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REDUCIBLE EPIDEMICS: CHOOSING YOUR SADDLE

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1. Introduction. The propagation of infection in spatial models for a deterministic n-type $S \to I \to R$ epidemic with a nonreducible infection matrix is now fairly well understood. Under certain conditions the equations admit wave solutions travelling with each speed c greater than or equal to a minimum speed c_0. The conditions are that the Perron-Frobenius root of the infection matrix is greater than one, and that the contact distributions are exponentially dominated in the forward tail. The solution at each speed $c > c_0$ has been proved to be unique modulo translation. For the critical case $c = c_0$, when c_0 is positive, the wave solution has been proved, except in an exceptional case, to be unique modulo translation.

A saddle-point approximation can be used to give an indication of the asymptotic speed of propagation. The result for the one-type simple epidemic was obtained by Daniels [**2**]. A rigorized approach to the saddle-point method, and the result for the speed of propagation for the n-type epidemic are given in Radcliffe and Rass [**6**]. This suggests that the asymptotic speed of propagation is in fact c_0, the minimum speed for which wave solutions exist. That this is the case has been proved by exact analytic methods. The one-type case appears in Aronson [**1**], Diekmann [**3**] and Thieme [**9**], and the n-type cases appears in Radcliffe and Rass [**7**]. Note that these results all assume radially symmetric contact distributions.

Recently the authors (Radcliffe and Rass [**8**]) have investigated the possible wave solutions for an $S \to I \to R$ model with a reducible infection matrix. Some interesting results were obtained. In particular, under certain conditions, more than one wave solution exists at a particular speed; these wave solutions affecting both types and having different behavior in their tails.

It is shown in Section 2 that the spatial models for the $S \to I \to R$ and $S \to I \to S$ epidemics both lead to the same equations for the spread of infection in the forward tail of the epidemic. Thus, the

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results of our paper [**6**], which were derived for the $S \to I \to R$ epidemic, apply equally well to the $S \to I \to S$ epidemic. In this paper we apply the saddle-point method to the deterministic n-type $S \to I \to R$ and $S \to I \to S$ epidemics with contact distributions which are not necessarily radially symmetric, and with a reducible infection matrix. The velocity of propagation of infection is determined in the various subgroups for any specified direction. There are, of course, conditions under which the saddle-point method can be applied. These are discussed in Section 5.

The analogous results for the n-type stochastic epidemic and the contact birth process are also briefly mentioned. It is hoped to investigate the velocity of propagation of the reducible $S \to I \to R$ epidemic along the lines of Radcliffe and Rass [**7**] shortly.

2. The models. The model for the n-type reducible deterministic $S \to I \to R$ epidemic is identical to the model described in Radcliffe and Rass [**6**], except that the infection matrix is no longer nonreducible and we do not restrict the epidemic to occur on the real line $\mathbf{R}$ and be symmetric about zero. There is an analogous $S \to I \to S$ model which is described in (ii) below. When the infection matrix is nonreducible, any infectious individual of any type can, possibly through a sequence of infections, infect any susceptible individual of any type. For a reducible epidemic, there is at least one pair of types i, j where a type i infectious individual cannot, even through a sequence of infections, infect a type j susceptible individual.

(i) The $S \to I \to R$ model. Consider n types of individuals, each type having uniform density in N-dimensional space R^N, and consisting of susceptible, infectious and removed individuals. Denote the proportions of susceptible, infectious and removed individuals of type i at position $\mathbf{s}$ and time t by $x_i(\mathbf{s}, t), y_i(\mathbf{s}, t)$ and $z_i(\mathbf{s}, t)$, respectively, so that $x_i(\mathbf{s}, t) + y_i(\mathbf{s}, t) + z_i(\mathbf{s}, t) = 1$. The density of type i individuals is σ_i. Let λ_{ij} be the rate of infection of susceptible individuals of type i by infectious individuals of type j. The contact distribution representing the distance $\mathbf{r}$ over which infection occurs has density $p_{ij}(\mathbf{r})$. The removal rate for infectious individuals of type i is μ_i.

The epidemic occurring over all real t is described by the equations

(1)
$$\partial x_i(\mathbf{s},t)/\partial t = -x_i(\mathbf{s},t)\sum_{j=1}^{n}\lambda_{ij}\sigma_j\int_{R^N} p_{ij}(\mathbf{r})y_j(\mathbf{s}-\mathbf{r},t)\,d\mathbf{r},$$
$$\partial y_i(\mathbf{s},t)/\partial t = x_i(\mathbf{s},t)\sum_{j=1}^{n}\lambda_{ij}\sigma_j\int_{R^N} p_{ij}(\mathbf{r})y_j(\mathbf{s}-\mathbf{r},t)\,d\mathbf{r} - \mu_i y_i(\mathbf{s},t),$$
$$\partial z_i(\mathbf{s},t)/\partial t = \mu_i y_i(\mathbf{s},t), \qquad i = 1,\dots,n.$$

(ii) The $S \to I \to S$ model. Consider n types of individuals, each type having uniform density in N-dimensional space R^N and consisting of susceptible and infectious individuals. Denote the proportions of susceptible and infectious individuals of type i at position $\mathbf{s}$ and time t by $x_i(\mathbf{s},t)$ and $y_i(\mathbf{s},t)$, respectively, so that $x_i(\mathbf{s},t)+y_i(\mathbf{s},t)=1$. The density of type i individuals is σ_i. Let λ_{ij} be the rate of infection of susceptible individuals of type i by infectious individuals of type j. The contact distribution representing the distance $\mathbf{r}$ over which infection occurs has density $p_{ij}(\mathbf{r})$. The rate at which infectious individuals of type i recover and re-enter the susceptible state is μ_i.

The equations for this model are

(2)
$$\frac{\partial x_i(\mathbf{s},t)}{\partial t} = -x_i(\mathbf{s},t)\sum_{j=1}^{n}\lambda_{ij}\sigma_j\int_{R^N} p_{ij}(\mathbf{r})y_j(\mathbf{s}-\mathbf{r},t)\,dr + \mu_i y_i(\mathbf{s},t),$$
$$\frac{\partial y_i(\mathbf{s},t)}{\partial t} = x_i(\mathbf{s},t)\sum_{j=1}^{n}\lambda_{ij}\sigma_j\int_{R^N} p_{ij}(\mathbf{r})y_j(\mathbf{s}-\mathbf{r},t)\,dr - \mu_i y_i(\mathbf{s},t),$$
$$i = 1,\dots,n.$$

Note that equations (2) also hold if births and deaths are included in the model, which are balanced so that the population size stays constant. Then μ_i includes the death rate for infectives as a component.

For both models we restrict each $p_{ij}(\mathbf{r})$ so that the joint Laplace transform, $P^*_{ij}(\theta)$, exists in an open region of $\theta = 0$, where $P^*_{ij}(\theta) = \int_{R^N} e^{-\theta'\mathbf{r}}p_{ij}(\mathbf{r})\,d\mathbf{r}$.

We wish to consider the speed of spread of the forward front of the epidemic in a specific direction. In the extreme forward region, where

there is little infection, we may approximate $x_i(\mathbf{s},t)$ by one, so that in both models (i) and (ii) $y_i(s,t)$ approximately satisfies the equation
(3)
$$\frac{\partial y_i(\mathbf{s},t)}{\partial t} = \sum_{j=1}^{n} \lambda_{ij}\sigma_j \int_{R^N} p_{ij}(\mathbf{r}) y_j(s-\mathbf{r},t)\,dr - \mu_i y_i(\mathbf{s},t), \qquad i=1,\dots,n.$$

Let $\Lambda = (\lambda_{ij})$ be reducible. For simplicity of exposition, we will discuss the following simple reducible case. The population of n types is taken to consist of two groups of m_1 and m_2 types, respectively, where $m_1 + m_2 = n$. Within a group, infection can occur between any types, possibly through a sequence of infections. No type in group 2 can infect a type in group 1; but at least one type in group 1 can infect some type in group 2. Hence, by reordering the types so that Λ is in normal form (see Gantmacher [**4**]), Λ may be partitioned so that

$$\Lambda = \begin{pmatrix} \Lambda_{11} & 0 \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix},$$

where Λ_{11} and Λ_{22} are nonnegative, nonreducible square matrices of sizes m_1 and m_2, respectively, and Λ_{21} is nonnegative and is not identically zero. The extension to the general reducible case is discussed in Section 7.

The nonreducible n-type contact birth process and n-type $S \to I \to R$ stochastic epidemic, described in Radcliffe and Rass [**6**], also have reducible analogues. There is also an analagous stochastic $S \to I \to S$ reducible epidemic. These can be treated in a similar manner and lead us to consider the same equation (3).

For the contact birth process, spreading in R^N, we can immediately restrict attention to the spread in a specific direction. This, of course, is not true for an epidemic process. The projection of an N-dimensional contact birth process in a given direction is a 1-dimensional contact birth process, with its contact distribution the marginal distribution of the N-dimensional contact distribution in the specified direction. A full description of this process is given in Radcliffe and Rass [**6**], Section 3. For the contact birth process, let $U(t)$ be the position of furthest spread in that direction from 0 at time t. Then

$$y_i(s,t) = P\big(U(t) > s \mid \text{one type } i \text{ individual at position 0 at time } t=0\big).$$

Note that $y_i(s,t)$ for the projection of the contact birth process satisfies equation (3) with $N = 1$ and $p_{ij}(r)$ the marginal contact distribution in that direction. The saddle-point method will give the asymptotic speed of translation of the distribution function of furthest spread of the contact birth process in that direction. In a similar way, the asymptotic speed of translation of the distribution function of furthest spread of a specific type k in a specific direction may be obtained.

For the stochastic epidemic, $y_i(\mathbf{s},t)$ is the expected proportion of infectives at position $\mathbf{s}$ and time t. The saddle-point method gives an approximation to the asymptotic expectation velocity in a specific direction.

3. An expression for the velocity of propagation obtained using the saddle-point method when the infection matrix is nonreducible. This section defines the velocity of propagation, indicates how the saddle-point method is applied, and gives an expression for the velocity of propagation when the contact distributions are symmetric and the infection matrix is nonreducible. For a detailed description and proofs of these results, the reader is referred to Radcliffe and Rass [**6**].

The velocity of propagation considered in that paper is in fact the speed of spread of the forward tail of the epidemic when the population is of uniform density on the real line $\mathbf{R}$. This can be defined for each specific type j. Consider a small amount η of infection ahead of position $\mathbf{s}$ at time t for a specific type j. If η remains constant as t increases, then the position $\mathbf{s}$ is defined by $\int_s^\infty y_j(u,t)\,du = \eta$.

The speed of spread of the forward tail of infection for population j is then $\lim_{t\to\infty} s/t$. This limit is assumed to exist and to be positive. The saddle-point method enables us to find this limit; and to verify that it is the same for all types when the infection matrix is nonreducible.

The method uses the approximate equations (3) with $N = 1$, which are valid in the tail of the epidemic; and works with the Laplace transform of $y_i(s,t)$, since the differential equations for the Laplace transform have a simple solution which can be inverted.

Let $L_i(\lambda,t) = \int_{-\infty}^\infty e^{\lambda s} y_i(s,t)\,ds$, $\{\mathbf{L}(\lambda,t)\}_i = L_i(\lambda,t)$ and $P_{ij}(\lambda) =$

$\int_{-\infty}^{\infty} e^{\lambda r} p_{ij}(r)\, dr$. Then Laplace transforming equation (3), we obtain

$$\frac{\partial \mathbf{L}(\lambda,t)}{\partial t} = (\mathbf{A}(\lambda) - \mu \mathbf{I})\mathbf{L}(\lambda,t),$$

where $\mu = \max(\mu_1, \dots, \mu_n)$ and $\{\mathbf{A}(\lambda)\}_{ij} = \sigma_j \lambda_{ij} P_{ij}(\lambda) + \delta_{ij}(\mu - \mu_i)$. Note that

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Then $\mathbf{L}(\lambda,t) = e^{(\mathbf{A}(\lambda)-\mu\mathbf{I})t}\mathbf{u}(\lambda)$ where $\mathbf{u}(\lambda) = \mathbf{L}(\lambda, 0)$.

Let $\Delta_{V_{ij}}$ be the abscissa of convergence of $P_{ij}(\lambda)$ in the positive half of the complex plane; and let $\Delta_V = \min_{i,j} \Delta_{V_{ij}}$. Conditions are imposed on $P_{ij}(\lambda)$ and $L_i(\lambda, 0)$. It is assumed that the $p_{ij}(r)$ are symmetric about zero with $\lim_{\lambda \uparrow \Delta_{V_{ij}}} P_{ij}(\lambda) = \infty$ for all i, j; and that for any $[\theta_1, \theta_2] \subset (0, \Delta_V)$, there exists a $k_{ij}(y)$ with $|P_{ij}(\theta + iy)| \leq k_{ij}(y)$ for all $\theta \in [\theta_1, \theta_2]$ with $\int_{-\infty}^{\infty} k_{ij}(y)\, dy < \infty$. Each $L_i(\lambda, 0)$ is taken to be the Laplace transform of a function $y_i(s, 0)$ of bounded support, the Laplace transform being analytic for all λ.

The transform $L_j(\lambda, t)$ is inverted so that, for a suitable choice of $\theta(t)$,

$$y_j(u,t) = \frac{1}{2\pi i} \int_{\theta(t)-i\infty}^{\theta(t)+i\infty} e^{-u\lambda} \{\mathbf{L}(\lambda,t)\}_j \, d\lambda.$$

Integrating over u from $\mathbf{s}$ to ∞ gives an equation for η. The dominant term in $\{\mathbf{L}(\lambda,t)\}_j$ comes from considering the eigenvalue of $\mathbf{A}(\lambda)$ with largest real part. When λ is real, this eigenvalue is the Perron-Frobenius eigenvalue $\rho(\mathbf{A}(\lambda))$. This can be extended to complex λ close to the real line, so that $\mathrm{Re}\,(\rho(\mathbf{A}(\lambda))$ is the maximum of the real parts of the eigenvalues of $\mathbf{A}(\lambda)$.

$\theta(t)$ is chosen to be the saddle-point of $\mathrm{Re}\,(g(\lambda))$, where $g(\lambda) = [\rho(\mathbf{A}(\lambda))t - \lambda s]$. Note that $g(\lambda)$ is a convex function of λ, for λ real, with a unique minimum at $\lambda = \theta(t)$, where $\rho'(A(\theta(t))) = s/t$. That $\lambda = \theta(t)$ is a saddle-point of $\mathrm{Re}\,(g(\lambda))$ can be seen by observing that, from the proof of Lemma 5 by Radcliffe and Rass [**6**], $\mathrm{Re}\,(g(x+iy)) < \mathrm{Re}\,(g(x))$ for $y \neq 0$.

Note that the symmetry condition imposed on the $p_{ij}(\mathbf{r})$ ensures that the derivatives of $\rho(\mathbf{A}(\theta))$ with respect to θ, denoted by $\rho'(\mathbf{A}(\theta))$, takes all positive values for $\theta \in (0, \Delta_v)$. This is easily seen by noting that $\rho(A(\theta))$ is a convex function of θ for $\theta \in (0, \Delta_v)$, with

$$\rho'(\mathbf{A}(\theta)) = \frac{\sum_i \sum_j \mathbf{A}'_{ij}(\theta)\{\mathrm{Adj}\,(\mathbf{A}(\theta) - \rho(\mathbf{A}(\theta))\mathbf{I})\}_{ij}}{\mathrm{trace}\,\mathrm{Adj}\,(\mathbf{A}(\theta) - \rho(\mathbf{A}(\theta))\mathbf{I})},$$

so that $\rho'(\mathbf{A}(0)) = 0$. Observe that, with the additional assumption that $\lim_{\lambda \uparrow \Delta_{v_{ij}}} P_{ij}(\lambda) = \infty$, the corollary to Lemma 5 of our paper [**6**] also shows that $\lim_{\lambda \uparrow \Delta v} \rho'(\mathbf{A}(\lambda)) = \infty$.

The symmetry condition also ensures that if $\rho(\mathbf{A}(0)) > \mu$, then $\rho(\mathbf{A}(\lambda)) > \mu$ for all $\lambda \in (0, \Delta_v)$. This ensures that $\rho(\mathbf{A}(\theta(t))) > \mu$, which is required in the proof of Theorem 1 of Radcliffe and Rass [**6**].

This theorem establishes that, given any small $\delta > 0$, for t sufficiently large

$$\eta \simeq \frac{1}{2\pi i} \int_{\theta(t)-i\delta}^{\theta(t)+i\delta} \frac{1}{\lambda} e^{[(\rho(\mathbf{A}(\lambda))-\mu)t - \lambda s]} \{\mathbf{E}(\lambda)\mathbf{u}(\lambda)\}_j \, d\lambda,$$

where $\mathbf{E}(\lambda)$ is the idempotent of $\mathbf{A}(\lambda)$ corresponding to $\rho(\mathbf{A}(\lambda))$. It then follows that

$$\begin{aligned}
&\eta e^{-[(\rho(\mathbf{A}(\theta(t)))-\mu)t - s\theta(t)]} \\
&\quad \simeq \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{1}{\theta(t)} \{\mathbf{E}(\theta(t))\mathbf{u}(\theta(t)))\}_j \; e^{iy\rho'(A(\theta(t)))t - \rho''(\theta(t))y^2 t/2 - iys} \, dy \\
&\quad = \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{1}{\theta(t)} \{\mathbf{E}(\theta(t))\mathbf{u}(\theta(t))\}_j \; e^{-\rho''(A(\theta(t)))y^2 t/2} \, dy \\
&\quad \simeq \{\mathbf{E}(\theta(t))\mathbf{u}(\theta(t))\}_j / [(2\pi)^{1/2} \theta(t) (\rho''(A(\theta(t)))t)^{1/2}].
\end{aligned}$$

Therefore, $f = \lim_{t\to\infty} s/t = \lim_{t\to\infty} (\rho(\mathbf{A}(\theta(t))) - \mu)/\theta(t)$. If $\lim_{t\to\infty} \theta(t) = \theta_0$, then $f = (\rho(\mathbf{A}(\theta_0)) - \mu)/\theta_0$. Note that $\rho'(\mathbf{A}(\theta(t))) = s/t$ so that $f = \rho'(\mathbf{A}(\theta_0))$. Observe that f does not depend on the type j. Note that, in order to derive this result, we need to use the analyticity of $\rho(\mathbf{A}(\lambda))$ in an open region of $\lambda = \theta_0$.

Now consider $f(\lambda) = (\rho(\mathbf{A}(\lambda)) - \mu)/\lambda$. This is a continuous function of λ, for $\lambda \in (0, \Delta_v)$, with $\lim_{\lambda \uparrow \Delta_v} f(\lambda) = \infty$. Also, from Lemma 6 part (ii) of our paper [**6**], any positive solution λ to $f'(\lambda) = 0$ is a minimum of $f(\lambda)$. Hence, there can be at most one such solution.

When $\rho(\mathbf{A}(0)) > \mu$, $f(\lambda) > 0$ for $\lambda \in (0, \Delta_v)$ and $f(0) = \infty$. There is then a unique $\lambda \in (0, \Delta_v)$ such that $f'(\lambda) = 0$, this being at the unique minimum of the function $f(\lambda)$. Hence, $f'(\theta_0) = (\rho'(\mathbf{A}(\theta_0)) - f(\theta_0))/\theta_0 = 0$, so that $f = f(\theta_0) = \min_{\lambda\in(0,\Delta_v)} f(\lambda)$. This then gives an explicit expression for the speed of propagation f of the forward tail of the epidemic.

When $\rho(\mathbf{A}(0)) < \mu$, $\lim_{\lambda\downarrow 0} f(\lambda) = -\infty$. Hence, there is no $\lambda \in (0, \Delta_v)$ such that $f'(\lambda) = 0$. When $\rho(\mathbf{A}(0)) = \mu$, $\lim_{\lambda\downarrow 0} f(\lambda) = 0$; and $f(\lambda) > 0$ for $\lambda \in (0, \Delta_v)$. Hence, there is no $\lambda \in (0, \Delta_v)$ with $f'(\lambda) = 0$. Theorem 1 of our paper [**6**] requires $\rho(\mathbf{A}(\theta(t))) > \mu$. The implication is then that there cannot exist a positive limit f to $\mathbf{s}/t$ with $f > \rho(\mathbf{A}(\theta^*))$ where $\theta^* \in (0, \Delta_v)$ is such that $\rho(\mathbf{A}(\theta^*)) = \mu$. When $\rho(\mathbf{A}(0)) = \mu$, this implies $f = 0$. This suggests that $f = 0$ for $\rho(\mathbf{A}(0)) \le \mu$.

4. Connecting the functions $K(c, \lambda)$ and $f(\lambda)$ to show that $f = c_0$. Let $K(c, \lambda) = \rho(\mathbf{V}_c(\lambda))$ and $\mathbf{V}_c(\lambda) = \Lambda(\lambda)(\mathbf{F}(c, \lambda))^{-1}$, where $\{\Lambda(\lambda)\}_{ij} = \sigma_j \lambda_{ij} P_{ij}(\lambda)$ and $\mathbf{F}(c, \lambda) = \operatorname{diag}(\mu_1 + c\lambda, \dots, \mu_n + c\lambda)$. Note that $K(c, 0) = \rho(\Gamma)$ for all $c > 0$, where $\{\Gamma\}_{ij} = \sigma_j \lambda_{ij}/\mu_j$.

In the analysis of the $S \to I \to R$ epidemic, the function $K(c, \lambda)$ is central to the problem of the existence and nonexistence of wave solutions to equation (1) at different speeds c (see Radcliffe and Rass [**5**]). Note that in that paper the contact distributions were not restricted to be symmetric about zero. When $\rho(\Gamma) > 1$, the minimum speed c_0 for which wave solutions exist has an explicit expression in terms of $K(c, \lambda)$; namely,

$$c_0 = \inf\{c > 0 : K(c, \lambda) = 1 \text{ for some } \lambda \in (0, \Delta_v)\}.$$

When $\rho(\Gamma) \le 1$, no wave solutions exist at any speed $c > 0$. In this case we define $c_0 = 0$.

The properties of $K(c, \lambda)$ are discussed in our paper [**6**]. In particular, for each fixed positive speed c, $K(c, \lambda)$ is a convex function of λ. Also, $K(c, \lambda)$ is a decreasing function of c for each $\lambda \in (0, \Delta_v)$.

The saddle-point method suggests that when $\rho(\mathbf{A}(0)) \le \mu$, the speed of propagation $f = 0$. When $\rho(\mathbf{A}(0)) > \mu$, it identifies the speed of propagation f as $\min_{\lambda\in(0,\Delta_v)} f(\lambda)$, where $f(\lambda) = (\rho(\mathbf{A}(\lambda)) - \mu)/\lambda$. Note that in this case, with the restriction imposed in Section 3, $f(\lambda)$

is a continuous function of λ which tends to infinity as $\lambda \downarrow 0$ and as $\lambda \uparrow \Delta_v$.

We now prove two lemmas. They are proved in more generality than is needed in this section so that they may be used in the latter part of the paper where less restrictions are placed on the contact distributions. The only condition required is that each $P_{ij}(\lambda)$ exists for some open region about $\lambda = 0$.

Lemma 1. *Let* $\mathbf{A}(0)$ *be a nonreducible matrix so that* Γ *is also nonreducible. Then* $\rho(\mathbf{A}(0)) \leq \mu$ *if and only if* $\rho(\Gamma) \leq 1$.

Proof. If $\rho(\mathbf{A}(0)) \leq \mu$, since the Perron-Frobenius eigenvalue of a nonreducible nonnegative matrix is a continuous increasing function of its entries, there exists a $\mathbf{B} \geq 0$ such that $\rho(\mathbf{A}(0) + \mathbf{B}) = \mu$. Hence, there is a $\mathbf{u} > 0$ such that

$$\mathbf{u}'(\mathbf{A}(0) + \mathbf{B} - \mu\mathbf{I}) = 0'.$$

Hence,

$$\mathbf{u}'((\sigma_j \lambda_{ij}) + \mathbf{B} - \operatorname{diag}(\mu_1, \dots, \mu_n)) = 0'.$$

Then

$$u'(\Gamma + \mathbf{B}^* - \mathbf{I}) = 0',$$

where $\mathbf{B}^* = \mathbf{B}(\operatorname{diag}(\mu_1, \dots, \mu_n))^{-1} \geq 0$. Therefore, $\rho(\Gamma + \mathbf{B}^*) = 1$, and hence, $\rho(\Gamma) \leq 1$.

Similarly, if $\rho(\Gamma) \leq 1$, there exists a $\mathbf{B}^* \geq 0$ such that $\rho(\Gamma + \mathbf{B}^*) = 1$. We may then reverse the steps and show that $\rho(\mathbf{A}(0)) \leq \mu$.

Hence, $\rho(\mathbf{A}(0)) \leq \mu$ if and only if $\rho(\Gamma) \leq 1$. □

Lemma 2. *When* $\rho(\mathbf{A}(0)) > \mu$ *or, equivalently,* $\rho(\Gamma) > 1$, *then* $f = c_0$ *where* $f = \max(0, \mathit{inf}_{\lambda \in (0,\Delta_v)} f(\lambda))$ *and* $c_0 = \mathit{inf}\{c > 0 : K(c, \lambda) = 1$ *for some* $\lambda \in (0, \Delta_v)\}$.

Proof. Consider $K(c, \lambda)$. It is a convex function of λ which is monotone decreasing in c for each λ and $K(c, 0) = \rho(\Gamma) > 1$. For $\lambda \in (0, \Delta_v)$, $\lim_{c \to \infty} K(c, \lambda) = 0$.

For fixed $c > c_0$, $K(c, \lambda) = 1$ has either two distinct roots $\alpha(c)$ and $\alpha^*(c)$ or a single root $\alpha(c)$. The latter case cannot occur if each $P_{ij}(\lambda)$ becomes infinite at its abscissa of convergence. If $c_0 > 0$, there is only one root $\alpha(c_0)$. Note that $c_0 = 0$ occurs when each contact distribution is one-sided, so that $\lim_{\lambda\to\infty} P_{ij}(\lambda) = 0$ for all i, j. Note that, for $c > c_0$, $\alpha(c) < \alpha(c_0) < \alpha^*(c)$. For fixed $c > 0$, let $\lambda = \alpha(c)$ if $c = c_0$ and $\lambda = \alpha(c)$ or $\alpha^*(c)$ if $c > c_0$. Then there exists a positive vector $\mathbf{u}$ such that

$$\mathbf{u}'(\Lambda(\lambda)(\mathbf{F}(c, \lambda))^{-1} - \mathbf{I}) = 0',$$

i.e.,

$$\mathbf{u}'(\Lambda(\lambda) - \mathbf{F}(c, \lambda)) = 0',$$

i.e.,

$$\mathbf{u}'(\mathbf{A}(\lambda) - (\mu + c\lambda)\mathbf{I}) = 0'.$$

Hence, as $\mathbf{u} > 0$, $\rho(\mathbf{A}(\lambda)) = \mu + c\lambda$ so that $c = f(\lambda)$.

Thus, for $c > c_0$, if $\lambda = \alpha(c)$ and $\lambda = \alpha^*(c)$ are roots of $K(c, \lambda) = 1$, then $f(\alpha(c)) = c = f(\alpha^*(c))$. There may of course be only one root so that $f(\alpha(c)) = c$. When $c = c_0$, with $c_0 > 0$, so that $\lambda = \alpha(c_0)$ is the only root of $K(c_0, \lambda) = 1$, then $f(\alpha(c_0)) = c_0$.

Hence, $\inf_{\lambda\in(0,\Delta_v)} f(\lambda) \leq c_0$, and so $f \leq c_0$.

Now if we consider any λ such that $c = f(\lambda) > 0$, we have $\rho(\mathbf{A}(\lambda)) = \mu + c\lambda$. By reversing the steps, we have $K(c, \lambda) = 1$ and hence $c \geq c_0$. Hence, $f = \max(0, \inf_{\lambda\in(0,\Delta_v)} f(\lambda)) \geq c_0$.

Therefore, $f = c_0$. □

5. Constraints on the applicability of the saddle-point method in the reducible case. This section acts as a prelude to Section 7, in which the saddle-point method is used to obtain the velocity of propagation of the forward front of the epidemic in a specific direction. It will be seen in Section 7 that we again need to consider $f(\lambda) = (\rho(\mathbf{A}(\lambda)) - \mu)/\lambda$, where $\mathbf{A}(\lambda)$ is defined as in Section 3 but is now reducible and $P_{ij}(\lambda)$ is the Laplace transform of the appropriate marginal contact distribution in a specific direction. We may specify the direction by α, where $\alpha'\alpha = 1$. Then $P_{ij}(\lambda) = P^*_{ij}(\lambda\alpha)$. The specific value of α used specifies the direction of the front under consideration.

Conditions are imposed on the $P_{ij}(\lambda)$ and the $y_i(\mathbf{s}, 0)$. We assume that, for any $[\theta_1, \theta_2] \subset (0, \Delta_v)$, there exists a $k_{ij}(y)$ with $|P_{ij}(\theta + iy)| \leq$

$k_{ij}(y)$ for all $\theta \in [\theta_1, \theta_2]$ where $\int_{-\infty}^{\infty} k_{ij}(y)\,dy < \infty$. Also, we assume that each $y_i(\mathbf{s}, 0)$ is a function of bounded support, with its Laplace transform analytic for all values of its entries.

The matrix $\mathbf{A}(\lambda)$ is now reducible and can be expressed in the form

$$\mathbf{A}(\lambda) = \begin{pmatrix} \mathbf{A}_{11}(\lambda) & 0 \\ \mathbf{A}_{21}(\lambda) & \mathbf{A}_{22}(\lambda) \end{pmatrix},$$

where $\mathbf{A}_{11}(\lambda)$ and $\mathbf{A}_{22}(\lambda)$ are nonreducible, of sizes $m_1 \times m_1$ and $m_2 \times m_2$, respectively, and $\mathbf{A}_{21}(\lambda)$ has at least one nonzero element. The function

$$f(\lambda) = (\rho(\mathbf{A}(\lambda)) - \mu)/\lambda = (\max(\rho(\mathbf{A}_{11}(\lambda)), \rho(\mathbf{A}_{22}(\lambda))) - \mu)/\lambda$$

may be written in the form $f(\lambda) = \max(f_1(\lambda), f_2(\lambda))$, where $f_i(\lambda) = (\rho(\mathbf{A}_{ii}(\lambda)) - \mu)/\lambda$ for $i = 1, 2$.

The results for types in group 1 follow, with some adaptation to allow for increased generality, from our paper [**6**]. However, for types in group 2 the situation is more intricate. The aim of this section is to obtain conditions under which the saddle-point method can be applied to obtain results for types in group 2. In order to accomplish this, it is necessary, when $\rho(\mathbf{A}_{ii}(\lambda)) > \mu$, for $i = 1, 2$, to classify the possible joint configurations of $f_1(\lambda)$ and $f_2(\lambda)$. Consideration also needs to be given to the case when $\rho(\mathbf{A}_{ii}(\lambda)) \leq \mu$ for some $i = 1, 2$.

In order to apply the saddle-point method to obtain the speed of propagation for types in group 2, we need $f(\lambda)$ to have a minimum at some point $\lambda = \theta_0 \in (0, \Delta_v)$, with $f(\theta_0) > 0$, and $\rho(\mathbf{A}(\lambda))$ to be analytic in some open region of $\lambda = \theta_0$. We therefore prove Lemma 3 concerning the behavior of $f_i(\lambda)$ and then use this lemma to identify different cases.

Define Δ_{ij} to be the minimum of the abscissae of convergence, in the positive half of the complex plane, of each of the entries in $\mathbf{A}_{ij}(\lambda)$. We assume that $\Delta_{21} \geq \min(\Delta_{11}, \Delta_{22})$. This is sufficient to ensure that $\mathbf{A}(\lambda)$ exists for $\mathrm{Re}\,(\lambda) = \theta_0 + \delta$ for some small positive δ. In order for the saddle-point method to be applied, in fact, the only restriction needed is that $\Delta_{21} > \theta_0$.

Lemma 3. *When $\rho(\mathbf{A}_{ii}(0)) > \mu$, the function $f_i(\lambda) = (\rho(\mathbf{A}_{ii}(\lambda)) - \mu)/\lambda$ either has a unique minimum at $\lambda = \theta_i \in (0, \Delta_{ii})$ at which point*

$f_i'(\theta_i) = 0$ or it is monotone decreasing for $\lambda \in (0, \Delta_{ii})$. In addition, if $\rho'(\mathbf{A}_{ii}(0)) \geq 0$, then there is a minimum at $\lambda = \theta_i \in (0, \Delta_{ii})$, $f_i(\theta_i) > 0$.

When $\rho(\mathbf{A}_{ii}(0)) \leq \mu$, $f_i(\lambda)$ is monotone increasing for $\lambda \in (0, \Delta_{ii})$.

Proof. We use the convexity of $\rho(\mathbf{A}_{ii}(\lambda))$ established in our paper [**6**]. We first show that any solution $\lambda \in (0, \Delta_{ii})$ to $f_i'(\lambda) = 0$ must be a minimum of $f_i(\lambda)$. Now $f_i'(\lambda) = (\rho'(\mathbf{A}_{ii}(\lambda)) - f_i(\lambda))/\lambda$ and $f_i^2(\lambda) = (\rho^2(\mathbf{A}_{ii}(\lambda)) - 2f_i'(\lambda))/\lambda$.

Hence, for any $\lambda = \theta_i \in (0, \Delta_{ii})$ such that $f_i'(\theta_i) = 0$ we have $f_i^2(\theta_i) = \rho^2(\mathbf{A}_{ii}(\theta_i))/\theta_i \geq 0$ with equality only possible if $\rho'(\mathbf{A}_{ii}(\theta_i)) = 0$. Note that $\rho'((\mathbf{A}_{ii}(\theta_i)) = 0$ implies that $f_i(\theta_i) = 0$ so that $\rho(\mathbf{A}_{ii}(\theta_i)) = \mu$. The convexity of $\rho(\mathbf{A}(\lambda))$ then gives $\rho(\mathbf{A}_{ii}(\lambda)) > \mu$ for $\lambda \neq \theta_i$. Thus, any solution $\lambda \in (0, \Delta_{ii})$ to $f_i'(\lambda) = 0$ must be at a minimum of $f_i(\lambda)$.

Therefore, there is either a unique minimum of $f_i(\lambda)$ at $\lambda = \theta_i \in (0, \Delta_{ii})$ or the function $f_i(\lambda)$ is monotone for $\lambda \in (0, \Delta_{ii})$.

When $\rho(\mathbf{A}_{ii}(0)) > \mu$, $\lim_{\lambda \downarrow 0} f_i(\lambda) = \infty$. Hence, if there is no minimum of $f_i(\lambda)$ in the range $(0, \Delta_{ii})$, $f_i(\lambda)$ must be monotone decreasing in that range.

If $\rho(\mathbf{A}_{ii}(0)) > \mu$ and $\rho'(\mathbf{A}_{ii}(0)) \geq 0$ then, from the convexity of $\rho(\mathbf{A}_{ii}(\lambda))$, $\rho(\mathbf{A}_{ii}(\lambda)) > 0$ for $\lambda \in (0, \Delta_{ii})$. Hence, $f_i(\lambda) \geq (\rho(\mathbf{A}_{ii}(0)) - \mu)/\lambda$ for all λ in that range. Then if $f_i(\lambda)$ has a minimum at $\lambda = \theta_i \in (0, \Delta_{ii})$, $f_i(\theta_i) > 0$. Note that if $\rho'(\mathbf{A}_{ii}(0)) < 0$, $f_i(\theta_i)$ can be negative.

When $\rho(\mathbf{A}_{ii}(0)) < \mu$, $\lim_{\lambda \downarrow 0} f_i(\lambda) = -\infty$. Since $f_i(\lambda)$ can only have zero derivative at a minimum, $f_i(\lambda)$ must be monotone increasing for $\lambda \in (0, \Delta_{ii})$.

Finally, consider the case when $\rho(\mathbf{A}_{ii}(0)) = \mu$. Using l'Hopital's Rule, $\lim_{\lambda \downarrow 0} f_i(\lambda) = \rho'(\mathbf{A}_{ii}(0))$ and $\lim_{\lambda \downarrow 0} f_i'(\lambda) = \rho^2(\mathbf{A}_{ii}(0))/2 \geq 0$ with equality only possible if $\rho'(\mathbf{A}_{ii}(0)) = 0$. The convexity of $\rho(\mathbf{A}_{ii}(\lambda))$ implies that if $\rho'(\mathbf{A}_{ii}(0)) = 0$, $\rho(\mathbf{A}_{ii}(\lambda)) > \mu$ for $\lambda > 0$, so that $f_i(\lambda) > 0$ for $\lambda > 0$. As $f_i(\lambda)$ can only have a minimum in the range $(0, \Delta_{ii})$, this implies that when $\rho(\mathbf{A}_{ii}(0)) = \mu$, $f_i(\lambda)$ is monotone increasing in that range. □

$\rho(\mathbf{A}_{ii}(0)) \geq \mu$ **for** $i = 1, 2$**.** There are four cases.

Case 1. For any i such that $f_i(\lambda)$ has a minimum at $\lambda = \theta_i \in (0, \Delta_{ii})$ with $f_i(\theta_i) > 0$, we have $f_i(\theta_i) < f_{1-i}(\theta_i)$.

If $f_i(\lambda)$ does not have a minimum for each $i = 1, 2$, then by Lemma 3 each $f_i(\lambda)$ is monotone decreasing. Hence, $f(\lambda)$ does not have a minimum in $(0, \Delta_v)$.

If $f_i(\lambda)$ has a minimum at $\lambda = \theta_i$, but $f_{1-i}(\lambda)$ does not have a minimum, then either $f_{1-i}(\theta_i) > f_i(\theta_i)$ or $f_{1-i}(\theta_i) \leq f_i(\theta_i) \leq 0$. In the former case $f(\lambda)$ either does not have a minimum, or has a minimum at $\lambda = \theta_0 > \theta_i$ at which point $\rho(\mathbf{A}(\lambda))$ is not differentiable. In the latter case $f(\lambda)$ has a minimum at $\lambda = \theta_0 = \theta_i$ at which $f(\theta_0) \leq 0$.

Finally, if $f_i(\lambda)$ has a minimum at $\lambda = \theta_i$ for $i = 1, 2$, we have (for each i) either $f_{1-i}(\theta_i) > f_i(\theta_i)$ or $f_{1-i}(\theta_i) \leq f_i(\theta_i) \leq 0$. If the latter holds for at least one i, then $\min f(\lambda) \leq 0$. If the former holds for both i, then $f(\lambda)$ has a minimum at $\lambda = \theta_0$, where θ_0 lies strictly between θ_1 and θ_2, and $f(\lambda)$ is not differentiable at $\lambda = \theta_0$.

Hence, either $f(\lambda)$ does not have a minimum at which it is positive, or $f(\lambda)$ has a minimum at $\lambda = \theta_0$ at which $f(\theta_0) > 0$ but $\rho(\mathbf{A}(\lambda))$ is not analytic in an open region of $\lambda = \theta_0$.

Case 2. $f_1(\lambda)$ has a minimum at $\lambda = \theta_1$ with $f_1(\theta_1) > 0$ and $f_1(\theta_1) > f_2(\theta_1)$.

In this case $f(\lambda)$ has a minimum at $\lambda = \theta_0 = \theta_1$, and $\rho(\mathbf{A}(\theta_0)) = \rho(\mathbf{A}_{11}(\theta_0)) > \rho(\mathbf{A}_{22}(\theta_0))$. Since $\rho(\mathbf{A}_{11}(\lambda))$ is analytic, there is an open region of $\lambda = \theta_0$ in the complex plane for which $\rho(\mathbf{A}(\lambda))$ is analytic. Also $f(\theta_0) > 0$.

Case 3. $f_2(\lambda)$ has a minimum at $\lambda = \theta_2$ with $f_2(\theta_2) > 0$ and $f_2(\theta_2) > f_1(\theta_2)$.

In this case $f(\lambda)$ has a minimum at $\lambda = \theta_0 = \theta_2$, and $\rho(\mathbf{A}(\theta_0)) = \rho(\mathbf{A}_{22}(\theta_0)) > \rho(\mathbf{A}_{11}(\theta_0))$. There is therefore an open region of $\lambda = \theta_0$ in the complex plane for which $\rho(\mathbf{A}(\lambda))$ is analytic. Also, $f(\theta_0) > 0$.

Case 4. $f_1(\lambda) = f_2(\lambda) > 0$ for at least one of $\lambda = \theta_1$ and $\lambda = \theta_2$.

Note that equality is only possible at both $\lambda = \theta_1$ and $\lambda = \theta_2$ if $\theta_1 = \theta_2$. If equality occurs at $\lambda = \theta_1$, but either $f_2(\lambda)$ does not have a minimum or equality does not occur at $\lambda = \theta_2$, then $\theta_0 = \theta_1$. The function $f(\lambda)$, and hence $\rho(\mathbf{A}(\lambda))$, is not differentiable at $\lambda = \theta_0$ for λ real. Hence, $\rho(\mathbf{A}(\lambda))$ is not analytic in any open region of $\lambda = \theta_0$ in the complex plane. A similar result holds if equality occurs at $\lambda = \theta_2$ but either $f_1(\lambda)$ does not have a minimum or equality does not occur at $\lambda = \theta_1$.

Now consider $f_1(\lambda) = f_2(\lambda)$ for $\lambda = \theta_1 = \theta_2$. Then $\theta_0 = \theta_1 = \theta_2$. Note that $f_1'(\theta_0) = f_2'(\theta_0) = 0$. Hence, $\rho(\mathbf{A}(\theta_0)) = \rho(\mathbf{A}_{11}(\theta_0)) = \rho(\mathbf{A}_{22}(\theta_0))$ and the derivatives of $\rho(\mathbf{A}_{ii}(\lambda))$, for $i = 1, 2$, with respect to λ are equal at $\lambda = \theta_0$. Consider the Taylor expansion of each $\rho(\mathbf{A}_{ii}(\lambda))$ about $\lambda = \theta_0$, and let $\rho^s(\mathbf{A}_{ii}(\lambda))$ denote the s^{th} derivative of $\rho(\mathbf{A}_{ii}(\lambda))$ with respect to λ. Suppose there exists a positive integer r such that $\rho^s(\mathbf{A}_{11}(\theta_0)) = \rho^s(\mathbf{A}_{22}(\theta_0))$ for $s < r$, and $\rho^r(\mathbf{A}_{11}(\theta_0)) \neq \rho^r(\mathbf{A}_{22}(\theta_0))$. If $\rho^r(\mathbf{A}_{ii}(\theta_0)) > \rho^r(\mathbf{A}_{1-i,1-i}(\theta_0))$, then $\mathrm{Re}\,(\rho(\mathbf{A}_{ii}(\theta_0 + x))) > \mathrm{Re}\,(\rho(\mathbf{A}_{1-i,1-i}(\theta_0 + x)))$ for x a small positive real, whereas $\mathrm{Re}\,(\rho(\mathbf{A}_{ii}(\theta_0 + z))) < \mathrm{Re}\,(\rho(\mathbf{A}_{1-i,1-i}(\theta_0 + z)))$ for $z = \mathrm{R}e^{\,\pi/(2r)}$, where R is a small positive real. Hence, $\rho(\mathbf{A}(\lambda))$ cannot be analytic at $\lambda = \theta_0$. This implies that $\rho(\mathbf{A}(\lambda))$ is not analytic in an open region of $\lambda = \theta_0$ unless $\rho(\mathbf{A}_{11}(\lambda)) \equiv \rho(\mathbf{A}_{22}(\lambda))$.

Hence, when $\rho(\mathbf{A}_{ii}(0)) > \mu$ for $i = 1, 2$, we can only apply the saddle point method in cases 2 and 3 and in the special situation in case 4 when $\rho(\mathbf{A}_{11}(\lambda)) \equiv \rho(\mathbf{A}_{22}(\lambda))$. This special situation in case 4 will clearly occur if $m_1 = m_2$ and $\mathbf{A}_{11}(\lambda) \equiv \mathbf{A}_{22}(\lambda)$. As this is the only situation of interest where this is likely to arise, we will restrict ourselves in case 4 to $\mathbf{A}_{11}(\lambda) \equiv \mathbf{A}_{22}(\lambda)$.

$\rho(\mathbf{A}_{ii}(0)) \leq \mu$ **for at least one of** $i = 1, 2$**.** We can partition Γ in a similar way to $\mathbf{A}(\lambda)$, so that

$$\Gamma = \begin{pmatrix} \Gamma_{11} & 0 \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}.$$

There are five cases.

Case 5. $\rho(\mathbf{A}_{ii}(0)) \leq \mu$ for $i = 1, 2$, (and hence $\rho(\Gamma) \leq 1$).

There is no $\lambda \in (0, \Delta_v)$ such that $f'(\lambda) = 0$.

Case 6. $\rho(\mathbf{A}_{11}(0)) > \mu$ and $\rho(\mathbf{A}_{22}(0)) \leq \mu$, (and hence $\rho(\Gamma_{11}) > 1$

and $\rho(\Gamma_{22}) \leq 1$). Also, $f_1(\lambda)$ has a minimum at $\lambda = \theta_1$ with $f_1(\theta_1) > 0$ and $f_1(\theta_1) > f_2(\theta_1)$.

In this case $\theta_0 = \theta_1$ and $\rho(\mathbf{A}(\theta_0)) = \rho(\mathbf{A}_{11}(\theta_0)) > \rho(\mathbf{A}_{22}(\theta_0))$, so that $\rho(\mathbf{A}(\lambda))$ is analytic in an open region of $\lambda = \theta_0$. Also, $f(\theta_0) = f_1(\theta_0) > 0$.

Case 7. $\rho(\mathbf{A}_{11}(0)) > \mu$ and $\rho(\mathbf{A}_{22}(0)) \leq \mu$, (and hence $\rho(\Gamma_{11}) > 1$ and $\rho(\Gamma_{22}) \leq 1$). Also either $f_1(\lambda)$ does not have a minimum in $(0, \Delta_{11})$, has a minimum at $\lambda = \theta_1$ with $f_1(\theta_1) \leq 0$, or has a minimum at $\lambda = \theta_1$ with $f_1(\theta_1) \leq f_2(\theta_1)$.

Either $f(\lambda)$ has no minimum in $(0, \Delta_v)$, or (if $f_2(\theta_1) < f_1(\theta_1) \leq 0$) it has a minimum at $\lambda = \theta_0 = \theta_1$ with $f(\theta_0) = f_1(\theta_0) \leq 0$, or (if $f_2(\theta_1) \geq f_1(\theta_1)$) it has a minimum at $\lambda = \theta_0 < \theta_1$ at which $f(\lambda)$ is not differentiable. Hence either $f(\lambda)$ does not have a minimum at which $f(\lambda)$ is positive, or it has such a minimum at $\lambda = \theta_0$ but $\rho(\mathbf{A}(\lambda))$ is not analytic in an open region of $\lambda = \theta_0$.

Case 8. $\rho(\mathbf{A}_{22}(0)) > \mu$ and $\rho(\mathbf{A}_{11}(0)) \leq \mu$, (and hence $\rho(\Gamma_{22}) > 1$ and $\rho(\Gamma_{11}) \leq 1$). Also $f_2(\lambda)$ has a minimum at $\lambda = \theta_2$ with $f_2(\theta_2) > 0$ and $f_2(\theta_2) > f_1(\theta_2)$.

In this case $\theta_0 = \theta_2$ and $\rho(\mathbf{A}(\theta_0)) = \rho(\mathbf{A}_{22}(\theta_0)) > \rho(\mathbf{A}_{11}(\theta_0))$, so that $\rho(\mathbf{A}(\lambda))$ is analytic in an open region of $\lambda = \theta_0$. Also $f(\theta_0) = f_2(\theta_0) > 0$.

Case 9. $\rho(\mathbf{A}_{22}(0)) > \mu$ and $\rho(\mathbf{A}_{11}(0)) \leq \mu$, (and hence $\rho(\Gamma_{22}) > 1$ and $\rho(\Gamma_{11}) \leq 1$). Also either $f_2(\lambda)$ does not have a minimum in $(0, \Delta_{22})$, has a minimum at $\lambda = \theta_2$ with $f_2(\theta_2) \leq 0$, or has a minimum at $\lambda = \theta_2$ with $f_2(\theta_2) \leq f_1(\theta_2)$.

As in Case 7, either $f(\lambda)$ does not have a minimum at which $f(\lambda)$ is positive, or it has such a minimum at $\lambda = \theta_0 < \theta_2$, but $\rho(\mathbf{A}(\lambda))$ is not analytic in an open region of $\lambda = \theta_0$.

Hence, when $\rho(\mathbf{A}_{ii}(0)) \leq \mu$ for at least one of $i = 1, 2$, the saddle-point method can only be applied in Cases 6 and 8.

6. Linking wave solution existence and saddle-point constraints. It is of interest to see how the cases described in Section 5, in which the saddle-point can be applied, are linked with the existence of wave solutions to the reducible $S \to I \to R$ epidemic obtained in our paper [**8**]. For an epidemic in R^N, the waves considered are those travelling in a fixed direction, the proportions of susceptible, infected and removed individuals being constant on the orthogonal $(N-1)$-dimensional hyperplane. The contact distributions in that paper are therefore the marginal distributions of each $p_{ij}(\mathbf{r})$ in that specific direction; the Laplace transforms being the $P_{ij}(\lambda)$ defined in Section 5 of this paper. Note that in [**8**], for simplicity of exposition, we assumed that each $P_{ij}(\lambda)$ became infinite at its abscissa of convergence. This restriction is therefore also imposed for this section only.

Let $\mathbf{V}_c(\lambda)$, as defined in Section 4, be partitioned by groups so that

$$\mathbf{V}_{\mathbf{c}}(\lambda) = \begin{pmatrix} \mathbf{V}_{11}(c,\lambda) & 0 \\ \mathbf{V}_{21}(c,\lambda) & \mathbf{V}_{22}(c,\lambda) \end{pmatrix}.$$

If $\rho(\Gamma_{ii}) > 1$, $c_i = \inf\{c > 0 : \rho(\mathbf{V}_{ii}(c,\lambda)) = 1 \text{ for some } \lambda \in (0, \Delta_{ii})\}$. Then for $c \geq c_i$ we define $\lambda = \alpha_i(c)$ to be the smallest positive root of $\rho(\mathbf{V}_{ii}(c,\lambda)) = 1$. This is the only root if $c = c_i > 0$. If $c > c_i$ there is a second positive root which we define to be $\alpha_i^*(c)$. If $c_i = 0$ there may be only one root, or there may be a second positive root $\alpha_i^*(0)$.

If $\rho(\Gamma_{ii}) \leq 1$, $c_i = 0$, and for each $c > 0$ there is a single root of $\rho(\mathbf{V}_{ii}(c,\lambda) = 1$. We define this root as $\alpha_i^*(c)$.

$\rho(\Gamma_{11}) > 0$ and $\rho(\Gamma_{22}) > 1$. Case 2 corresponds to $c_1 > c_2$ with $\alpha_2(c_1) < \alpha_1(c_1) < \alpha_2^*(c_1)$. In our paper [**8**] we showed (Theorem 11 part (d)) that in this case wave solutions affecting both groups exist. There is an infinite number of such solutions. However, there is a unique wave solution of a particular type, namely with the same behavior in the forward tail for both groups determined by $\alpha_1(c_1)$. For precise details, see our paper [**8**]. This wave was interpreted as being generated purely by a wave in group 1 which induces a wave in group 2.

Case 3 corresponds to $c_2 > c_1$ with $\alpha_1(c_2) < \alpha_2(c_2)$. In our paper [**8**] we showed (Theorem 11 part (c)) that no wave solution is possible for both groups. A wave in the second group only at speed c_2 exists, which is unique modulo translation.

Case 4 corresponds to a critical case in our paper [**8**] with $c_1 = c_2$.

The existence or nonexistence of waves was not established (see the comment after Theorem 11 of that paper).

$\rho(\Gamma_{11}) > 1$ and $\rho(\Gamma_{22}) \leq 1$. Case 6 corresponds to $f_1(\theta_1) > f_2(\theta_1)$ with $c_1 > c_2 = 0$ and $\alpha_1(c_1) < \alpha_2^*(c_1)$. In this case there is a unique wave solution for both groups at speed c_1 with its tail behavior determined by $\alpha_1(c_1)$.

$\rho(\Gamma_{11}) \leq 1$ and $\rho(\Gamma_{22}) > 1$. Case 8 corresponds to $f_2(\theta_2) > f_1(\theta_2)$ with $c_2 > c_1 = 0$. Only a wave solution in the second group is possible at speed c_2. There is a unique wave solution in group 2 at speed c_2 with its tail behavior determined by $\alpha_2(c_2)$.

7. Choosing the saddle in the reducible case. If we consider the speed of spread of the forward region in a specific direction (determined by α such that $\alpha'\alpha = 1$) for the j^{th} type in equation (3), we may proceed in an analagous manner to our paper [**6**]; this being outlined in Section 3 of this paper. For fixed j and each t we define $\mathbf{s}$ so that, for fixed small positive η,

$$\int_{\mathbf{x}:\alpha'\mathbf{x}\geq s} y_j(\mathbf{x},t)\,d\mathbf{x} = \eta.$$

This specifies that the total proportion of infectives beyond the hyperplane $\alpha'\mathbf{x} = s$ is η.

Define $P_{ij}^*(\lambda)$ as in Section 2. Let $\{\mathbf{A}^*(\lambda)\}_{ij} = \sigma_j\lambda_{ij}P_{ij}^*(\lambda) + \delta_{ij}(\mu - \mu_i)$ and $\{\mathbf{A}(\lambda)\}_{ij} = \sigma_j\lambda_{ij}P_{ij}(\lambda) + \delta_{ij}(\mu - \mu_i)$, where μ and δ_{ij} are as defined in Section 3 and $P_{ij}(\lambda) = P_{ij}^*(\lambda\alpha)$. Partition $\mathbf{A}(\lambda)$ as in Section 5. Also take an equivalent partition of $\mathbf{A}^*(\lambda)$. Take Δ_v to be the minimum of the abscissae of convergence of the $P_{ij}(\lambda)$ in the right hand half of the complex plane.

We assume that the limit of s/t as t tends to infinity, f, exists and is positive. This limit is then identified for different values of j.

Since $\rho(\mathbf{A}(\lambda)) = \max(\rho(\mathbf{A}_{11}(\lambda)), \rho(\mathbf{A}_{22}(\lambda)))$ we have two possible saddle-points, namely that of each of $g_i(\lambda) = \operatorname{Re}(\rho(\mathbf{A}_{ii}(\lambda))t - \lambda s)$, for $i = 1, 2$, on the real axis. The saddle-point chosen will depend on whether the type is in group 1 or group 2. For group 2 types it will depend on the initial conditions, specifically whether or not there is some initial infection in group 1. When there is initial infection in group 1 the choice of the appropriate saddle-point is more complicated, and Cases 2, 3, 4, 6, and 8, listed in Section 5, will need to be considered separately.

Define $L_i^*(\lambda,t) = \int_{R^N} e^{\lambda'\mathbf{s}} y_i(\mathbf{s},t)\,d\mathbf{s}$ and $\{\mathbf{L}^*(\lambda,t)\}_i = L_i^*(\lambda,t)$. Also partition $(\mathbf{L}^*(\lambda,t))' = ((\mathbf{L}_1^*(\lambda,t))', (\mathbf{L}_2^*(\lambda,t))')$, where $\mathbf{L}_1^*(\lambda,t)$ and $\mathbf{L}_2^*(\lambda,t)$ are vectors of length m_1 and m_2, respectively.

Transforming equation (3), we obtain

$$\begin{aligned} \frac{\partial \mathbf{L}_1^*(\lambda,t)}{\partial t} &= (\mathbf{A}_{11}^*(\lambda) - \mu\mathbf{I})\mathbf{L}_1^*(\lambda,t), \\ \frac{\partial \mathbf{L}_2^*(\lambda,t)}{\partial t} &= \mathbf{A}_{21}^*(\lambda)\mathbf{L}_1^*(\lambda,t) + (\mathbf{A}_{22}^*(\lambda) - \mu\mathbf{I})\mathbf{L}_2^*(\lambda,t). \end{aligned} \tag{4}$$

Let $\mathbf{u}^*(\lambda) = \mathbf{L}^*(\lambda,0)$. Then the solution to equation (4) is given by

$$\mathbf{L}^*(\lambda,t) = e^{(\mathbf{A}^*(\lambda)-\mu\mathbf{I})t}\mathbf{u}^*(\lambda).$$

Let $\mathbf{L}(\lambda,t) = \mathbf{L}^*(\lambda\alpha,t)$ and $\mathbf{u}(\lambda) = \mathbf{u}^*(\lambda\alpha) \equiv \mathbf{L}^*(\lambda\alpha,0) = \mathbf{L}(\lambda,0)$. We may partition $(\mathbf{u}(\lambda))' = ((\mathbf{u}_1(\lambda))', (\mathbf{u}_2(\lambda))')$ where $\mathbf{u}_i(\lambda) = \mathbf{L}_i^*(\lambda\alpha,0)$. Then we obtain

$$\mathbf{L}(\lambda,t) = e^{(\mathbf{A}(\lambda)-\mu\mathbf{I})t}\mathbf{u}(\lambda).$$

Note that $\{\mathbf{u}(\lambda)\}_i$ is the Laplace transform of the function $y_i(\mathbf{x},0)$ which has been integrated over the hyperplane $\alpha'\mathbf{x} = s$.

We then invert $\{\mathbf{L}(\lambda,t)\}_j$ and integrate the inversion from $\mathbf{s}$ to ∞ to obtain an equation for η. This is done in a similar manner to our paper [**6**], since $\eta = \int_{\mathbf{x}:\alpha'\mathbf{x}\geq s} y_j(\mathbf{x},t)\,d\mathbf{x}$.

Hence, for t sufficiently large,

$$\eta = \frac{1}{2\pi i}\int_{\theta(t)-i\infty}^{\theta(t)+i\infty} \lambda^{-1}e^{-\lambda s}(\{\mathbf{L}(\lambda,t)\}_j - e^{-\operatorname{diag}(\mu_1,\dots,\mu_n)t}\{\mathbf{u}(\lambda)\}_j)\,d\lambda.$$

No initial infection in group 1. First consider the case where the infection starts amongst individuals in the second group of types only, so that $\mathbf{u}_1(\lambda) \equiv 0$. In this case $\mathbf{L}_1(\lambda,t) \equiv 0$ and

$$\mathbf{L}_2(\lambda,t) = e^{(\mathbf{A}_{22}(\lambda)-\mu\mathbf{I})t}\mathbf{u}_2(\lambda).$$

Clearly, there is no spread of infection amongst the first group of types so that the speed of propagation of the forward tail of the epidemic for types in group 1 is zero. For types in group 2 we may proceed in an identical fashion to our paper [**6**] using the saddle-point of $g_2(\lambda)$. Note that we can only prove Theorem 1 of that paper if there exists a $\lambda \in (0, \Delta_{22})$ with $\rho(\mathbf{A}_{22}(\lambda)) > \mu_2$ and $\rho'(\mathbf{A}_{22}(\lambda)) = \theta(t)/t$ for t sufficiently large.

When $\rho(\mathbf{A}_{22}(0)) > \mu$, i.e., $\rho(\Gamma_{22}) > 1$, this establishes the speed of propagation of the forward tail of the epidemic for each type in group 2 as f_2, provided $f_2 = \inf_{\lambda \in (0,\Delta_{22})}(\rho(\mathbf{A}_{22}(\lambda)) - \mu)/\lambda$ is positive with the inf achieved for $\lambda \in (0, \Delta_{22})$. Note that, from the results in Section 4, f_2 is the minimum speed c_2 for which wave solutions exist to the $S \to I \to R$ epidemic when the epidemic is amongst types in group 2 only.

When $\rho(\mathbf{A}_{22}(0)) \le \mu$, i.e., $\rho(\Gamma_{22}) \le 1$, there cannot exist a positive speed of propagation f_2 with $f_2 > \inf \rho'(\mathbf{A}_{22}(\lambda))$, where the inf is over $\lambda \in (0, \Delta_{22})$ such that $\rho(\mathbf{A}_{22}(\lambda)) \ge \mu$. This tends to suggest the speed of propagation is zero.

The exact saddle-point results are summarized in Theorem 1.

Theorem 1. *Consider a fixed direction determined by* α *and a specific type* j *in group 2. For some small* $\eta > 0$*, let* $\mathbf{s}$ *be defined by* $\eta = \int_{\mathbf{x}:\alpha'\mathbf{x} \ge s} y_j(\mathbf{x}, t)\, d\mathbf{x}$*. Also let* $\rho(\Gamma_{22}) > 1$ *and* $f_2(\lambda)$ *have a minimum for some* $\lambda = \theta_0 \in (0, \Delta_{22})$ *with* $f_2(\theta_0) > 0$*. Suppose the following conditions hold*:

(i) $f_2 = \lim_{t\to\infty} s/t$ *exists and is positive with* $\rho'(\mathbf{A}_{22}(\lambda)) = f_2$ *for some* $\lambda \in (0, \Delta_{22})$.

(ii) $\mathbf{A}_{22}(\theta_0)$ *has distinct eigenvalues.*

(iii) $P^*_{ij}(\theta)$ *exists in some open region of* $\theta = 0$ *for all* i, j *from* $m_1 + 1$ *to* n.

(iv) *For any interval* $[\theta_1, \theta_2] \subset (0, \Delta_{22})$ *and for all* $\theta \in [\theta_1, \theta_2]$*, there exists a* $k_{ij}(y)$ *such that* $|P_{ij}(\theta + iy)| \le k_{ij}(y)$ *and* $\int_{-\infty}^{\infty} k_{ij}(y)\, dy < \infty$ *for all* i, j *from* $m_1 + 1$ *to* n.

(v) *Each* $y_i(\mathbf{s}, 0)$ *is a function of bounded support for* $i = m_1 + 1, \dots, n$ *and* $y_i(\mathbf{s}, 0) \equiv 0$ *for* $i \le m_1$.

The speed of spread of the forward front in equation (3) *is* $c_2 = \min_{\theta\in(0,\Delta_{22})} f_2(\theta)$. *This is the minimum velocity for which wave solutions exist in the model for the* $S \to I \to R$ *deterministic epidemic for types in group* 2 *only.*

Initial infection in group 1: *speed of spread for types in group* 1. In this case $\mathbf{u}_1(\lambda) \neq \mathbf{0}$. The first m_1 of equation (4) gives

$$\mathbf{L}_1(\lambda, t) = e^{(\mathbf{A}_{11}(\lambda)-\mu\mathbf{I})t}\mathbf{u}_1(\lambda).$$

Hence the speed of propagation of the forward front for types in group 1 can be obtained exactly as in our paper [**6**] using the saddle-point of $g_1(\lambda)$. Again we have to assume that there exists a $\lambda \in (0, \Delta_{11})$ such that $\rho(\mathbf{A}_{11}(\lambda)) > \mu$ and $\rho'(\mathbf{A}_{11}(\lambda)) = \theta(t)/t$ for t sufficiently large. When $\rho(\mathbf{A}_{11}(0)) > \mu$, i.e., $\rho(\Gamma_{11}) > 1$, this establishes the speed as f_1, provided $f_1 = \inf_{\lambda\in(0,\Delta_{11})}(\rho(\mathbf{A}_{11}(\lambda)) - \mu)/\lambda$ is positive with the inf achieved for $\lambda \in (0, \Delta_{11})$. Note that, from the results in Section 4, f_1 is the minimum speed c_1 for which wave solutions exist for an $S \to I \to R$ epidemic amongst types in group 1 only, the infection of types in group 2 being ignored. If $\rho(\mathbf{A}_{11}(0)) \leq \mu$, i.e., $\rho(\Gamma_{11}) \leq 1$, there again are restrictions on the possible positive speed of propagation. This tends to suggest, as in the previous case considered, that the speed of propagation is zero.

The exact saddle-point results may be summarized in Theorem 2.

Theorem 2. *Consider a fixed direction determined by* α *and a specific type* j *in group* 1. *For some small* $\eta > 0$, *let* $\mathbf{s}$ *be defined by* $\eta = \int_{\mathbf{x}:\alpha'\mathbf{x}\geq s} y_j(\mathbf{x}, t)\,d\mathbf{x}$. *Also let* $\rho(\Gamma_{11}) > 1$ *and* $f_1(\lambda)$ *have a minimum for some* $\lambda = \theta_0 \in (0, \Delta_{11})$ *with* $f_1(\theta_0) > 0$. *Suppose the following conditions hold*:

(i) $f_1 = \lim_{t\to\infty} s/t$ *exists and is positive with* $\rho'(\mathbf{A}_{11}(\lambda)) = f_1$ *for some* $\lambda \in (0, \Delta_{11})$.

(ii) $\mathbf{A}_{11}(\theta_0)$ *has distinct eigenvalues.*

(iii) $P^*_{ij}(\theta)$ *exists in some open region of* $\theta = 0$ *for all* i, j *from* 1 *to* m_1.

(iv) *For any interval* $[\theta_1, \theta_2] \subset (0, \Delta_{11})$ *and for all* $\theta \in [\theta_1, \theta_2]$ *there exists a* $k_{ij}(y)$ *such that* $|P_{ij}(\theta + iy)| \leq k_{ij}(y)$ *and* $\int_{-\infty}^{\infty} k_{ij}(y)\,dy < \infty$ *for all* i, j *from* 1 *to* m_1.

(v) *Each* $y_i(\mathbf{s}, 0)$ *is a function of bounded support for* $i = 1, \dots, m_1$ *and* $y_i(\mathbf{s}, 0) \equiv 0$ *for* $i > m_1$.

The speed of propagation of the forward front in equation (3) *is* $c_1 = \min_{\theta \in (0, \Delta_{11})} f_1(\theta)$. *This is the minimum velocity for which wave solutions exist in the model for the* $S \to I \to R$ *deterministic epidemic for types in group* 1 *only.*

Initial infection in group 1*: speed of spread for types in group* 2. Again $\mathbf{u}_1(\lambda) \neq 0$. We consider the speed of spread of the forward front for type j, where $m_1 + 1 \leq j \leq n$. The saddle-point at $\lambda = \theta(t)$ of $g(\lambda) = \mathrm{Re}\,(\rho(\mathbf{A}(\lambda))t - \lambda s)$ is used. We need to assume that there exists a $\lambda \in (0, \Delta_v)$ such that $\rho'(\mathbf{A}(\lambda)) = f$. Notice that in Section 5, when $\rho(\mathbf{A}_{ii}(0)) > \mu$ for $i = 1, 2$, we have listed four cases and observed that the method can only be applied in Cases 2 and 3 and in a special situation in Case 4. The situation we consider in Case 4 is when $\mathbf{A}_{11}(\lambda) \equiv \mathbf{A}_{22}(\lambda)$. In addition, when $\rho(\mathbf{A}_{ii}(0)) \leq \mu$ for at least one i, we have listed five cases and observed that the saddle-point method can only be applied in Cases 6 and 8.

Case 2. In this case, the saddle-point of $g(\lambda)$ that we use is that of $g_1(\lambda)$. The proof of Theorem 1 proceeds as in our paper [**6**] and shows that for any type j from $m_1 + 1$ to n, for δ sufficiently small and t sufficiently large,

$$\eta e^{\mu t - \gamma_1(t)} \simeq \frac{1}{2\pi i} e^{-\gamma_1(t)} \int_{\theta(t) - i\delta}^{\theta(t) + i\delta} \frac{1}{\theta(t)} \{\mathbf{E}(\lambda)\mathbf{u}(\lambda)\}_j e^{\rho(\mathbf{A}_{11}(\lambda))t - \lambda s}\, d\lambda,$$

where $\gamma_i(t) = \rho(\mathbf{A}_{ii}(\theta(t))) - s\theta(t)$, and in an open region of $\lambda = \theta_0$, $\mathbf{E}(\lambda)$ is the idempotent of $\mathbf{A}(\lambda)$ corresponding to $\rho(\mathbf{A}(\lambda)) \equiv \rho(\mathbf{A}_{11}(\lambda))$. Note that

$$\mathbf{E}(\lambda) = \begin{pmatrix} \mathbf{E}_1(\lambda) & 0 \\ (\rho(\mathbf{A}_{11}(\lambda))\mathbf{I} - \mathbf{A}_{22}(\lambda))^{-1}\mathbf{A}_{21}(\lambda)\mathbf{E}_1(\lambda) & 0 \end{pmatrix},$$

where $\mathbf{E}_1(\lambda) > 0$ is the idempotent of $\mathbf{A}_{11}(\lambda)$ corresponding to $\rho(\mathbf{A}_{11}(\lambda))$.

This idempotent is easily obtained from the right and left eigenvectors of $\mathbf{A}(\lambda)$ corresponding to $\rho(\mathbf{A}(\lambda)) \equiv \rho(\mathbf{A}_{11}(\lambda))$ and demonstrates that the dominant term in the integral involves only $\mathbf{u}_1(\lambda)$ and not $\mathbf{u}_2(\lambda)$.

We then proceed in an identical fashion to our paper [**6**] to establish that the speed of propagation of the forward front of infection for each type in group 2 is f_1. Note that, from the results of section 4, f_i is the minimum speed c_i for which wave solutions exist for the $S \to I \to R$ epidemic if we consider an epidemic amongst group i types only, the infection amongst group $(1-i)$ types being ignored. In this case $f_1 > f_2$ so that, using the results in Section 4, $f_1 = c_1 = \max(c_1, c_2)$.

Case 3. The saddle-point of $g(\lambda)$ that we use is that of $g_2(\lambda)$. For any type j, from $m_1 + 1$ to n, for δ sufficiently small and t sufficiently large,

$$\eta e^{\mu t - \gamma_2(t)} \simeq \frac{1}{2\pi i} e^{-\gamma_2(t)} \int_{\theta(t)-i\delta}^{\theta(t)+i\delta} \frac{1}{\lambda} \{\mathbf{E}(\lambda)\mathbf{u}(\lambda)\}_j e^{\rho(\mathbf{A}_{22}(\lambda))t - \lambda s}\, d\lambda,$$

where $\mathbf{E}(\lambda) > 0$ is the idempotent of $\mathbf{A}(\lambda)$ corresponding to $\rho(\mathbf{A}(\lambda)) \equiv \rho(\mathbf{A}_{22}(\lambda))$. Note that

$$\mathbf{E}(\lambda) = \begin{pmatrix} 0 & 0 \\ \mathbf{E}_2(\lambda)\mathbf{A}_{22}(\lambda)(\rho(\mathbf{A}_{22}(\lambda))I - \mathbf{A}_{11}(\lambda))^{-1} & \mathbf{E}_2(\lambda) \end{pmatrix},$$

where $\mathbf{E}_2(\lambda) > 0$ is the idempotent of $\mathbf{A}_{22}(\lambda)$ corresponding to $\rho(\mathbf{A}_{22}(\lambda))$.

The idempotent is obtained as in case 2 and demonstrates that the dominant term in the integral involves only $\mathbf{u}_2(\lambda)$.

Proceeding in an identical fashion to our paper [**6**] we obtain the speed of propagation for each type in group 2 as f_2. Note that in this case $f_1 > f_2$. Using the results in Section 4 we have $f_2 = c_2 = \max(c_1, c_2)$. Hence, the speed of propagation for types in group 2 is $\max(c_1, c_2)$.

Case 4. We consider case 4 when $m_1 = m_2$ and $\mathbf{A}_{11}(\lambda) \equiv \mathbf{A}_{22}(\lambda)$ ($\equiv \mathbf{B}(\lambda)$ say). Then $e^{\mathbf{A}(\lambda)t}$ has dominant term $(\mathbf{E}(\lambda) + \mathbf{N}(\lambda)t)e^{\rho(\mathbf{B}(\lambda))t}$, where (in the case where the eigenvalues of $\mathbf{B}(\lambda)$ are all distinct), the nilpotent $\mathbf{N}(\lambda)$ and the idempotent $\mathbf{E}(\lambda)$ are given by

$$\mathbf{N}(\lambda) = \begin{pmatrix} 0 & 0 \\ \mathbf{E}_1(\lambda)\mathbf{A}_{21}(\lambda)\mathbf{E}_1(\lambda) & 0 \end{pmatrix},$$

and

$$\mathbf{E}(\lambda) = \begin{pmatrix} \mathbf{E}_1(\lambda) & 0 \\ \sum_{j\neq 1} \frac{1}{\rho(\mathbf{B}(\lambda))-\lambda_j}(\mathbf{E}_j(\lambda)\mathbf{A}_{21}(\lambda)\mathbf{E}_1(\lambda)+\mathbf{E}_1(\lambda)\mathbf{A}_{21}(\lambda)\mathbf{E}_j(\lambda)) & \mathbf{E}_1(\lambda) \end{pmatrix},$$

where $\rho(\mathbf{B}(\lambda)), \lambda_2, \dots, \lambda_{m_1}$ are the eigenvalues of $\mathbf{B}(\lambda)$ and $\mathbf{E}_1(\lambda), \dots, \mathbf{E}_{m_1}(\lambda)$ are the corresponding idempotents.

The form of idempotent and nilpotent can be obtained by considering the spectral expansion of $(\mathbf{A}(\lambda) - \alpha\mathbf{I})^{-1}$ as α tends to $\rho(\mathbf{A}(\lambda))$.

Hence for any type j, from $m_1 + 1$ to n, for δ sufficiently small and t sufficiently large,

$$\begin{aligned} &\eta e^{\mu t-\gamma(t)} \\ &\quad\simeq \frac{1}{2\pi i}e^{-\gamma(t)}\int_{\theta(t)-i\delta}^{\theta(t)+i\delta}\frac{1}{\lambda}\{(\mathbf{E}(\lambda)+t\mathbf{N}(\lambda))\mathbf{u}(\lambda)\}_j e^{\rho(\mathbf{B}(\lambda))t-\lambda s}\,d\lambda, \\ &\quad\simeq \frac{1}{2\pi i}e^{-\gamma(t)}\int_{\theta(t)-i\delta}^{\theta(t)+i\delta}\frac{t}{\lambda}\{\mathbf{E}_1(\lambda)\mathbf{A}_{21}(\lambda)\mathbf{E}_1(\lambda)\mathbf{u}(\lambda)\}_{j-m_1} e^{\rho(\mathbf{B}(\lambda))t-\lambda s}\,d\lambda, \end{aligned}$$

where $\gamma(t) = \rho(\mathbf{B}(\theta(t)))t - s\theta(t)$. Thus the dominant term in the integral involves only $\mathbf{u}_1(\lambda)$.

The proof then follows as in our paper [**6**] to identify the speed of propagation of the forward front of infection for any type in group 2 as $f_1 = f_2$. Using the results of Section 4, we have $c_1 = f_1 = f_2 = c_2$. Hence the speed of propagation for types in group 2 is again $\max(c_1, c_2)$.

Case 6. As in Case 2, the saddle-point of $g(\lambda)$ that we use is that of $g_1(\lambda)$. In an identical fashion to Case 2, the speed of propagation is identified as being f_1 for each type in group 2. Note that, from the results of Section 4, $f_1 = c_1 > 0$ and, since $\rho(\Gamma_{22}) \leq 1$, $c_2 = 0$. Hence the speed of propagation of the forward front of infection for all types in group 2 is $\max(c_1, c_2)$.

Case 8. As in Case 3, the saddle-point of $g(\lambda)$ that we use is that of $g_2(\lambda)$. The speed of propagation is identified as being f_2 for each type in group 2. Again, from the results of Section 4, $f_2 = c_2 > 0$ and, since

$\rho(\Gamma_{11}) \leq 1$, $c_1 = 0$. Hence, the speed of propagation of the forward front of infection for all types in group 2 is $\max(c_1, c_2)$.

Hence, when there is initial infection in group 1, in all cases that we are able to treat by the saddle-point method we obtain the same result for types in group 2; namely that the speed of spread of the forward front for type 2 individuals is $\max(c_1, c_2)$.

The exact saddle-point results are summarized in Theorem 3.

Theorem 3. *Consider a fixed direction determined by α and a specific type j in group 2. For some small $\eta > 0$, let $\mathbf{s}$ be defined by $\eta = \int_{\mathbf{x}:\alpha'\mathbf{x}\geq s} y_j(\mathbf{x}, t)\, d\mathbf{x}$. Also let $\rho(\Gamma) > 1$ and $f(\lambda)$ have a minimum for some $\lambda = \theta_0 \in (0, \Delta_v)$ with $f(\theta_0) > 0$. Let $\rho(\mathbf{A}(\lambda))$ be analytic in an open region about $\lambda = \theta_0$. Suppose the following conditions hold*:

(i) *$f = \lim_{t\to\infty} s/t$ exists and is positive with $\rho'(\mathbf{A}(\lambda)) = f$ for some $\lambda \in (0, \Delta_v)$.*

(ii) *If $\mathbf{A}_{11}(\lambda) \not\equiv \mathbf{A}_{22}(\lambda)$ then let $\mathbf{A}(\theta_0)$ have distinct eigenvalues. In the special case where $\mathbf{A}_{11}(\lambda) \equiv \mathbf{A}_{22}(\lambda)$, then let $\mathbf{A}_{11}(\theta_0)$ have distinct eigenvalues.*

(iii) *$P^*_{ij}(\theta)$ exists in some open region of $\theta = 0$ for all i, j.*

(iv) *For any interval $[\theta_1, \theta_2] \subset (0, \Delta_v)$ and for all $\theta \in [\theta_1, \theta_2]$ there exists a $k_{ij}(y)$ such that $|P_{ij}(\theta + iy)| \leq k_{ij}(y)$ and $\int_{-\infty}^{\infty} k_{ij}(y)\, dy < \infty$ for all i, j.*

(v) *Each $y_i(\mathbf{s}, 0)$ is a function of bounded support.*

The speed of spread of the forward front for equation (3) is then $c_0 = \max(c_1, c_2)$, where $c_i = 0$ if $\rho(\Gamma_{ii}) \leq 1$ and $c_i = \max\{0, inf_{\lambda\in(0,\Delta_{ii})} f_i(\lambda))$ if $\rho(\Gamma_{ii}) > 1$. Note that if $\rho(\Gamma_{ii}) > 1$, then c_i is the infimum of the positive speeds at which wave solutions exist for an $S \to I \to R$ epidemic involving types in group i only.

8. Conclusion and extension to a general reducible epidemic. For each $i = 1, 2$ we make the following definition. Define $c_i = 0$ if $\rho(\Gamma_{ii}) \leq 1$. If $\rho(\Gamma_{ii}) > 1$, c_i is defined to be the infimum of the positive speeds for which wave solutions exist in a specific direction when we consider an $S \to I \to R$ epidemic amongst group i individuals only, group $(1 - i)$ individuals being ignored.

The saddle-point approximation suggests the following results for the reducible $S \to I \to R$ and $S \to I \to S$ epidemics with two groups.

(i) If the initial infection affects group 1 individuals, then the speed of propagation in the specific direction for any type in group 1 is c_1. The speed of propagation for any type in group 2 is, in this case, $\max(c_1, c_2)$.

(ii) If the initial infection affects group 2 individuals only, then there is no infection amongst group 1 types. The speed of propagation for any type in group 2 is c_2.

The values of c_1 and c_2 depend upon the direction considered, as determined by the value of α, where $\alpha'\alpha = 1$.

Let $f_i(\lambda; \alpha)$ denote the function $f_i(\lambda)$ for a specific direction given by α, and let $c_i(\alpha)$ and $\Delta_{ii}(\alpha)$ denote the corresponding values of c_i and Δ_{ii}.

We can easily show that $c_i(\alpha)$ is a continuous function of α if $\rho(\Gamma_{ii}) > 1$ and $f_i(\lambda; \alpha)$ has a minimum within the range $(0, \Delta_{ii}(\alpha))$ for each α. Note that $c_i(\alpha) = 0$ for all α if $\rho(\Gamma_{ii}) \le 1$.

Theorem 4. *If $\rho(\Gamma_{ii}) > 1$ and $f_i(\lambda; \alpha^*)$ has a minimum at $\lambda = \theta_i \in (0, \Delta_{ii}(\alpha^*))$, then $c_i(\alpha)$ is a continuous function of α at $\alpha = \alpha^*$.*

Proof. Note that $\rho(\mathbf{A}_{ii}(\lambda))$ is a continuous function of the entries of $\mathbf{A}_{ii}(\lambda)$; these being continuous functions of α. Hence, $f_i(\lambda; \alpha)$ is a continuous function of α and λ.

Let $\beta_i = (\Delta_{ii}(\alpha^*) + \theta_i)/2$ if $\Delta_{ii}(\alpha^*)$ is finite, and $\beta_i = 3\theta_i/2$ if $\Delta_{ii}(\alpha^*)$ is infinite. Take $b = \min(f_i(\theta_i/2; \alpha^*), f_i(\beta_i; \alpha^*)) - f_i(\theta_i; \alpha^*)$. Using the continuity of $f_i(\lambda; \alpha)$ in α at $\lambda = \theta_i/2$, $\lambda = \beta_i$ and $\lambda = \theta_i$, there exists a $\delta_1 > 0$ such that

$$|f_i(\lambda; \alpha) - f_i(\lambda; \alpha^*)| < b/2 \qquad \text{for } (\alpha - \alpha^*)'(\alpha - \alpha^*) \le \delta_1$$

for each of $\lambda = \theta_i/2$, β_i and θ_i.

This shows that $\Delta_{ii}(\alpha) \ge \beta_i$ for $(\alpha - \alpha^*)'(\alpha - \alpha^*) \le \delta_1$. Also in this range of α, $f_i(\theta_i; \alpha) < \min(f_i(\theta_i/2; \alpha), f_i(\beta_i; \alpha))$. Hence, using the properties of $f_i(\lambda; \alpha)$ obtained in Lemma 3, $f_i(\lambda; \alpha)$ has a minimum within the range $(\theta_i/2, \beta_i)$ for $(\alpha - \alpha^*)'(\alpha - \alpha^*) \le \delta_1$.

Take any $\varepsilon > 0$ and let $\varepsilon_1 = \min(\varepsilon, b/2)$. Now $f_i(\lambda;\alpha)$ is uniformly continuous for $\lambda \in [\theta_i/2, \beta_i]$ and $(\alpha-\alpha^*)'(\alpha-\alpha^*) \leq \delta_1$. Therefore, there exists a δ with $0 < \delta < \delta_1$ such that

$$|f_i(\lambda;\alpha) - f_i(\lambda;\alpha^*)| < \varepsilon_1 \qquad \text{for } (\alpha-\alpha^*)'(\alpha-\alpha^*) < \delta$$

and $\lambda \in [\theta_0/2, \beta_i]$.

Then, because $f_i(\lambda;\alpha^*)$ has its only minimum at $\lambda = \theta_i$,

$$\Big|\min_{\lambda\in(0,\Delta_{ii}(\alpha))} f_i(\lambda;\alpha) - \min_{\lambda\in(0,\Delta_{ii}(\alpha^*))} f_i(\lambda;\alpha^*)\Big| < \varepsilon_1 \leq \varepsilon \quad \text{for } (\alpha-\alpha^*)'(\alpha-\alpha^*) < \delta.$$

But $c_i(\alpha) = \max(0, \inf_{\lambda\in(0,\Delta_{ii}(\alpha))} f_i(\lambda;\alpha))$, when $f_i(\lambda;\alpha)$ has a minimum in $(0, \Delta_{ii}(\alpha))$. Therefore,

$$|c_i(\alpha) - c_i(\alpha^*)| < \varepsilon \qquad \text{for } (\alpha-\alpha^*)'(\alpha-\alpha^*) < \delta.$$

Hence $c_i(\alpha)$ is a continuous function of α at $\alpha = \alpha^*$. □

These results may be extended to cover the more general reducible epidemic. By reordering the types, Γ may be expressed in normal form (see Gantmacher [**4**, p. 75]), i.e.,

$$\Gamma = \begin{pmatrix} \Gamma_{1,1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \Gamma_{2,2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \Gamma_{g,g} & 0 & \cdots & 0 \\ \Gamma_{g+1,1} & \Gamma_{g+1,2} & \cdots & \Gamma_{g+1,g} & \Gamma_{g+1,g+1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Gamma_{s,1} & \Gamma_{s,2} & \cdots & \Gamma_{s,g} & \Gamma_{s,g+1} & \cdots & \Gamma_{s,s} \end{pmatrix},$$

where Γ_{ii} is nonreducible for $i = 1, \dots, s$; and for each $i = g+1, \dots, s$ there exists a $j < i$ such that $\Gamma_{j,i} \neq 0$.

Consider the speed of propagation of an epidemic in a specific direction. For each $i = 1, \dots, g$, define $c_i = 0$ if $\rho(\Gamma_{ii}) \leq 1$. For each $i = 1, \dots, g$, if $\rho(\Gamma_{ii}) > 1$, c_i is defined to be the infimum of the positive speeds for which wave solutions in that direction exist when we consider an $S \to I \to R$ epidemic amongst group i individuals only, all other groups of individuals being excluded from the epidemic.

Infection can be transmitted from the j^{th} group to the i^{th} group if there exists a sequence $\{i_r, r = 1, \dots, k\}$ of distinct elements such that $i_1 = j$ and $i_k = i$ and $\Gamma_{i_r, i_{r+1}} \neq 0$ for $r = 1, \dots, k-1$.

Infection will occur in the i^{th} group if either there is infection present in i^{th} group at time $t = 0$ or if there is infection present at time $t = 0$ in another group from which it can be transmitted, possibly through a sequence of infections, to the i^{th} group.

Then the speed of propagation in a specific direction for types in group i is $\max\{c_j\}$, where the max is taken over all j such that the initial infection causes some infection in group j and the j^{th} group can infect the i^{th} group, perhaps through a sequence of infections. Again, the speed is a continuous function of the direction of propagation considered.

Equivalent results can be obtained for the mean expectation velocity of the stochastic epidemic and the asymptotic speed of translation of the distribution function of furthest speed in the contact birth process.

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