ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 2, Spring 1993

STABILITY ANALYSIS OF IMPULSIVE SYSTEM VIA PERTURBING FAMILIES OF LYAPUNOV FUNCTIONS

XINZHI LIU

ABSTRACT. We extend, in this paper, Lyapunov's second method to impulsive systems and prove various stability criteria in a unified set up employing perturbing families of piecewise continuous Lyapunov functions. Examples are discussed which demonstrate the advantage of the technique and the effect of impulses.

1. Introduction. It is well known that Lyapunov's second method is an interesting and fruitful technique that has gained increasing significance and has given decisive impetus for modern development of stability theory of differential equations. A manifest advantage of this method is that it does not demand the knowledge of solutions and therefore has great power in applications. A stability property can be considered as a family of properties depending on some parameters. Consequently, when we employ a single Lyapunov function to prove a given stability property, the Lyapunov function used is assumed to play the role for every choice of these parameters. As a result, if we utilize a family of Lyapunov functions instead of one, it is natural to expect that each member of the family has to satisfy weaker requirements. This is precisely the idea of using a family of Lyapunov functions [**11**]. An interesting idea of perturbing Lyapunov functions is introduced in [**6**] which is useful in the study of nonuniform stability properties under weaker conditions. These ideas are further utilized and extended recently in the study of stability properties of nonautonomous ordinary differential systems [**8**]. Since in many problems of nonlinear mechanics [**1**], biology [**3**] and control theory [**2**], solutions experience jump discontinuities at certain moments of the evolution process, the study of impulsive dynamical systems has been assuming a greater importance lately [**5, 10**]. Employing piecewise continuous Lyapunov

Received by the editors on April 9, 1991.
This work was partially supported by a grant from the University of Waterloo which is gratefully acknowledged.

functions and the theory of impulsive differential inequalities, we wish to extend Lyapunov's method in the light of [**8**] to impulsive differential systems. In Section 2, we give some basic notions and introduce the concepts of (h_0, h)-stability which enable us to unify a variety of stability notions found in the literature and offer a general framework for investigation. Consequently, our results developed in Section 3 include many interesting special cases. Also in Section 3, we discuss some examples which clearly demonstrate the crucial dependence on the parameters of the Lyapunov function and the differential inequalities. The advantage of using one parameter family of Lyapunov functions is further revealed in the case of uniform stability properties (Theorems 3.2–3.3). It is important to note that impulse effects do contribute to yield stability properties even when the corresponding differential system without impulses does not enjoy any stability behavior at all. See Remark 1 following the proof of Theorem 3.1 in Section 3 for a discussion of this point.

2. Preliminaries. Consider the impulsive differential system

$$\begin{cases} x' = f(t,x), & t \neq t_k, \\ \Delta x = I_k(x), & t = t_k, \ k = 1, 2, \dots, \end{cases} \tag{2.1}$$

where $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $t_k \to \infty$ as $k \to \infty$; $f : R_+ \times R^n \to R^n$ is continuous on $(t_{k-1}, t_k] \times R^n$ and

$$\lim_{\substack{(t,y)\to(t_k,x) \\ t>t_k}} f(t,y) = f(t_k^+, x) \qquad \text{exists;}$$

and $I_k : R^n \to R^n$ is continuous for each $k = 1, 2, \dots$.

Let us list the following classes of functions for convenience.

$$\begin{aligned} PC = [\sigma : R_+ \to R, \ &\text{continuous on } (t_{k-1}, t_k] \text{ and } \lim_{t\to t_k^+} \sigma(t) \\ &= \sigma(t_k^+) \text{ exists}]. \\ K = [\sigma \in C[R_+, R_+], \ &\text{strictly increasing and } \sigma(0) = 0]. \end{aligned}$$

$$PCK = [\sigma : R_+ \times R_+ \to R_+, \sigma(\cdot, u) \in PC \text{ for each } u \in R_+ \text{ and } \sigma(t, \cdot) \in K \text{ for each } t \in R_+].$$

$$\Gamma = [h : R_+ \times R^n \to R_+, h(\cdot, x) \in PC \text{ for each } x \in R^n, h(t, \cdot) \in C[R^n, R_+] \text{ for each } t \in R_+ \text{ and } \inf h(t, x) = 0].$$

$$\nu_0 = [V : R_+ \times R^n \to R_+, \text{ continuous on } (t_{k-1}, t_k] \times R^n, \text{ locally Lipschitz in } x \text{ and } \lim_{\substack{(t,y)\to(t_k,x)\\ t>t_k}} V(t, y) = V(t_k^+, x) \text{ exists}].$$

Definition 2.1. $V \in \nu_0$. Then for $(t, x) \in (t_{k-1}, t_k) \times R^n$, the upper right derivative $V(t, x)$ with respect to the impulsive differential system (2.1) is defined as

$$D^+V(t, x) = \lim_{\delta\to 0^+} \sup(1/\delta)[V(t + \delta, x + \delta f(t, x)) - V(t, x)].$$

Definition 2.2. Let $h_0, h \in \Gamma$. Then we say that (i) h_0 is finer than h if there exists a $\delta > 0$ and a function $\phi \in K$ such that $h_0(t, x) < \delta$ implies $h(t, x) \in \phi(h_0(t, x))$; (ii) h_0 is weakly finer than h if there exists a $\delta > 0$ and a function $\psi \in PCK$ such that $h_0(t, x) < \delta$ implies $h(t, x) \leq \psi(t, h_0(t, x))$.

Definition 2.3. Let $V \in \nu_0$ and $h_0, h \in \Gamma$. Then $V(t, x)$ is said to be

(i) h-positive definite if there exists a constant $\rho > 0$ and a function $b \in K$ such that $h(t, x) < \rho$ implies $b(h(t, x)) \leq V(t, x)$;

(ii) h_0-decrescent if there exists a constant $\delta > 0$ and a function $a \in K$ such that $h_0(t, x) < \delta$ implies $V(t, x) \leq a(h_0(t, x))$;

(iii) weakly h_0-decrescent if there exists a constant $\delta > 0$ and a function $d \in PCK$ such that $h_0(t, x) < \delta$ implies $V(t, x) \leq d(t, h_0(t, x))$.

Definition 2.4. Let $\lambda : R_+ \to R_+$ be a measurable function. Then $\lambda(t)$ is said to be integrally positive if $\int_I \lambda(s)\, ds = \infty$ whenever $I = U_{i=1}^{\infty}[\alpha_i, \beta_i]$, $\alpha_i < \beta_i < \alpha_{i+1}$ and $\beta_i - \alpha_i \geq \delta > 0$. It is easy to see that λ is integrally positive if and only if $\lim_{t\to\infty} \inf \int_t^{t+\gamma} \lambda(s)\, ds > 0$ for every $\gamma > 0$ [**4**].

Definition 2.5. The impulsive system (2.1) is said to be

(i) (h_0, h)-stable if given $\varepsilon > 0$ and $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon$, $t \geq t_0$, for any solution $x(t) = x(t, t_0, x_0)$ of (2.1) with initial value $x(t_0) = x_0$;

(ii) (h_0, h)-uniformly stable if δ in (i) is independent of t_0;

(iii) (h_0, h)-asymptotically stable if it is (h_0, h)-stable and for $t_0 \in R_+$ there exists $\alpha = \alpha(t_0) > 0$ such that $h_0(t_0, x_0) < \alpha$ implies $\lim_{t\to\infty} h(t, x(t)) = 0$ for any solution $x(t) = x(t, t_0, x_0)$ of (2.1);

(iv) (h_0, h)-uniformly asymptotically stable if it is (h_0, h)-uniformly stable and for any $\varepsilon > 0$ there exist two positive numbers α and $T = T(\varepsilon)$ such that for $t_0 \in R_+$, $h_0(t_0, x_0) < \alpha$ implies $h(t, x(t)) < \varepsilon$, $t \geq t_0 + T$, for any solution $x(t) = x(t, t_0, x_0)$ of (2.1).

The concepts of (h_0, h)-stability enable us to unify a variety of stability notions found in the literature, such as stability of the trivial solution, partial stability, stability of an invariant set, and conditional stability, to name a few. See [**7, 8, 9**] for a discussion of this point.

We denote by $[\alpha]_+$ and $[\alpha]_-$ the positive and negative parts of the real number α, respectively, i.e., $[\alpha]_+ = \max\{0, \alpha\}$, $[\alpha]_- = \max\{0, -\alpha\}$ [**4**]. We assume that the solutions of (2.1) exist for all $t \geq 0$.

3. Main results.

We state and prove our main results in this section. Let us start with proving a nonuniform stability result under weaker assumptions which also shows that, in those cases where the Lyapunov function found does not satisfy the desired conditions, it is fruitful to perturb it rather than to discard it.

Theorem 3.1. *Assume that*

(i) h_0, $h \in \Gamma$ *and* h_0 *is weakly finer than* h;

(ii) *for any* $\beta > 0$ *there exists* $\alpha > 0$ *such that* $h_0(t_k, x) < \alpha$ *implies* $h_0(t_k^+, x + I_k(x)) < \beta$ *for* $k = 1, 2, \ldots$;

(iii) *there exists* $0 < \rho_0 < \rho$ *such that* $h(t_k, x) < \rho_0$ *implies* $h(t_k^+, x + I_k(x)) < \rho$ *for* $k = 1, 2, \ldots$;

(iv) $V_1 \in \nu_0 \cdot V_1(t,x)$ *is weakly* h_0*-decrescent and satisfies*

$$D^+V_1(t,x) \le g_1(t, V_1(t,x)), \qquad (t,x) \in S(h,\rho),\ t \ne t_k, \tag{3.1}$$

$$V_1(t_k^+, x + I_k(x)) \le \psi_k^{(1)}(V_1(t_k,x)), \qquad (t_k,x) \in S(h,\rho),\ k = 1,2,\dots, \tag{3.2}$$

where $g_1 : R_+ \times R_+ \to R$ *is continuous on* $(t_{k-1}, t_k]$, $g_1(t,0) \equiv 0$ *and* $\lim_{\substack{(t,v)\to(t_k,u)\\ t>t_k}} g_1(t,v) = g_1(t_k^+, u)$ *exists* $\psi_k^{(1)} : R_+ \to R_+$ *is nondecreasing and* $\psi_k^{(1)}(0) = 0$ *for all* $k = 1,2,\dots;$

(v) *for every* $\eta > 0$ *there exists* $V_{2\eta} \in \nu_0$ *such that*

$$\begin{gathered} b(h(t,x)) \le V_{2\eta}(t,x) \le a(h_0(t,x)), \\ (t,x) \in S(h,\rho) \cap S^C(h_0,\eta), \end{gathered} \tag{3.3}$$

$$\begin{gathered} D^+V_1(t,x) + D^+V_{2\eta}(t,x) \le g_2(t, V_1(t,x) + V_{2\eta}(t,x)), \\ (t,x) \in S(h,\rho) \cap S^C(h_0,\eta),\ t \ne t_k, \end{gathered} \tag{3.4}$$

$$\begin{gathered} V_1(t_k^+, x + I_k(x)) + V_{2\eta}(t_k^+, x + I_k(x)) \le \psi_k^{(2)}(V_1(t_k,x) + V_{2\eta}(t_k,x)), \\ (t,x) \in S(h,\rho) \cap S^C(h_0,\eta),\ k = 1,2,\dots, \end{gathered} \tag{3.5}$$

where $a, b \in K$, $g_2 : R_+ \times R_+ \to R$ *is continuous on* $(t_{k-1}, t_k]$, $g_2(t,0) \equiv 0$ *and* $\lim_{\substack{(t,v)\to(t_k,u)\\ t>t_k}} g_2(t,v) = g_2(t_k^+, u)$ *exists,* $\psi_k^{(2)} : R_+ \to R_+$ *is nondecreasing and* $\psi_k^{(2)}(0) = 0$ *for all* $k = 1,2,\dots;$

(vi) *The trivial solution of*

$$\begin{cases} u' = g_1(t,u), & t \ne t_k, \\ u(t_k^+) = \psi_k^{(1)}(u(t_k)), & k = 1,2,\dots, \\ u(t_0^+) = u_0 \ge 0, \end{cases} \tag{3.6}$$

is stable and the trivial solution of

$$\begin{cases} w' = g_2(t,w), & t \ne t_k, \\ w(t_k^+) = \psi_k^{(2)}(w(t_k)), & k = 1,2,\dots, \\ w(t_0^+) = w_0 \ge 0 \end{cases} \tag{3.7}$$

is uniformly stable.

Then the system (2.1) *is* (h_0, h)*-stable.*

Proof. Let $0 < \varepsilon < \rho_0$ and $t_0 \in R_+$ be given. Without loss of generality, we assume that $t_0 < t_k$, $k = 1, 2, \dots$. Since the trivial solution of (3.7) is uniformly stable, there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that

$$w(t) < b(\varepsilon), \qquad t \geq t_0, \qquad \text{if} \quad w_0 < \delta_0, \tag{3.8}$$

$w(t) = w(t, t_0, w_0)$ being any solution of (3.7).

The stability of the trivial solution of (3.6) implies that, given $\delta_0/2 > 0$ and $t_0 \in R_+$, there exists $\delta^* = \delta^*(t_0, \varepsilon) > 0$ such that

$$u(t) < \delta_0/2, t \geq t_0 \qquad \text{provided} \quad u_0 < \delta^*, \tag{3.9}$$

where $u(t) = u(t, t_0, u_0)$ is any solution of (3.6). Since $V_1(t, x)$ is weakly h_0-decrescent, there exist constant $\sigma_0 > 0$ and function $\phi_0 \in PCK$ such that

$$V_1(t, x) \leq \phi_0(t, h_0(t, x)), \qquad \text{if} \quad h_0(t, x) < \sigma_0. \tag{3.10}$$

Also, the fact that h_0 is weakly finer than h implies that there exist constant σ_1, $0 < \sigma_1 < \sigma_0$, and function $\phi_1 \in PCK$ such that

$$h(t, x) \leq \phi_1(t, h_0(t, x)), \qquad \text{provided} \quad h_0(t, x) < \sigma, \tag{3.11}$$

and

$$\phi_1(t, \sigma_1) < \rho_0. \tag{3.12}$$

Since $a \in K$ and $\phi_1 \in PCK$, we can find a constant $\delta_1 = \delta_1(t_0, \varepsilon)$, $0 < \delta_1 < \sigma_1$, such that

$$a(\delta_1) < \delta_0/2 \quad \text{and} \quad \phi_1(t_0, \delta_1) < \varepsilon. \tag{3.13}$$

By assumption (ii), there exists constant $\alpha > 0$ such that

$$h_0(t_k, x) < \alpha \quad \text{implies} \quad h_0(t_k, x + I_k(x)) < \delta_1. \tag{3.14}$$

Choose $u_0 = V_1(t_0, x_0)$. Since $\phi_0 \in PCK$ and inequality (3.10) holds, there exists constant $\delta_2 = \delta_2(t_0, \varepsilon)$, $0 < \delta_2 < \min(\rho_0, \delta_1, \alpha, \sigma_1)$, such that

$$h_0(t_0, x_0) < \delta_2 \quad \text{implies} \quad V_1(t_0, x_0) \le \phi_0(t_0, h_0(t_0, x_0)) < \delta^*. \tag{3.15}$$

We set $\delta = \delta_2$ and suppose that $h_0(t_0, x_0) < \delta$. Then we have from (3.11) and (3.13),

$$h(t_0, x_0) \le \phi_1(t_0, h_0(t_0, x_0)) \le \phi_1(t_0, \delta_1) < \varepsilon. \tag{3.16}$$

We claim that

$$h(t, x(t)) < \varepsilon, \qquad t \ge t_0$$

for any solution $x(t) = x(t, t_0, x_0)$ of (2.1) with $h_0(t_0, x_0) < \delta$. If this is not true, then there exists a solution $x(t) = x(t, t_0, x_0)$ with $h_0(t_0, x_0) < \delta$, and $t_0 < t^* < \bar{t}^*$ such that $t_k < t^* \le t_{k+1}$, $t_m < \bar{t}^* \le t_{m+1}$ for some $k, m, 0 \le k \le m$, satisfying

$$\delta \le h_0(t^*, x(t^*)), \qquad h_0(t, x(t)) < \delta, \qquad t \in [t_0, t_k], \tag{3.17}$$
$$\varepsilon \le h(\bar{t}^*, x(\bar{t}^*)), \qquad h(t, x(t)) < \varepsilon, \qquad t \in [t_0, t_m]. \tag{3.18}$$

Since $0 < \delta < \alpha$ and $0 < \varepsilon < \rho_0$, it follows from (3.14), (3.17), (3.18) and assumptions (ii)–(iii) that

$$\begin{aligned} h_0(t_k^+, x_k^+) &= h_0(t_k^+, x_k + I_k(x_k)) < \delta_1, \\ h(t_m^+, x_m^+) &= h(t_m^+, x_m + I_m(x_m)) < \rho, \end{aligned}$$

where $t_k = x(t_k)$, $x_m = x(t_m)$. Hence, we can find t^0 and $\bar{t}^0$ such that $t_k < t^0 \le t^*$, $t_m < \bar{t}^0 \le \bar{t}^*$ and

$$\delta \le h_0(t^0, x(t^0)) < \delta_1, \quad \text{and} \quad h_0(t, x(t)) < \delta_1, \qquad t \in [t_0, t^0]. \tag{3.19}$$
$$\varepsilon \le h(\bar{t}^0, x(\bar{t}^0)) < \rho \quad \text{and} \quad h(t, x(t)) < \rho, \qquad t \in [t_0, \bar{t}^0]. \tag{3.20}$$

Thus, we have

$$(t, x(t)) \in S(h, \rho) \cap S^C(h_0, \delta), \qquad t \in [t^0, \bar{t}^0]. \tag{3.21}$$

Setting $\eta = \delta$ we see by (v) that there exists a $V_{2\eta} \in \nu_0$ satisfying (3.3)–(3.5). Hence, letting $m(t) = V_1(t, x(t)) + V_{2\eta}(t, x(t))$ for $t \in [t^0, \bar{t}^0]$, we obtain from (3.4)–(3.5),

$$\begin{cases} D^+ m(t) \le g_2(t, m(t)), & t \in [t^0, \bar{t}^0], t \ne t_j \\ m(t_j^+) \le \psi_j^{(2)}(m(t_j)), & t_j \in [t^0, \bar{t}^0]. \end{cases} \tag{3.22}$$

Thus by the comparison theorem [**5**], we get

$$m(t) \le \gamma_2(t, t^0, m(t^0)) \qquad t \in [t^0, \bar{t}^0], \tag{3.23}$$

where $\gamma_2(t, t^0, m(t^0))$ is the maximal solution of (3.7). Also, we can obtain similarly the estimate

$$V_1(t, x(t)) \le \gamma_1(t, t_0, V_1(t_0, x_0)), \qquad t \in [t_0, t^0],$$

$\gamma_1(t, t_0, V_1(t_0, x_0))$ being the maximal solution of (3.6). Hence, by (3.9) and (3.15), we have

$$V_1(t^0, x(t^0)) < \delta_0/2.$$

Also, by (v) and (3.13), we get

$$V_{2\eta}(t^0, x(t^0)) \le a(h_0(t^0, x(t^0)) < a(\delta_1) < \delta_0/2.$$

Hence, it follows that $m(t^0) < \delta_0$ and therefore (3.8) and (3.23) imply that

$$m(\bar{t}^0) \le \gamma_2(\bar{t}^0, t^0, m(t^0)) < b(\varepsilon).$$

But $m(\bar{t}^0) \ge V_{2\eta}(\bar{t}^0, x(\bar{t}^0)) \ge b(h(\bar{t}^0, x(\bar{t}^0)) \ge b(\varepsilon)$, which leads to a contradiction. Hence, the proof is complete. □

Remark 1. The function $g_1(t, u) = p(t)\phi(u)$, where $p \in C[R_+, R_+]$ and $\phi \in K$, is admissible provided for some $\rho_0 > 0$ and each $\sigma \in (0, \rho_0)$,

$$\int_{t_k}^{t_{k+1}} p(s)\, ds + \int_{\sigma}^{\psi_k^{(1)}(\sigma)} \frac{ds}{\phi(s)} \le 0, \qquad k = 1, 2, \dots . \tag{3.24}$$

In fact, let $t_0 \in (t_j, t_{j+1}]$ and $0 < \varepsilon < \rho_0$ be given. Choose $\delta > 0$ such that $\delta < \min(\varepsilon, \psi_k^{(1)}(\varepsilon))$ and suppose that $0 \le u_0 < \delta$. We claim that $u(t) < \varepsilon$, $t \ge t_0$, $t \in (t_0, t_{j+1}]$, where $u(t) = u(t, t_0, u_0)$ is any solution of (3.6). If this is not true, we have $u(t^*) \geq \varepsilon$ for some $t^* \in (t_0, t_{j+1}]$. Then we get

$$\int_{\psi_j^{(1)}(\varepsilon)}^{\varepsilon} \frac{ds}{\phi(s)} < \int_{\delta}^{\varepsilon} \frac{ds}{\phi(s)} \leq \int_{u_0}^{\varepsilon} \frac{ds}{\phi(s)} \leq \int_{u_0}^{u(t^*)} \frac{ds}{\phi(s)}$$
$$\leq \int_{t_0}^{t^*} p(s)\, ds \leq \int_{t_j}^{t_{j+1}} p(s)\, ds,$$

which implies

$$\int_{t_j}^{t_{j+1}} p(s)\, ds + \int_{\varepsilon}^{\psi_j^{(1)}(\varepsilon)} \frac{ds}{\phi(s)} > 0,$$

contradicting (3.24).

Hence, $u(t) < \varepsilon$ for $t \in [t_0, t_{j+1}]$. Let $i \geq j+2$ and assume that $u(t) < \varepsilon$ for $t \in (t_{j+1}, t_i]$. Then for $t \in (t_i, t_{i+1}]$, we have

$$\int_{u(t_i^+)}^{u(t)} \frac{ds}{\phi(s)} \leq \int_{t_i}^{t} p(s)\, ds \leq \int_{t_i}^{t_{i+1}} p(s)\, ds. \tag{3.25}$$

Since $u(t_i^+) = \psi_i^{(1)}(u(t_i))$, it follows that

$$\int_{u(t_i)}^{u(t_i^+)} \frac{ds}{\phi(s)} = \int_{u(t_i)}^{\psi_i^{(1)}(u(t_i))} \frac{ds}{\phi(s)},$$

which, together with (3.25), implies

$$\int_{u(t_i)}^{u(t)} \frac{ds}{\phi(s)} \leq \int_{t_i}^{t_{i+1}} p(s)\, ds + \int_{u(t_i)}^{\psi_i^{(1)}(u(t_i))} \frac{ds}{\phi(s)} \leq 0.$$

Hence, $u(t) \leq u(t_i) < \varepsilon$ for $t \in (t_j, t_{j+1}]$ and it then follows by induction that $u(t) < \varepsilon$ for $t \geq t_0$. We thus have stability of the trivial solution of (3.6). If, in particular, $p(t) = 1/t$, $t \geq 1$, $\phi(u) = 2u$, $\psi_k^{(1)} = (1+\alpha_k)^2$ with $|1+\alpha_k| \leq k/(k+1)$ for all $k = 2, 3, 4, \ldots$, then it is easy to check that condition (3.24) is satisfied and hence $u = 0$ of (3.6) is stable. We note that for the corresponding differential equation $x' = 2x/t$, the trivial solution is not stable. This shows that impulse effects do contribute in stabilization of unstable systems.

Remark 2. If $V_1 \equiv 0$ and $g_1 \equiv 0$ in Theorem 3.1, then we get the following result which shows the advantage of utilizing a family of Lyapunov functions in proving uniform stability.

Theorem 3.2. *Assume that conditions* (*ii*) *and* (*iii*) *of Theorem* 3.1 *hold. Suppose further that*

(i) $h_0, h \in \Gamma$ *and* h_0 *is finer than* h;

(ii) *for every* $\eta > 0$, *there exists a* $V_\eta \in \nu_0$ *such that*

$$
\begin{aligned}
b(h(t,x)) \le V_\eta(t,x) \le a(h_0(t,x)), &\qquad (t,x) \in S(h,\rho) \cap S^C(h_0,\eta),\\
D^+V_\eta(t,x) \le g(t, V_\eta(t,x)), &\qquad (t,x) \in S(h,\rho) \cap S^C(h_0,\eta),\\
&\qquad t \ne t_k,\\
V_\eta(t_k^+, x + I_k(x)) \le \psi_k(V_\eta(t_k,x)), &\qquad (t_k,x) \in S(h,\rho) \cap S^C(h_0,\eta),\\
&\qquad k = 1,2,\dots,
\end{aligned}
$$

where $a, b \in K$, $g : R_+ \times R_+ \to R$ *is continuous on* $(t_{k-1}, t_k]$, $g(t,0) \equiv 0$ *and* $\lim_{\substack{(t,v)\to(t_k,u)\\ t>t_k}} g(t,v) = g(t_k^+, u)$ *exists*, $\psi_k : R_+ \to R_+$ *is nondecreasing and* $\psi_k(0) = 0$ *for all* $k = 1, 2, \dots$;

(iii) *the trivial solution of*

$$
\begin{cases}
u' = g(t,u), & t \ne t_k,\\
u(t_k^+) = \psi_k(u(t_k)), & k = 1,2,\dots,\\
u(t_0^+) = u_0 \ge 0
\end{cases}
$$

is uniformly stable.

Then the system (2.1) *is* (h_0, h)*-uniformly stable.*

In the following result, the functions g and ψ_k are given so that (h_0, h)-uniform asymptotic stability is obtained.

Theorem 3.3. *Assume that conditions* (ii)–(iii) *of Theorem* 3.1 *hold. Suppose further that*

(i) $h_0, h \in \Gamma$ *and* h_0 *is finer than* h;

(ii) *for every* $\eta > 0$, *there exist an integrally positive function* $\lambda_\eta(t)$ *and* $V_\eta \in \nu_0$ *such that for* $a, b \in K$,

$$b(h(t,x)) \le V_\eta(t,x) \le a(h_0(t,x)), \qquad (t,x) \in S(h,\rho) \cap S^C(h_0,\eta),$$
$$D^+V_\eta(t,x) \le -\lambda_\eta(t), \qquad (t,x) \in S(h,\rho) \cap S^C(h_0,\eta), \quad t \ne t_k,$$
$$V_\eta(t_k^+, x + I_k(x)) \le V_\eta(t_k,x), \qquad (t_k,x) \in S(h,\rho) \cap S^C(h_0,\eta), \quad k = 1,2,\dots .$$

Then the system (2.1) *is* (h_0,h)*-uniformly asymptotically stable.*

Proof. It follows from Theorem 3.2 that the system (2.1) is (h_0,h)-uniformly stable. Then, taking $\varepsilon = \rho_0$, we set $\delta_0 = \delta(\rho_0)$. Let $t_0 \in R_+$ and $h_0(t_0,x_0) < \delta_0$. Then for any solution $x(t) = x(t,t_0,x_0)$ of (2.1), we have

$$h(t,x(t)) < \rho_0, \qquad t \ge t_0. \tag{3.26}$$

Let $0 < \varepsilon < \rho_0$ and $\delta = \delta(\varepsilon)$ corresponding to (h_0,h)-uniform stability. Let us prove that there exists a $t_0^* > t_0$ such that

$$h_0(t_0^*, x(t_0^*)) < \delta(\varepsilon). \tag{3.27}$$

If there is no such t_0^*, we would have

$$\delta \le h_0(t,x(t)), \qquad t \ge t_0.$$

Then using the assumptions of the theorem, it follows that

$$V_\eta(t,x(t)) \le V_\eta(t_0,x_0) - \int_{t_0}^{t} \lambda_\eta(s)\,ds, \qquad t \ge t_0, \tag{3.28}$$

and hence $V_\eta(t,x(t)) \to -\infty$ as $t \to \infty$. This contradicts the nonnegativeness of V_η and therefore there exists t_0^* such that (3.27) holds. We thus have

$$h(t,x(t)) < \varepsilon, \qquad t \ge t_0^*,$$

$x(t) = x(t,t_0,x_0)$ being any solution of (2.1).

Now choose $T > 0$ such that

$$\int_0^T \lambda_\eta(s)\,ds \geq 2a(\delta_0).$$

Then it follows from (3.28) that the interval $[t_0, t_0 + T]$ contains a number t_0^* such that (3.27) holds. We thus conclude that the system (2.1) is (h_0, h)-uniformly asymptotically stable. The proof is complete. □

As an application of Theorem 3.3, we consider the following example.

Example 3.1. Consider the impulsive differential equation

$$\begin{cases} x'' + e(t)x' + \beta x + \gamma x^3 = 0, & t \neq t_k, \\ \Delta x = -b_k x, & t = t_k, \\ \Delta x' = 0, & t = t_k, \;\; k = 1, 2, \ldots \end{cases} \tag{3.29}$$

where $e \in C[R_+, R_+]$, $0 < e_1 \leq e(t) \leq e_2$, $\beta > 0$, $\gamma > 0$, $0 < b_k < 1$ for all $k = 1, 2, \ldots$.

Setting $y = x'$, (3.29) becomes

$$\begin{cases} x' = y, & t \neq t_k, \\ y' = -e(t)y - \beta x - \gamma x^3, & t \neq t_k, \\ \Delta x = -b_k x, & t = t_k, \\ \Delta y = 0, & t = t_k, \;\; k = 1, 2, \ldots . \end{cases} \tag{3.30}$$

Now, for any $0 < \eta < 1$, choose a function $\phi_\eta \in C^1[R_+, R_+]$ such that

$$\phi_\eta(s) = \begin{cases} 1, & 0 \leq s \leq \eta/2, \\ 0, & s \geq \eta. \end{cases} \tag{3.31}$$

Next we define functions ψ_η^+ and ψ_η^- by

(3.32)

$$\psi_\eta^+(x, y) = \begin{cases} \phi_n(|y|), & \text{if } \eta \leq \sqrt{x^2 + y^2} \leq 1 \text{ and } e(t)y + \beta x + \gamma x^3 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(3.33)

$$\psi_\eta^-(x, y) = \begin{cases} \phi_\eta(|y|), & \text{if } \eta \leq \sqrt{x^2 + y^2} \leq 1,\; e(t)y + \beta x + \gamma x^3 < 0 \\ 0, & \text{otherwise.} \end{cases}$$

Construct a Lyapunov function $V_\eta(x,y)$ by

$$V_\eta(x,y) = \frac{y^2}{2} + \frac{1}{2}\beta x^2 + \frac{1}{4}\gamma x^4 + u(\psi_\eta^+(x,y) - \psi_\eta^-(x,y))y, \tag{3.34}$$

where $u > 0$ is a constant to be determined later.

Let $\beta_0 = \min\{1/4, \beta/4\}$ and $\alpha_0 = 2\max\{1/2, \beta/2+\gamma/4, \beta_0/2\}$. Then it is not difficult to verify that for $u \le \beta_0\eta/2$,

$$\beta_0(x^2+y^2) \le V_\eta(x,y) \le \alpha_0(x^2+y^2), \qquad \eta \le \sqrt{x^2+y^2} \le 1. \tag{3.35}$$

For $t \ne t_k$ we obtain from (3.34) that

$$\begin{aligned} D^+V_\eta(x,y) \le{}& -e(t)y^2 - u(\psi_\eta^+(x,y) - \psi_\eta^-(x,y))(e(t)y + \beta x + \gamma x^3) \\ &+ u(D^+\psi_\eta^+(x,y) - D_+\psi_\eta^-(x,y))y. \end{aligned} \tag{3.36}$$

1^0. If $|y| > \eta$, then it follows from (3.36) that

$$D^+V_\eta(x,y) \le -e_1\eta^2, \qquad \eta \le \sqrt{x^2+y^2} \le 1. \tag{3.37}$$

2^0. Suppose that $\eta/2 \le |y| \le \eta$. Then if $u \le (e_1\eta^2)/(\delta M)$, where

$$\begin{aligned} M &= \max_{(x,y)\in\Omega} |D^+\psi_\eta^+(x,y)| + \max |D_+\psi_\eta^-(x,y)|, \\ \Omega &= \{(x,y) \in R^2; \eta \le \sqrt{x^2+y^2} \le 1, \eta/2 \le |y| \le \eta\}, \end{aligned}$$

we have from (3.36)

$$D^+V_\eta(x,y) \le -e_1\eta^2/\delta, \qquad \eta \le \sqrt{x^2+y^2} \le 1. \tag{3.38}$$

3^0. In case $|y| < \eta/2$ holds, then $D^+\psi_\eta^+(x,y) = D_+\psi_\eta^-(x,y) \equiv 0$ and

$$\begin{aligned} D^+V_\eta(x,y) &\le -e(t)\eta^2 - u|e(t)y + \beta x + \gamma x^3| \\ &\le -e_1 y^2 - u\min\{\max(0, e_1|y| - |\beta x + \gamma x^3|), \\ &\qquad\qquad \max(0, |\beta x + \gamma x^3| - e_2|y|)\}. \end{aligned}$$

It is easy to see that the function

$$\begin{aligned} F(x,y) = e_1 y^2 + u\min\{&\max(0, e_1|y| - |\beta x + \gamma x^3|), \\ &\max(0, |\beta x + \gamma x^3| - e_2|y|)\} \end{aligned}$$

is continuous and positive on the set $\eta \le \sqrt{x^2+y^2} \le 1$. Set $u = \min\{\beta_0\eta/2, (e_1\eta^2)/(\delta M)\}$. Then there exists $N > 0$ such that

$$N = \max_{\eta \le \sqrt{x^2+y^2} \le 1} F(x,y)$$

Thus we have

$$D^+V_\eta(x,y) \le -N, \qquad \eta \le \sqrt{x^2+y^2} \le 1. \tag{3.39}$$

Let $\lambda_\eta = \min\{e_1\eta^2, (e_1/\delta)\eta^2, N\}$. Then it follows from (3.37), (3.38) and (3.39) that

$$D^+V_\eta(x,y) \le -\lambda_\eta, \qquad \eta \le \sqrt{x^2+y^2} \le 1,\ t \ne t_k. \tag{3.40}$$

Since $0 < b_k < 1$ for all $k = 1, 2, \dots$, a direct calculation from (3.34) gives

$$V_\eta(x^+, y^+) \le V_\eta(x,y), \qquad t = t_k,\ k = 1, 2, \dots . \tag{3.41}$$

Setting $h_0 = h = \sqrt{x^2+y^2}$, $b(s) = \beta_0 s^2$ and $a(s) = \alpha_0 s^2$, we see that all conditions of Theorem 3.3 are met and therefore we conclude that the trivial solution $(x,y) = (0,0)$ of (3.30) is uniformly asymptotically stable.

We shall next consider a result on asymptotic stability in the same spirit as that of Theorem 3.1.

Theorem 3.4. *Let assumptions* (i)–(iii) *of Theorem* 3.1 *hold. Suppose further that*

(i) $V_1 \in \nu_0$, $V_1(t,x)$ *is weakly* h_0*-decrescent and there exists* $V_2 \in \nu_0$ *such that* $V_2(t,x)$ *is* h*-positive definite and*

$$D^+V_1(t,x) \le -\lambda(t)\phi(V_2(t,x)), \qquad (t,x) \in S(h,\rho),\ t \ne t_k, \tag{3.42}$$

$$V_i(t_k^+, x + I_k(x)) \le V_i(t_k, x), \qquad (t_k, x) \in S(h,\rho),\ i = 1, 2,\ k = 1, 2, \dots , \tag{3.43}$$

where $\phi \in K$ *and* $\lambda(t)$ *is integrally positive*;

(ii) *for every solution* $x(t)$ *of* (2.1) *such that* $(t, x(t)) \in S(h, \rho)$ *the function*

$$\int_0^t [D^+V_3(s, x(s))] \pm ds, \qquad \textit{where} \quad V_3 = V_1 - V_2, \tag{3.44}$$

is uniformly continuous on R_+. *Moreover, if* $[\cdot]_+$ *stands in* (3.44), *then*

$$V_3(t_k^+, x + I_k(x)) \le V_3(t_k, x), \qquad (t_k, x) \in S(h, \rho),\ k = 1, 2, \dots , \tag{3.45}$$

and if $[\cdot]_-$ *stands in* (3.44), *then*

$$V_3(t_k^+, x + I_k(x)) \ge V_3(t_k, x), \qquad (t_k, x) \in S(h, \rho),\ k = 1, 2, \dots ; \tag{3.46}$$

(iii) *for every* $\eta > 0$, *there exists* $V_\eta \in \nu_0$ *such that*

$$\begin{gathered} b(h(t,x)) \le V_\eta(t,x) \le a(h_0(t,x)), \qquad (t,x) \in S(h,\rho) \cap S^C(h_0,\eta), \\ D^+V_1(t,x) + D^+V_\eta(t,x) \le g(t, V_1(t,x) + V_\eta(t,x)), \\ (t,x) \in S(h,\rho) \cap S^C(h_0,\eta),\ t \ne t_k, \\ V_1(t_k^+, x + I_k(x)) + V_\eta(t_k^+, x + I_k(x)) \le \psi_k(V_1(t_k,x) + V_\eta(t_k,x)), \\ (t_k,x) \in S(h,\rho) \cap S^C(h_0,\eta),\ k = 1, 2, \dots , \end{gathered}$$

where $a, b \in K$, $g : R_+ \times R_+ \to R$ *is continuous on* $(t_{k-1}, t_k]$, $g(t, 0) \equiv 0$ *and* $\lim_{\substack{(t,v)\to(t_k,u) \\ t>t_k}} g(t,v) = g(t_k^+, u)$ *exists,* $\psi_k : R_+ \to R_+$ *is nondecreasing and* $\psi_k(0) = 0$ *for all* $k = 1, 2, \dots$;

(iv) *the trivial solution of*

$$\begin{cases} u' = g(t,u), & t \ne t_k, \\ u(t_k^+) = \psi_k(u(t_k)), & k = 1, 2, \dots, \\ u(t_0^+) = u_0 \ge 0 \end{cases}$$

is uniformly stable.

Then the system (2.1) *is* (h_0, h)*-asymptotically stable and* $\lim_{t\to\infty} V_3(t, x(t))$ *exists and is finite for any solution of* (2.1).

Proof. Since (3.42) implies that $D^+V_1(t,x) \le 0$, it follows from Theorem 3.1 that the system (2.1) is (h_0,h)-stable. Choosing $\varepsilon = \rho_0$ and designating by $\delta_0 = \delta(t_0,\rho_0)$, it is clear that we have

$$h(t_0,x_0) < \delta_0 \quad \text{implies} \quad h(t,x(t)) < \rho_0, \quad \text{for } t \ge t_0, \tag{3.47}$$

where $x(t) = x(t,t_0,x_0)$ is any solution of (2.1) with $h(t_0,x_0) < \delta_0$.

Let $x(t)$ be any solution of (2.1) satisfying (3.47). Define the functions $m_i(t) = V_i(t,x(t))$, $i = 1,2,3$, so that $m_1(t) = m_2(t) + m_3(t)$. Assumption (i) implies that $m_1(t)$ is nonincreasing and bounded from below, and therefore $\lim_{t\to\infty} m_1(t) = \sigma < \infty$. We claim that $\lim_{t\to\infty} \inf m_2(t) = 0$. If this is not true, then there exist constants $\delta > 0$ and $T_1 > t_0$ such that $m_2(t) \ge \delta$, $t \ge T_1$. It follows from (3.42) and (3.47) that

$$D^+m_1(t) \le -\lambda(t)\phi(m_2(t)) \le -\lambda(t)\phi(\delta), \qquad t \ge T_1,\ t \ne t_k,$$

which, together with (3.43), implies

$$\lim_{t\to\infty} m_1(t) \le m_1(T_1) - \phi(\delta)\int_{T_1}^{\infty} \lambda(s)\,ds = -\infty.$$

This is a contradiction.

Suppose now that $\lim_{t\to\infty} \sup m_2(t) > 0$. Then there exists a constant $\gamma > 0$ such that $\lim_{t\to\infty} \sup m_2(t) > 3\gamma$. Since $\lim_{t\to\infty} m_1(t) = \sigma$ and $m_1(t)$ is nonincreasing, there exists a constant $T_2 > t_0$ such that

$$\sigma \ge m_1(t) \le \sigma + \gamma, \qquad t \ge T_2. \tag{3.48}$$

For definiteness, suppose that $[\cdot]_+$ stands in (3.44) and consequently (3.45) holds. Thus we can choose a sequence

$$T_2 < \alpha_1 < \beta_1 < \cdots < \alpha_j < \beta_j < \cdots$$

such that for $j = 1,2,\dots$,

$$m_2(\alpha_j) = 3\gamma, \qquad m_2(\beta_j) = \gamma, \qquad \gamma \le m_2(t) \le 3\gamma,\ t \in [\alpha_j,\beta_j]. \tag{3.49}$$

From (3.48)–(3.49), it is easy to see that, for $j = 1,2,\dots,$

$$m_1(\alpha_j) - m_2(\alpha_j) \le \sigma - 2\gamma, \qquad m_1(\beta_j) - m_2(\beta_j) \ge \sigma - \gamma. \tag{3.50}$$

Since $m_1(t) = m_2(t) + m_3(t)$, it follows from (3.50) that

$$0 < \gamma \le m_3(\beta_j) - m_3(\alpha_j) \le \int_{\alpha_j}^{\beta_j} [D^+V_3(s, x(s))]_+ \, ds,$$

which shows by the uniform continuity of (3.44) that there exists a constant $\sigma > 0$ such that

$$\beta_j - \alpha_j \ge \delta, \qquad j = 1, 2, \dots . \tag{3.51}$$

By (3.42), (3.43), (3.49) and (3.51), we then get

$$\begin{aligned} \lim_{t\to\infty} m_1(t) &\le m_1(T_2) - \int_{T_2}^{\infty} \lambda(s)\phi(m_2(s)) \, ds \\ &\le m_1(T_2) - \phi(\gamma) \int_I \lambda(s) \, ds = -\infty, \end{aligned}$$

where $I = \cup_{j=1}^{\infty}[\alpha_j, \beta_j]$. This contradiction implies that $\lim_{t\to\infty} m_2(t) = 0$. When $[\cdot]_-$ stands in (3.44) and (3.46) holds, the proof is similar. Since $V_2(t,x)$ is h-positive definite, we get in turn $\lim_{t\to\infty} h(t, x(t)) = 0$. Thus we conclude that the system (2.1) is (h_0, h)-asymptotically stable. Since $\lim_{t\to\infty} m_1(t) = \sigma$ and $\lim_{t\to\infty} m_2(t) = 0$, it follows that $\lim_{t\to\infty} m_3(t) = \sigma$ and this proves the last assertion of the theorem. The proof is therefore complete. $\square$

Remark. If there exist measurable functions $q_1(t) \le 0$ and $q_2(t) \ge 0$ such that $\int_0^t q_i(s) \, ds$ is uniformly continuous on R_+,

$$\begin{aligned} q_1(t) &\le D^+V_3(t, x), \qquad t \ne t_k, \\ V_3(t_K^+, x + I_k(x)) &\ge V_3(t_k, x), \end{aligned}$$

or

$$\begin{aligned} D^+V_3(t, x) &\le q_2(t) \\ V_3(t_K^+, x + I_k(x)) &\le V_3(t_k, x), \end{aligned}$$

then condition (ii) is satisfied, where

$$D^+V_3(t, x) = \lim_{\delta\to 0^+} \sup(1/\delta)[V_3(t + \delta, x + \delta f(t, x)) - V_3(t, x)].$$

If we set $V_\eta \equiv 0$, $g \equiv 0$ and assume that $V_3 \in \nu_0$ in Theorem 3.4, then we have the following result.

Theorem 3.5. *Assume that conditions* (i)–(iii) *of Theorem* 3.1 *and conditions* (i)–(ii) *of Theorem* 3.4 *hold. Suppose further that* $V_3 \in \nu_0$ *in Theorem* 3.4. *Then the system* (2.1) *is* (h_0, h)*-asymptotically stable.*

Proof. Since $V_3 \in \nu_0$ implies that $V_1(t,x)$ is h-positive definite, it then follows from Theorem 3.1 in [**9**] that the system (2.1) is (h_0, h)-stable. The rest of the proof is the same as that of Theorem 3.4. $\square$

We conclude our paper by discussing another example.

Example 3.2. Consider the generalized Liénard equation with impulse effects

$$\begin{cases} x'' + a(t)g(x,x')x' + b(t)f(x) = 0, & t \neq t_k, \\ \Delta x = 0, & t = t_k, \\ \Delta x' = -b_k x', & t = t_k,\ k = 1,2,\dots, \end{cases} \tag{3.52}$$

where $a : R_+ \to R_+$ is integrally positive, $g : R^2 \to R_+$, $g(x,x') \geq L > 0$ if $\sqrt{x^2 + x'^2} \leq \rho$, $\rho > 0$, $b : R_+ \to R_+$ is continuously differentiable, nonincreasing and $\int_0^t b(s)\,ds$ is uniformly continuous on R_+, $f : R \to R$ is continuous and $f(x)x \geq 0$ if $x \neq 0$, $0 < b_k < 1$, $k = 1,2,\dots$.

Setting $y = x'$, (3.52) becomes

$$\begin{cases} x' = y, & t \neq t_k, \\ y' = -a(t)g(x,y)y - b(t)f(x), & t \neq t_k, \\ \Delta x = 0, & t = t_k, \\ \Delta y = -b_k y, & t = t_k,\ k = 1,2,\dots. \end{cases} \tag{3.53}$$

Since $f(x)x \geq 0$, $x \neq 0$ implies $f(0) = 0$, it follows that (3.53) admits the trivial solution $(x,y) = (0,0)$. Let $V_1(t,x,y) = y^2/2 + b(t)\int_0^x f(s)\,ds$ and $V_2(t,x,y) = y^2/2$. Then for $t \neq t_k$,

$$\begin{aligned} D^+V_1(t,x,y) &= -2a(t)g(x,y)V_2(t,x,y) + b'(t)\int_0^x f(s)\,ds \\ &\leq -2La(t)V_2(t,x,y), \qquad \sqrt{x^2 + y^2} \leq \rho, \end{aligned} \tag{3.54}$$

and

$$V_1(t_k^+, x+\Delta x, y+\Delta y) = (1-b_k)^2 y^2/2 + b(t)\int_0^x f(s)\,ds \\ \leq V_1(t_k, x, y), \qquad k = 1, 2, \dots . \tag{3.55}$$

Let $V_3(t,x,y) = V_1(t,x,y) - V_2(t,x,y) = b(t)\int_0^x f(s)\,ds$. Then $V_3(t,x,y) \geq 0$, $V_3(t_k^+, x+\Delta x, y+\Delta y) = V_3(t_k, x, y)$, for all $k = 1, 2, \dots$, and

$$D^+V_3(t,x,y) = b(t)\int_0^x f(s)\,ds + b(t)yf(x), \qquad t \neq t_k. \tag{3.56}$$

Since $f(x)$ is continuous, there exists constant $M > 0$ such that

$$f(x) \leq M, \qquad \text{if} \quad |x| \leq \rho.$$

Then by (3.56) we have

$$D^+V_3(t,x,y) \leq \rho M b(t), \qquad t \neq t_k,\ \sqrt{x^2+y^2} \leq \rho. \tag{3.57}$$

Clearly, the function $\int_0^t \rho M b(s)\,ds$ is uniformly continuous on R_+.

Set $h = |y|$, $h_0 = \sqrt{x^2+y^2}$. Then it is easy to see that V_2 is h-positive definite and V_1 is weakly h_0-decreasing. It thus follows from Theorem 3.5 that the trivial solution of (3.53) is asymptotically stable with respect to y, and for every solution of (3.53) the function $b(t)\int_0^{x(t)} f(s)\,ds$ has a finite limit.

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Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1