,$$

where $C_\lambda = 1 - \lambda/\lambda_1 > 0$. □

Remark . Note that, for $u \in M$, it follows that

$$\rho_K^{2^*}(u) \geq C_\lambda ||u||^2.$$

Also, since the proof of Lemma 1.1 shows that

$$\text{dist}(0, M)^2 = \inf_{u \in M} ||u||^2 \geq C_\lambda^{2/(2^*-2)} S_K^{2^*/(2^*-2)} > 0,$$

we obtain from Lemma 1.3 that

$$I_\lambda := \inf_M \phi = \frac{1}{n} \inf_M \rho_K^{2^*}(u) \geq \frac{1}{n} C_\lambda \inf_M ||u||^2 \geq \frac{1}{n}(C_\lambda S_K)^{2^*/(2^*-2)},$$

that is,

$$I_\lambda \geq \frac{1}{n}(C_\lambda S_K)^{n/2} > 0.$$

In the next lemma, we compute I_λ explicitly in terms of the number

$$S_{\lambda,K} := \inf\{||v||^2 - \lambda|v|_2^2 \mid \rho_K^{2^*}(v) = 1\}.$$

Lemma 1.4. $I_\lambda := \inf_M \phi = (1/n) S_{\lambda,K}^{n/2} > 0.$

Proof. Let v_ε, $\varepsilon > 0$, be such that $\rho_K^{2^*}(v_\varepsilon) = 1$ and

$$q_\lambda(v_\varepsilon) := ||v_\varepsilon||^2 - \lambda|v_\varepsilon|_2^2 = S_{\lambda,K} + o(1).$$

Let $u_{r,\varepsilon} = r v_\varepsilon$ and choose $r = r_\varepsilon = (S_{\lambda,K} + o(1))^{1/(2^*-2)}$ so that

$$r_\varepsilon^{2^*} = r_\varepsilon^2 q_\lambda(v_\varepsilon),$$

that is,

$$\rho_K^{2^*}(u_{r_\varepsilon,\varepsilon}) = q_\lambda(u_{r_\varepsilon,\varepsilon}).$$

Then, defining $u_\varepsilon = u_{r_\varepsilon,\varepsilon}$, we obtain

$$\begin{aligned} I_\lambda &= \inf\left\{\frac{1}{n} q_\lambda(u) \mid q_\lambda(u) = \rho_k^{2^*}(u)\right\} \leq \frac{1}{n} q_\lambda(u_\varepsilon) \\ &= \frac{1}{n} r_\varepsilon^2 q_\lambda(v_\varepsilon) = \frac{1}{n} q_\lambda(v_\varepsilon)^{1+2/(2^*-2)}, \end{aligned}$$

that is,

$$I_\lambda \leq \frac{1}{n} q_\lambda(v_\varepsilon)^{2^*/(2^*-2)} = \frac{1}{n}(S_{\lambda,K} + o(1))^{n/2},$$

hence, $I_\lambda \leq (1/n) S_{\lambda,K}^{n/2}$. Conversely, if $u_\varepsilon \neq 0$ is such that

$$q_\lambda(u_\varepsilon) = \rho_K^{2^*}(u_\varepsilon)$$

and
$$\frac{1}{n}q_\lambda(u_\varepsilon) = I_\lambda + o(1),$$
then, letting $v_\varepsilon = u_\varepsilon/\rho_k^{2^*}(u_\varepsilon)^{1/2^*}$, we obtain $\rho_k^{2^*}(v_\varepsilon) = 1$ and

$$S_{\lambda,K} \le q_\lambda(v_\varepsilon) = \frac{1}{\rho_K^{2^*}(u_\varepsilon)^{2/2^*}} q_\lambda(u_\varepsilon) = q_\lambda(u_\varepsilon)^{1-2/2^*},$$

that is,

$$S_{\lambda,K} \le n(I_\lambda + o(1))^{(2^*-2)/2^*} = n(I_\lambda + o(1))^{2/n}.$$

Therefore, we obtain
$$S_{\lambda,K} \le (nI_\lambda)^{2/n},$$

which, combined with the previously obtained $I_\lambda \le (1/n)S_{\lambda,K}^{n/2}$, finishes the proof. □

Proposition 1.5. *$\phi|M : M \to \mathbf{R}$ satisfies* $(\mathrm{PS})_c$ *for $c \in (0, (1/n)S_K^{n/2})$.*

Proof. We want to show that if a sequence $u_i \in M$ satisfies

$$\phi(u_i) \to c \in \left(0, \frac{1}{n}S_K^{n/2}\right), (\phi|M)'(u_i) \to 0 \quad \text{in } X^*,$$

then u_i possesses a convergent subsequence (still labelled u_i) in M : $u_i \to u_\infty \in M$. So, assume that

$$\frac{1}{n}(||u_i||^2 - \lambda|u_i|_2^2) = \frac{1}{n}\rho_K^{2^*}(u_i) \to c \in \left(0, \frac{1}{n}S_K^{n/2}\right), \tag{1.3}$$

$$\nabla\phi(u_i) - \left(\nabla\phi(u_i), \frac{\nabla\psi(u_i)}{||\nabla\psi(u_i)||}\right)\frac{\nabla\psi(u_i)}{||\nabla\psi(u_i)||} = o(1) \in X. \tag{1.4}$$

Then, (1.3) implies that

$$||u_i|| \quad \text{is bounded,} \tag{1.5}$$

so that, by passing to a subsequence, we have

$$u_i \to u_\infty \quad \text{weakly in } X = H_0^1$$
$$u_i \to u_\infty \quad \text{strongly in } L^p,\ 1 \le p < 2^*.$$

Next, we claim that

$$||\nabla\phi(u_i)||, \quad ||\nabla\psi(u_i)|| \quad \text{are bounded sequences.} \tag{1.6}$$

Indeed, we have $\nabla\phi(u) = u - \lambda A u - A(K(x)|u|^{2^*-2}u)$, where the operator $A = (-\Delta)^{-1} : L^{2^\dagger} \subset H^{-1} \to H_0^1 \subset L^{2^*}$, $2^\dagger = 2^*/(2^*-1)$, is bounded. Clearly, $A : H_0^1 \to H_0^1$ is also bounded. Therefore, we can estimate

$$||\nabla\phi(u)|| \le ||u|| + c_1||u|| + c_2|\ |u|^{2^*-2}u|_{2^\dagger} = (1+c_1)||u|| + c_2|u|_{2^*}^{2^*/2^\dagger},$$

where we used the fact that $2^\dagger$ is the conjugate exponent of 2^*. The above shows that $||\nabla\phi(u_i)||$ is bounded. Similarly, we obtain that $\nabla\psi(u_i) = 2(u_i - \lambda A u_i) - 2^* A(K(x)|u_i|^{2^*-2}u_i)$ is bounded. Thus (1.6) holds.

Now, using (1.6), we can take the inner product of (1.4) with $\nabla\phi(u_i)$ to get

$$||\nabla\phi(u_i)||^2 = \left(\nabla\phi(u_i), \frac{\nabla\psi(u_i)}{||\nabla\psi(u_i)||^2}\right) + o(1). \tag{1.7}$$

and, using $(\nabla\phi(u_i), u_i) = \psi(u_i) = 0$ (since $u_i \in M$), we take the inner product of (1.4) with u_i to get

$$\left(-\nabla\phi(u_i), \frac{\nabla\psi(u_i)}{||\nabla\psi(u_i)||}\right)\left(\frac{\nabla\psi(u_i)}{||\nabla\psi(u_i)||}, u_i\right) = o(1),$$

that is,

$$\frac{1}{||\nabla\psi(u_i)||}\left(\nabla\phi(u_i), \frac{\nabla\psi(u_i)}{||\nabla\psi(u_i)||}\right)\Big[2(||u_i||^2 - \lambda|u_i|_2^2) - 2^*\rho_K^{2^*}(u_i)\Big] = o(1),$$

or yet,

$$(2^*-2)\frac{||\nabla\phi(u_i)||}{||\nabla\psi(u_i)||}\rho_K^{2^*}(u_i) = o(1), \tag{1.8}$$

in view of (1.7) and the fact that $||u_i||^2 - \lambda|u_i|_2^2 = \rho_K^{2^*}(u_i)$.

Therefore, since $||\nabla\psi(u_i)||$ is bounded by (1.6) and $\rho_K^{2^*}(u_i) \to nc \neq 0$ by (1.3), we obtain from (1.8) that

$$||\nabla\phi(u_i)|| = o(1). \tag{1.9}$$

From here on, the proof goes exactly as in Lemma 2.3 to give that (a subsequence of) $u_i \to u_\infty$ strongly in H_0^1. □

Now, we show that the intermediate results in J. Escobar [**8**] for the constrained functional

$$\tilde{\phi}_\lambda(u) = \frac{\int_\Omega(|\nabla u|^2 - \lambda u^2)\,dx}{\big(\int_\Omega K|u|^{2^*}\,dx\big)^{\frac{2}{2^*}}} = \frac{q_\lambda(u)}{(\rho_K^{2^*}(u))^{\frac{2}{2^*}}}$$

(with $\rho_K^{2^*}(u) > 0$) imply that

$$S_{\lambda,K} \in (0, S_K).$$

Indeed, a typical result in [**8**] is the following theorem:

Let $\mathcal{M}$ be a four dimensional Riemannian manifold with boundary, which is locally conformally flat, and let K be a smooth function on $\mathcal{M}$ which is positive somewhere and attains its maximum at an interior point. Then, for any $\lambda \in (0, \lambda_1)$, there exists a solution of

$$\begin{cases} \Delta u + \lambda u + K(x)u^3 = 0 & \text{on } \mathcal{M}, \\ u > 0 & \text{on } \mathcal{M}, \\ u = 0 & \text{on } \partial\mathcal{M}. \end{cases}$$

Remark. In the above statement, the assumptions of $\mathcal{M}$ being locally conformally flat and K attaining its maximum at an interior point were included since they seem to be necessary for the proof in [**8**] to go through.

In the proof of the above theorem (and many other results in [**8**]) the main step was to show the strict inequality

$$(\max K)^{(n-2)/n} S_{\lambda,K} < S, \tag{1.10}$$

where $S = \inf\{||v||^2 \mid |v|_{2^*} = 1\}$ is the best constant for the Sobolev embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$. We notice now that (1.10) implies

$$S_{\lambda,K} < S_K. \tag{1.11}$$

Indeed, since $\rho_K^{2^*}(v) \leq \max K |v|_{2^*}^{2^*}$ and

$$S_K = \inf\{||v||^2/(\rho_K^{2^*}(v))^{2/2^*} | \rho_K^{2^*}(v) > 0\},$$

we clearly have

$$S_K \geq \frac{1}{(\max K)^{2/2^*}} \inf_{v \neq 0} \frac{||v||^2}{|v|_{2^*}^2} = \frac{1}{(\max K)^{(n-2)/n}} S,$$

which combined with (1.10) gives (1.11). Therefore, by Lemma 1.4, we obtain

$$0 < I_\lambda = \inf_M \phi_\lambda = \frac{1}{n} S_{\lambda,K}^{n/2} < \frac{1}{n} S_K^{n/2}, \tag{1.12}$$

which shows that I_λ falls into the range $(0, (1/n)S_K^{n/2})$ of validity of the $(\mathrm{PS})_c$ condition, cf. Proposition 1.5. Next, we recall the following basic result in the calculus of variations:

Let $\phi : M \to \mathbf{R}$ be C^1, bounded from below and satisfy $(\mathrm{PS})_c$ *for $c \in (a, \gamma)$. If*

$$a < I = \inf_M \phi < \gamma$$

then I is attained in M.

It follows from (1.12) and Lemmas 1.1, 1.3, 1.4 that there exists $u_0 \in M$ such that

$$0 < \phi(u_0) = \inf_M \phi,$$

hence $u_0 \neq 0$ is a critical point of $\phi|M$. By Lemma 1.2, u_0 is a critical point of the unconstrained functional ϕ, hence a (classical) solution of (1.1). We notice that, since $\phi(u) = \phi(|u|)$ for all $u \in X$ and $u \in M$ if and only if $|u| \in M$, we may assume, as usual, that $u_0 \geq 0$ and, hence, $u_0 > 0$ in Ω by the maximum principle.

2. A multiplicity result. Here, we consider the question of multiplicity of solutions for our problem

$$\begin{cases} -\Delta u = \lambda u + K(x)|u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{2.1}$$

where, as before, $\Omega \subset \mathbf{R}^n$ $(n \geq 3)$ is a bounded smooth domain, $2^* = 2n/(n-2)$ and $K \in C^\alpha(\overline{\Omega})$. This time we will assume that

$$K(x) > 0 \quad \text{a.e. in } \Omega, \tag{2.2}$$

and show that (2.1) has multiple (pairs of) solutions if λ is near (and to the left of) an eigenvalue λ_i of $-\Delta$ under Dirichlet boundary condition. More precisely, we prove

Theorem 2.1. *Assume $K \in C^\alpha(\overline{\Omega})$ satisfies condition* (2.2). *Then, for each $j \in \mathbf{N}$, there exists $\varepsilon_j > 0$ such that* (2.1) *has at least m_j (= multiplicity of λ_j) pairs of solutions $\pm u_k(\lambda)$, $k = 1, \dots, m_j$, for $\lambda \in (\lambda_j - \varepsilon_j, \lambda_j)$. Moreover, $||u_k(\lambda)|| \to 0$ as $\lambda \to \lambda_j$.*

This result and its proof are natural extensions of the ones in Cerami–Fortunato–Struwe [**7**] for $K(x) \equiv 1$, where the following variant is used due to Bartolo–Benci –Fortunato [**4**] of minimax results of Ambrosetti-Rabinowitz [**2**].

Theorem 2.2. [**4**]. *Let X be a Hilbert space and $\phi : X \to \mathbf{R}$ be C^1, even, and satisfy* $(\text{PS})_c$ *for $c \in (0, \beta)$. Assume that $\phi(0) = 0$ and there exist closed subspaces $V, W \subset X$ with* $\text{codim}V < +\infty$, $\text{codim}V < \text{dim}W$, *such that*

(i) $\sup_W \phi \leq \beta'$,

(ii) $\inf_{V \cap \partial B_r} \phi \geq \delta$ *for some $r > 0$,*

where $0 < \delta < \beta' < \beta$. Then, ϕ possesses at least

$$m = \dim W - \operatorname{codim} V$$

pairs of critical points with critical values in $[\delta, \beta']$.

In order to prove Theorem 2.1, we need to find a range of validity of the $(\text{PS})_c$ condition for the functional

$$\begin{aligned} \phi(u) &= \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx - \frac{1}{2^*} \int_\Omega K(x) |u|^{2^*}\, dx \\ &= \frac{1}{2}(||u||^2 - \lambda |u|_2^2) - \frac{1}{2^*} \rho_K^{2^*}(u), \\ &\qquad u \in H_0^1(\Omega) = X. \end{aligned}$$

Since K satisfies (2.2), we now have that $(\rho_K^{2^*}(u))^{1/2^*} := |u|_{2^*,K}$ is a weighted L^{2^*}-norm and, in this case, the constant

$$S_K = \inf_{u \in X \setminus \{0\}} \frac{||u||^2}{|u|^2_{2^*,K}} > 0 \tag{2.3}$$

is the best constant in the embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega, K\,dx)$.

It turns out that the counterpart of Lemma 2.1 [**7**] in our case (compare also with Lemma 1.5) is

Lemma 2.3. *$\phi : X \to \mathbf{R}$ satisfies* $(\mathrm{PS})_c$ *for $c \in (0, (1/n)S_K^{n/2})$.*

For completeness, we include a proof which is a modification of the one in [**7**].

Proof of Lemma 2.3. We want to show that if $\{u_i\}$ satisfies

$$\phi(u_i) \to c \in \left(0, \frac{1}{n} S_K^{n/2}\right), \tag{2.4}$$

$$\nabla\phi(u_i) \to 0 \quad \text{in } X, \tag{2.5}$$

then $\{u_i\}$ possesses a convergent subsequence (still labeled u_i), $u_i \to u_\infty$ in X. From (2.4), (2.5) it follows (as in [**5**], [**7**]) that

$$||u_i|| \quad \text{is bounded,} \tag{2.6}$$

so that, by passing to a subsequence, we have

$$u_i \to u_\infty \quad \text{weakly in } X \tag{2.7}$$

$$u_i \to u_\infty \quad \text{strongly in } L^p, \qquad 1 \le p < 2^*. \tag{2.8}$$

From (2.7), (2.8) it can be shown that, for any $\theta \in C_0^\infty(\Omega)$,

$$(\nabla\phi(u_\infty) - \nabla\phi(u_i), \theta) = o(1),$$

hence

$$(\nabla\phi(u_\infty), \theta) = 0,$$

in view of (2.5). Therefore, u_∞ is a weak solution of (2.1) and, hence, a classical solution of (2.1) (cf. [**12**]).

Now, let $v_i = u_i - u_\infty$ and take the inner product of (2.5) with v_i to get

(2.9)
$$o(1) = (\nabla\phi(u_i), v_i) = \int_\Omega \nabla u_\infty \nabla v_i + \int_\Omega |\nabla v_i|^2 - \int_\Omega \lambda(u_\infty + v_i)v_i \\ - \int_\Omega K(x)|u_\infty + v_i|^{2^*-2}(u_\infty + v_i)v_i.$$

In view of (2.7), (2.8), the first and third terms in the last inequality tend to zero, so that (2.9) becomes

(2.10)
$$||v_i||^2 = \int_\Omega K(x)|u_\infty + v_i|^{2^*-2}(u_\infty + v_i)v_i\,dx + o(1) := F(u_\infty + v_i, v_i) + o(1).$$

On the other hand, we can write

$$\begin{aligned}|F(u_\infty + v_i, v_i) - F(v_i, v_i)| \\ &= \left| \int_\Omega \int_0^{u_\infty(x)} \frac{\partial}{\partial\xi}[K|v_i + \xi|^{2^*-2}(v_i + \xi)v_i]\,d\xi\,dx \right| \\ &= (2^* - 1)\left| \int_\Omega \int_0^1 K|v_i + tu_\infty|^{2^*-2} v_i u_\infty\,dt\,dx \right| \\ &\le C \int_\Omega K(|u_\infty||v_i|^{2^*-1} + |u_\infty|^{2^*-1}|v_i|)\,dx,\end{aligned}$$

where the last term tends to zero in view of (2.8) and the fact that u_∞ is a smooth function. Therefore, (2.10) becomes

$$||v_i||^2 = F(v_i, v_i) + o(1) = |v_i|_{2^*,K}^{2^*} + o(1). \tag{2.11}$$

Next, from the fact that $(\nabla\phi(u_i), u_i) = o(1)$ by (2.5), (2.6), we obtain

$$|u_i|_{2^*,K}^{2^*} = ||u_i||^2 - \lambda|u_i|_2^2 + o(1),$$

which combined with the expression for $\phi(u_i)$ gives

$$\begin{aligned}\phi(u_i) &= \frac{1}{n}(||u_i||^2 - \lambda|u_i|_2^2) + o(1) \\ &= \frac{1}{n}(||u_\infty||^2 - \lambda|u_\infty|_2^2) + \frac{1}{n}||v_i||^2 + o(1).\end{aligned} \tag{2.12}$$

And, from the fact that u_∞ is a solution of (2.1), we obtain

$$||u_\infty||^2 - \lambda|u_\infty|_2^2 - |u_\infty|_{2^*,K}^{2^*} = (\nabla\phi(u_\infty), u_\infty) = 0,$$

hence $||u_\infty||^2 - \lambda|u_\infty|_2^2 = |u_\infty|_{2^*,K}^{2^*} \geq 0$. Therefore, (2.12) implies that

$$||v_i||^2 \leq n\phi(u_i) + o(1),$$

and so

$$||v_i||^2 \leq c_1 < S_K^{n/2} \tag{2.13}$$

for all i large, in view of (2.4).

Finally, using (2.11) and the definition (2.3) of S_K, we can write

$$S_K^{2^*/2}||v_i||^2 \leq ||v_i||^{2^*} + o(1),$$

that is,

$$||v_i||^2(S_K^{n/(n-2)} - ||v_i||^{4/(n-2)}) \leq o(1),$$

where we observe that the coefficient of $||v_i||^2$ in the above is strictly positive for i large, in view of (2.13). Thus, $v_i \to 0$ strongly in X, i.e., $u_i \to u_\infty$ strongly in X. □

Proof of Theorem 2.1. Let λ_j be given and assume that $\lambda_{j-1} < \lambda < \lambda_j$ ($0 < \lambda < \lambda_1$, if $j = 1$). Defined the subspaces

$$V = \overline{\bigoplus_{k\geq j} E_k}, \qquad W = \bigoplus_{k=1}^{j} E_k,$$

where E_k denotes the λ_k-eigenspace. Clearly, we have

$$\dim W - \operatorname{codim} V = \dim E_j = m_j, \tag{2.14}$$

the multiplicity of λ_j. In order to apply Theorem 2.2, we must verify conditions (i), (ii).

Given $u \in W$, we have the estimate

$$\phi(u) \leq \frac{1}{2}(\lambda_j - \lambda)|u|_2^2 - \frac{1}{2^*}|u|_{2^*,K}^{2^*} \leq \frac{1}{2}(\lambda_j - \lambda)a_j^{-1}|u|_{2^*,K}^2 - \frac{1}{2^*}|u|_{2^*,K}^{2^*},$$

where $a_j := \inf\{|u|^2_{2^*,K}/|u|_2^2 \mid 0 \neq u \in W\} > 0$. (Any two norms are equivalent in the finite-dimensional subspace W.) Therefore,

$$\sup_W \phi \leq \frac{1}{n}[(\lambda_j - \lambda)a_j^{-1}]^{n/2} := \beta'. \tag{2.15}$$

On the other hand, for $u \in V$ we have the estimate from below,

$$\begin{aligned}\phi(u) &\geq \left(1 - \frac{\lambda}{\lambda_j}\right)||u||^2 - \frac{1}{2^*}|u|^{2^*}_{2^*,K} \\ &\geq \left(1 - \frac{\lambda}{\lambda_j}\right)||u||^2 - \frac{1}{2^* S_K^{2^*/2}}||u||^{2^*} := \Phi(||u||),\end{aligned}$$

and it is clear that there exists an $r > 0$ such that

$$\inf_{V \cap \partial B_r} \phi \geq \delta, \tag{2.16}$$

where $0 < \delta < \beta'$ and β' is given by (2.15). Also, we must restrict β' so that $\beta' < \beta := (1/n)S_K^{n/2}$ (cf. Lemma 2.3), that is, $\lambda \in (\lambda_{j-1}, \lambda_j)$ must satisfy

$$\lambda_j - \lambda < a_j S_K.$$

Therefore, in view of (2.14)–(2.16) and Theorem 2.2, it follows that Theorem 2.1 holds true with $\varepsilon_j \leq a_j S_K$. We recall that

$$a_j := \inf_{W \setminus \{0\}} \frac{|u|^2_{2^*,K}}{|u|_2^2},$$

and, therefore, in the special case $K(x) \equiv 1$, an easy application of Hölder's inequality to $|u|_2^2$ shows that $a_j \geq (\text{vol } \Omega)^{-2/n}$. □

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