ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 2, Spring 1993

ON WEAKLY LINDELOF BANACH SPACES

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ABSTRACT. In this paper we define and investigate the properties of a proper class of Banach spaces each member of which is Lindelof in its weak topology; we call them weakly Lindelof determined (WLD) Banach spaces. The class of WLD Banach spaces extends the known class of WCD (weakly countably determined) Banach spaces and inherits some of its basic properties: (e.g., each WLD Banach space has a projectional resolution of identity, and it is also derived from a *small* weakly Lindelof subset, etc.).

We also present several examples, in our attempt to clarify the concept of *weakly countably determiness*, such as: (i) a WLD Banach space which is *dually* strictly convexifiable, but not WCD; (ii) a WLD Banach with an unconditional basis, which is not weak Asplund, whose dual space is strictly convexifiable; (iii) a dual weakly K-analytic Banach space which is not a subspace of a weakly compactly generated Banach space. On the grounds of these examples, we answer questions and problems of Gruenhage, Larman and Phelps, and Talagrand.

Introduction. The purpose of the present paper is to investigate Banach spaces related to Corson-compact spaces. Thus we define a new and wide class of weakly Lindelof Banach spaces (we call them weakly Lindelof determined-WLD Banach spaces), that extends the class of weakly countably determined (WCD) Banach spaces [**35, 21**], and we study its properties. This class is in a way a definite class with the features of WCD Banach spaces.

The paper is organized into three sections.

In Section 1, we study the general properties of WLD Banach spaces, and indicate similarities with WCD Banach spaces. We show in particular that each WLD Banach space admits a projectional resolution of identity (and hence an equivalent locally uniformly convex norm) (Theorem 1.4, Corollary 1.5), and also that is derived from a weakly Lindelof subset with a unique weak limit point (Theorem 1.6). We

Received by the editors on April 15, 1991.

also characterize in various ways those Banach spaces with an unconditional basis, which have the property being WLD (Theorem 1.7). An interesting Corollary is that every weak Asplund Banach space with an unconditional basis is WLD (Corollary 1.12).

In Section 2 we introduce a special category of Banach spaces with an unconditional basis, of some importance for the construction of suitable examples (which appear in the next section). We also establish some results (mainly) concerning Gateaux differentiability of the norm of Banach spaces belonging to this category (Proposition 2.7 and Theorem 2.9).

In the third section we present some examples of WLD Banach spaces answering several questions by Gruenhage Larman and Phelps, and also by Talagrand. In fact we have constructed: (a) a WLD Banach space (of the form $C(K)$) which admits an equivalent norm with strictly convex dual norm, though it is not WCD (Theorem 3.3). This space is of course weak Asplund (and smoothable). We answer with this example (of the compact K) a question by Gruenhage [**16**]. Furthermore (in contrast with the situation in the case of WCD Banach spaces) we present: (b) a WLD Banach space with an unconditional basis which is not weak Asplund, whose dual space is strictly convexifiable (Theorems 3.6 and 3.8). This example solves a problem of Larman and Phelps [**18**]. (c) A WLD Banach space with an unconditional basis, so that there is no equivalent strictly convex norm on its dual space (Theorem 3.12). This example is essentially due to R. Haydon with whom the authors discussed the related problem. We thank Professor Haydon for his valuable help.

Finally, we give a method of construction of dual WLD Banach spaces (Theorem 3.16); as a by-product we establish (d) a dual weakly K-analytic Banach space which is not a subspace of a weakly compactly generated Banach space (WCG) (Theorem 3.17). Note that this example solves a problem posed by Talagrand [**35**].

Notation and terminology. The cardinality of a set A is denoted by $|A|$. The (cardinality of the) set of natural numbers is denoted by ω; ω_1 is the first uncountable cardinal.

A topological space X is said to be *ccc* if every disjoint family of nonempty open subsets of X is at most countable, and is said to be scattered if every nonempty subset of X contains a relative isolated point.

For a topological space X we denote by $C(X)$ the linear space of continuous real valued functions defined on X, and by $C_p(X)$ the same space endowed with the topology of pointwise convergence.

Given a set Γ, the Σ-product of the real line is the subspace $\Sigma(\Gamma)$ of the Tychonoff product R^Γ consisting of points with all but countably many coordinates equal to 0 and let $c_0(\Gamma) = \{x \in R^\Gamma$: for every $\varepsilon > 0$ the set $\sigma_\varepsilon(x) = \{\gamma \in \Gamma : |x(\gamma)| \geq \varepsilon\}$ is finite$\}$. $l^\infty(\Gamma)$ is the Banach space of all bounded real-valued functions on Γ with the supremum norm; $l_c^\infty(\Gamma)$ denotes the subspace of $l^\infty(\Gamma)$ consisting of those x which have countable support (i.e., $\sigma(x) = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$ is at most countable). It is clear that $c_0(\Gamma)$, $l_c^\infty(\Gamma)$ are closed linear subspaces of the Banach space $l^\infty(\Gamma)$, and that $c_0(\Gamma) \subseteq l_c^\infty(\Gamma) \subseteq \Sigma(\Gamma)$. Also, $l^1(\Gamma)$ denotes the Banach space of all functions $f : \Gamma \to R$ such that $\sum_{\gamma\in\Gamma} |f(\gamma)| < +\infty$, with the obvious norm.

A compact space K is called a Corson (respectively, Eberlein) compact, if it is homeomorphic to a subset of the space $\Sigma(\Gamma)$ (respectively, $c_0(\Gamma)$). The class of Eberlein compacts coincides with the class of weakly compact subsets of Banach spaces [**2, 25**].

A (real) Banach space E is called:

(a) Weakly compactly generated-WCG, if E contains a weakly compact total subset [**2, 25**].

(b) Weakly countably determined-WCD (respectively, weakly $\mathcal{K}$-analytic) if E, endowed with the weak topology, is a continuous image of a closed subset of a space of the form $\mathcal{M} \times \mathcal{K}$, where $\mathcal{K}$ is a compact space and $\mathcal{M}$ a separable metric space (respectively, $\mathcal{M}$ a polish space) [**35, 25**].

Every WCG Banach space is weakly $\mathcal{K}$-analytic [**35, 25**], and clearly every weakly $\mathcal{K}$-analytic Banach space is WCD.

We also consider the following classes of compact spaces: Gulko (respectively, Talagrand) compacts, i.e., compact spaces K so that the Banach space $C(K)$ is WCD (respectively, weakly $\mathcal{K}$-analytic) (cf. [**25**]). It is known that K is an Eberlein compact if and only if $C(K)$ is WCG [**2, 25**]; it is also known that every Gulko compact is a Corson compact (cf. [**25**]).

A compact space K is said to be a Rosenthal compact, if it can be embedded in the space $B_1(M)$ of first Baire class real-valued functions on some Polish space M, equipped with the topology of pointwise convergence.

A set Γ in a Banach space E is called an unconditional basis for E if it is total in E, and there exists a constant $\lambda > 0$ so that for all n, $\gamma_1, \dots, \gamma_n$ in Γ, scalars $c_1, \dots, c_n$ and numbers $\varepsilon_1, \dots, \varepsilon_n$ with $\varepsilon_i = \pm 1$ for all i, $||\Sigma \varepsilon_i c_i \gamma_i|| \leq \lambda ||\Sigma c_i \cdot \gamma_i||$. The basis Γ is said to be boundedly complete, if for every choice $(c_\gamma)_{\gamma\in\Gamma}$ of scalars such that the family, $\{||\sum_{\gamma\in F} c_\gamma \cdot \gamma|| : F$ is a finite subset of $\Gamma\}$ is bounded, the series $\sum_{\gamma\in\Gamma} c_\gamma \cdot \gamma$ is unconditionally converging.

The Banach space E is said to be weak Asplund if every continuous convex function defined on an open and convex subset of E is Gateaux differentiable on a dense G_δ subset of its domain. We say that E is a Gateaux differentiability space (GDS) if every function as above is Gateaux differentiable on a dense subset (not necessarily a dense G_δ subset) of its domain. Suppose that K is a weak* compact and convex subset of the dual E^* of the Banach space E. A point $x^* \in K$ is called a weak* exposed point if there exists $x \neq 0$ in E such that $y^*(x) < x^*(x)$, for all y^* in K with $y^* \neq x^*$ (cf. [**18, 27**]).

A norm $||\cdot||$ of a Banach space E is said to be

(a) smooth, if it is Gateaux differentiable at every nonzero vector x of E.

(b) strictly convex if for all $x, y \in E$ with $||x|| = ||y|| = 1$, we have $||(x+y)/2|| < 1$, whenever $x \neq y$. A Banach space which admits an equivalent strictly convex (respectively, smooth) norm is said to be strictly convexifiable (respectively, smoothable).

(c) locally uniformly convex, if for every sequence $(x_n)_{n\in\omega}$ of points of E and every $x \in E$ with $||x_n|| = ||x|| = 1$, the condition $||(x_n+x)/2|| \to 1$ implies that $||x_n - x|| \to 0$.

Section 1. In this section we introduce the class of weakly Lindelof determined Banach spaces.

We begin by recalling some known results which inspired the definition of this new class of Banach spaces.

Theorem [**2**]. *A Banach space E is* WCG *if and only if there is a bounded linear one-to-one operator $T: E^* \to c_0(\Gamma)$ for some set Γ, which is weak* to pointwise continuous. This is the classical theorem of Amir-Lindenstrauss* [**2**].

Theorem [**15**]. *Let E be a* WCD *Banach space. Then there is a set Γ and a bounded linear one-to-one operator $T: E^* \to l_c^\infty(\Gamma)$, which is weak* to pointwise continuous.*

This deep result due to Gulko (stated in a slightly different form) motivates many of the results obtained in this area after Talagrand [**35**], Vasak [**38**] and Gulko [**15**] investigation of WCD Banach spaces.

Theorem [**21**]. *A Banach space E is* WCD *if and only if there are a separable metric space M, a compact space K, and a bounded linear one-to-one operator $T: E^* \to C_1(M \times K)$, which is weak* to pointwise continuous.* (*For a topological space X, $f \in C_1(X)$, if and only if f is bounded and for every $\varepsilon > 0$ the set $\sigma_\varepsilon(f) = \{t \in X : |f(t)| \geq \varepsilon\}$ is a closed and discrete subset of X. Note that $C_1(M \times K) \subseteq l_c^\infty(M \times K)$* [**21**].)

Theorem [**3**]. *For a Corson compact space K the following are equivalent*:

a) *K has property* (M) (*that is, every positive Radon measure on K has separable support*).

b) *$C(K)$ is weakly Lindelof.*

c) *There is a set Γ and a bounded linear one-to-one operator T : $C(K)^* = M(K) \to l_c^\infty(\Gamma)$, which is weak* to pointwise continuous.*

All the preceding results motivate to the following,

Definition 1.1. We call a Banach space E *weakly Lindelof* determined (briefly WLD), if there is a set Γ and a bounded linear one-to-one operator $T: E^* \to l_c^\infty(\Gamma)$, that is weak* to pointwise continuous.

It follows immediately from the above that, if K is a compact space, then the space $C(K)$ is WLD if and only if K is a Corson compact with property (M). We also notice the obvious: the property of being a space WLD is isomorphic invariant.

Note. It should be remarked that this class of spaces has been studied independently by M. Valdivia [**39**]. In fact, Valdivia considers that class of Banach spaces, each member of which has weak* Corson-compact dual unit ball. It is obvious that every WLD Banach space belongs to the class of Valdivia. It follows from the results of Valdivia [**39**] and also from a further result by J. Orihuela, W. Schachermayer and M. Valdivia (see Proposition 4.1 of [**26**]), that the converse implication is also true. Therefore, a Banach space E is WLD, if and only if the dual unit ball of E is weak* Corson-compact. We shall prove in Proposition 1.2 below a more general result than this, by repeating essentially the same argument as in the proof of Proposition 4.1 of [**26**].

We discuss in the sequel an alternative purely topological definition of the class of WLD Banach spaces, which is similar to the definition of WCD Banach spaces given in the terms of descriptive set theory [**38, 35, 25**].

For an infinite cardinal α we denote by $L(\alpha)$ the Lindelof space of cardinality α with the unique (possible) nonisolated point (see [**1**]). It is clear that $L(\alpha)$ is a Hausdorff completely regular space.

We notice that:

(i) If $\alpha = \omega$, then the space $L(\omega)^\omega$ is identified with the Baire space of irrationals; and

(ii) as Alster and Pol have proved, the product $M \times L(\alpha)^\omega$ is a Lindelof space for every separable metric space [**1**].

Now using some results of Alster and Pol [**1**] and Gulko [**29**], we can show the next,

Proposition 1.2. *Let E be a Banach space. Then the following are equivalent*:

(a) *E is a* WLD *Banach space*;

(b) *E in its weak topology is a continuous image of a closed subset of the space $L(\alpha)^\omega$, for some infinite cardinal α*;

(c) *there is a* (*bounded*) *total subset* X *of* E *so that* X *in its weak topology is a continuous image of a closed subset of the space* $L(\alpha)^\omega$ *for some infinite cardinal* α;

(d) *the dual unit ball of* E *is a Corson-compact space in its weak** *topology.*

Proof. (b) $\Rightarrow$ (c) and (a) $\Rightarrow$ (d) are obvious.

(c) $\Rightarrow$ (a). Let $\phi : Y \subseteq L(\alpha)^\omega \to (X, w)$ a continuous onto mapping, where Y is a closed subset of $L(\alpha)^\omega$. Then, by a result of Gulko (see Proposition 1.4 of [**29**]), there is a continuous linear one-to-one mapping $Q : C_p(Y) \to \Sigma(\Gamma)$, for some set Γ. Consider the operator $Q \circ R \circ S : E^* \to \Sigma(\Gamma)$, where $R : C_p(X) \to C_p(Y)$ is defined by $R(f) = f \circ \phi$ for $f \in C_p(X)$, $S : E^* \to C_p(X)$, by $S(x^*) = x^*|X$ for $x^* \in E^*$; then it is easy to verify that this operator is linear one-to-one and weak* to pointwise continuous. We set $Z = (Q \circ R \circ S)(E^*)$, and $\Omega = (Q \circ R \circ S)(B)$, where B is the unit ball of E^*, then clearly $Z = \cup_{n<\omega} n\Omega$; since Ω is a pointwise compact subset of $\Sigma(\Gamma)$ there exists $\varepsilon_\gamma > 0$ for every $\gamma \in \Gamma$ such that $|x(\gamma)| \leq \varepsilon_\gamma$ for all $x \in \Omega$ and $\gamma \in \Gamma$. We define a linear one-to-one mapping by the rule $\Phi(x)(\gamma) = x(\gamma)/\varepsilon_\gamma$, for $x \in Z$ and $\gamma \in \Gamma$, and easily verify that the operator $T = \Phi \circ (Q \circ R \circ S) : E^* \to l_c^\infty(\Gamma)$ is bounded linear one-to-one and weak* to pointwise continuous. (d) $\Rightarrow$ (b). Since the unit ball B of E^* is a Corson-compact space in its weak* topology, there exists by a result of Alster and Pol [**1**] a closed subset Y of $L(\alpha)^\omega$ for some infinite cardinal α, and a continuous onto mapping $\phi : Y \to C_p(B)$. Since the space (E, w) is identified (via the natural mapping $x \in E \to x|B \in C_p(B)$) with a closed linear subspace of $C_p(B)$, we have the conclusion.

The proof of the proposition is complete. $\square$

Remarks. 1) It is clear from the above Proposition that the class of WLD Banach spaces is closed under finite products and for closed linear subspaces.

2) Similar results hold for the classes of WCD and weakly $\mathcal{K}$-analytic Banach spaces (see, for instance, Section 1 and also Theorem 4.1 of [**21**]).

3) We don't know if a Banach space E, which in its weak topology is a continuous image of closed subset of a space of the form $K \times L(\alpha)^\omega$, where K is a compact Hausdorff space is WLD.

Using known results, we easily obtain the next

Theorem 1.3. a) *Every* WLD *Banach space is weakly Lindelof and admits a bounded linear one-to-one operator into some* $c_0(\Gamma)$*, and thus it is strictly convexifiable.*

b) *Every* WCD *Banach space is* WLD.

c) *There is a* WLD *Banach space which is not* WCD.

d) *There exists a weakly Lindelof Banach space which is not a* WLD *space.*

Proof. a) The closed unit ball $K = (B_{E^*}, w^*)$ of E^* is by definition a Corson-compact, hence the space $C(K)$ is pointwise Lindelof [**1**] and admits a bounded linear one-to-one operator into $c_0(\Gamma)$ [**3**]. So the claim (a) easily follows.

b) This claim is an immediate consequence of the definition and the characterization of WCD Banach spaces via their duals (see [**21**]).

c) Concerning this claim we notice that in [**3**] a variety of non-Gulko Corson-compact spaces with property (M) is given; for every such K, the Banach space $C(K)$ is WLD but not WCD.

d) Such an example has been constructed by R. Pol [**28**]. ◻

Since the dual unit ball of a WLD space is weak* Corson-compact, the following result is rather expected. Note that this result has also been obtained in [**39**]. We give here a rather simpler proof.

Theorem 1.4. *Every* WLD *Banach space E admits a projectional resolution of identity* (P.R.I.)*, that is a family $\{P_\alpha : \omega \le \alpha \le \mu\}$ of projections on E, where $\mu = \dim E$, with the following properties.*

a) $||P_\alpha|| = 1$, $\omega \le \alpha \le \mu$, $P_\mu = \mathrm{id}_E$,

b) $P_\alpha \circ P_\beta = P_\beta \circ P_\alpha = P_\beta$, $\omega \le \beta < \alpha \le \mu$,

c) $\dim P_\alpha(E) \leq |\alpha|$, $\omega \leq \alpha \leq \mu$, *and*

d) *for every* $x \in E$ *the function* $\alpha \in [\omega, \mu] \to P_\alpha(x) \in E$ *is continuous, whenever the interval* $[\omega, \mu]$ *has the order and the space* E *has the norm topology.*

Proof. We briefly describe the proof of this theorem.

Let T be the operator which makes E a WLD space and also let $||\cdot||$ be any equivalent norm on E. We set $X = T(E^*)$, $K = T(B_{E^*})$, and we consider the space X endowed with pointwise topology; hence K is a compact subset of X. We assume (without restriction of generality) that $\Gamma = \cup_{x\in X}\sigma(x)$, so $\dim E = W(K) = |\Gamma|$, and we set $\mu = |\Gamma|$. Since K is a compact subset of X, there exists a family $\{\Gamma_\alpha : \omega \leq \alpha \leq \mu\}$ of subsets of Γ with the following properties (see Lemma 1.3 of [**3**]).

a) $\Gamma_\alpha \subseteq \Gamma_\beta$ for $\omega \leq \alpha < \beta \leq \mu$, and $\Gamma_\mu = \Gamma$,

b) $|\Gamma_\alpha| \leq |\alpha|$ for $\omega \leq \alpha \leq \mu$,

c) if $\omega \leq \alpha \leq \mu$ is a limit ordinal, then $\Gamma_\alpha = \cup_{\beta<\alpha}\Gamma_\beta$.

d) $K|\Gamma_\alpha \subseteq K$, $\omega \leq \alpha \leq \mu$.

We set $r_\alpha : X \to X : r_\alpha(x) = x|\Gamma_a$ for $\omega \leq \alpha \leq \mu$; by using the above properties and the fact that $X = \cup_{n<\omega} nK$, we conclude that r_α is a retraction of the space X.

We define for $\omega \leq \alpha \leq \mu$ a norm one projection of the Banach space E in the following way: $P_\alpha : E \to E$ such that $P_\alpha(x)(x^*) = T^{-1}(T(x^*)|\Gamma_\alpha)(x)$ for $x \in E$ and $x^* \in E^*$, and we easily verify that the family $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ is a P.R.I. for the space E. □

Since the class of WLD Banach spaces contains the closed linear subspaces of its members (see the remarks after Proposition 1.2), the above theorem and the known *renorming* technique of Troyanski-Zizler (see [**36, 40**]) have as a consequence the next,

Corollary 1.5. *Every* WLD *Banach space admits an equivalent locally uniformly convex norm.*

Remark. Let E be a WLD Banach space. Then it is not difficult to show that the density character of E is equal to the weak* density character of its dual E^* (cf. Corollary 3.7 of [**3**], and also the proof of Theorem 1.4).

By using a result of Sokolov [**34**] we show that a WLD Banach space is derived from a rather simple set. In fact, we get the next characterization of WLD Banach spaces, which also justifies the term *weakly Lindelof determined*.

Theorem 1.6. *For a Banach space E, the following are equivalent:*

a) *E is* WLD.

b) *There is a bounded total subset L of E such that:*

(i) *the point $0 \in E$ is the only (possible) weak limit point of L, and*

(ii) *the set $L \cup \{0\}$ is weakly Lindelof.*

Proof. b) $\to$ a). We set $T : E^* \to l^\infty(L)$, so that $T(x^*) = (x^*(l))_{l \in L}$. It is obvious that T is bounded linear one-to-one and weak* to pointwise continuous. We shall show that the range of T is contained in the subspace $l_c^\infty(L)$ of $l^\infty(L)$. So let $x^* \in E^*$. We set $S = \{l \in L : x^*(l) \neq 0\}$, and for $n < \omega$, $S_n = \{l \in L : |x^*(l)| \geq 1/n\}$, hence $S = \cup_{n<\omega} S_n$.

Now suppose that for some $n_0 < \omega$ the set S_{n_0} is uncountable. It is then clear by our assumption that there is a net $(l_\delta)_{\delta \in \Delta} \subseteq S_{n_0}$ such that $l_\delta \to 0$ weakly, therefore $x^*(l_\delta) \to x^*(0) = 0$, which is absurd.

a) $\to$ b). Let $T : E^* \to l_c^\infty(\Gamma)$ be the operator of Definition 1.1, and suppose that $||T|| \leq 1$. We set $K = T(B_{E^*})$, and $x_\gamma : K \to R$: $x_\gamma(x) = x(\gamma)$ for $\gamma \in \Gamma$ and $x \in K$. By a result of [**34**] there is a partition $\{\Gamma_n : n < \omega\}$ of Γ so that, for every $n < \omega$, we have that:

(i) the set $L_n = \{x_\gamma : \gamma \in \Gamma_n\}$ is a discrete subset of the space $C_p(K)$, and

(ii) the set $L_n \cup \{0\}$ is a closed subset of $C_p(K)$, where $C_p(K)$ denotes the space $C(K)$ endowed with pointwise topology.

We set $L = \cup_{n=1}^\infty (1/n) L_n$; clearly this set L is contained in the unit ball of $C(K)$.

Claim. The set L has, as the only possible limit point in the space $C_p(K)$, the point $0 \in C_p(K)$.

Proof of the claim. Let f be a limit point of L in the space $C_p(K)$, so that $f \neq 0$. Let U, V be disjoints open neighborhood of $f, 0$, respectively, in the space $C_p(K)$, and also let $(f_\delta)_{\delta\in\Delta}$ be a net in L converging pointwise to f. It follows that there is $\delta_0 \in \Delta$, such that $f_\delta \in U$ for every $\delta \in \Delta$ with $\delta \geq \delta_0(1)$. Let $\varepsilon > 0$, such that $B(0,\varepsilon) \subseteq V$; since $||x_\gamma|| \leq 1$ for $\gamma \in \Gamma$, there exists $n_0 < \omega$ so that, if $n \geq n_0$, then $||x_\gamma||/n < \varepsilon$ for all $\gamma \in \Gamma$. Hence,

$$\bigcup_{n>n_0} (1/n)L_n \subseteq S(0,\varepsilon) \subseteq V. \tag{2}$$

From (1) and (2) we get that if $\delta \geq \delta_0$, then $f_\delta \in \cup_{n\leq n_0}(1/n)L_n$, hence by the properties of the partition $\{\Gamma_n : n < \omega\}$ we have that $f \equiv 0$. The proof of the claim is complete.

Now we consider the natural embedding $\Phi : E \to C(K)$ (we recall that K is the closed unit ball of E^*). Since $\Phi(E)$ is a pointwise closed subspace of $C_p(K)$ and since the set $\{x_\gamma : \gamma \in \Gamma\}$ is contained in $\Phi(E)$, we conclude that L is the desired set. □

We recall that not every weakly Lindelof Banach space is a WLD space (see Theorem 1.3d)). But, as the next theorem shows, this is the case for Banach spaces with an unconditional basis.

This theorem also makes the content of the previous theorem more clear (cf. also Proposition 1.3 of [**33**]).

Theorem 1.7. *Let E be a Banach space with an unconditional basis Γ, so that $||\gamma|| \leq 1$ for $\gamma \in \Gamma$. Then the following are equivalent*:

1) *The set $\{0\} \cup \Gamma$ is weakly Lindelof.*

2) *E is* WLD.

3) *E is weakly Lindelof.*

4) *$l^1(\omega_1)$ does not isomorphically embed in E.*

5) *If Δ is an uncountable subset of Γ, then Δ is not equivalent to the usual basis of $l^1(\Delta)$.*

6) *There exist $\omega^* G_\delta$ extreme points in E^* (that is, every ω^* compact and convex subset of the dual E^* of E contains at least one $\omega^* G_\delta$ extreme point).*

7) *There is a bounded linear one-to-one operator* $T : E^* \to l_c^\infty(\Delta)$ *for some set* Δ.

Proof. (1) $\leftrightarrow$ (2) is obvious by the previous theorem and the fact that the set $\{0\} \cup \Gamma$ is weakly closed in E.

(2) $\to$ (3), (3) $\to$ (4) and (4) $\to$ (5) are clear by definition and standard results.

We show (5) $\to$ (2) and (5) $\to$ (6) simultaneously.

It suffices to show that the set $\sigma(x^*) = \{\gamma \in \Gamma : x^*(\gamma) \neq 0\}$ is at most countable for any $x^* \in E^*$. Indeed, if $\sigma(x^*)$ was uncountable, then there is an uncountable subset Δ of $\sigma(x^*)$ and a positive δ, so that $|x^*(\gamma)| \geq \delta$ for all $\gamma \in \Delta$. Now since the basis Γ is unconditional the set Δ is equivalent to the usual basis of $l^1(\Delta)$, a contradiction. Now we define the operator $T : E^* \to l_c^\infty(\Gamma)$, in the obvious way, that is $T(x^*) = (x^*(\gamma))_{\gamma\in\Gamma}$, which clearly makes E a WLD space. So we have proved the implication (5) $\to$ (2).

We notice that the compact and convex subset $K = T(B_{E^*})$ of $\Sigma([-1,1]^\Gamma)$ has, by the unconditionality, the following property:

($*$) If $x \in K$ and $y \in l_c^\infty(\Gamma)$ such that $|y(\gamma)| \leq |x(\gamma)|$ for all $\gamma \in \Gamma$, then $y \in K$.

We shall need the following:

Lemma. *Let* Ω *be a compact subset of* $[-1,1]^\Gamma$ *with property* ($*$), *and let* K *be the closed convex hull of* Ω. *Then we have*:

a) *K also has property* ($*$);

b) *if x is an extreme point of K and $y \in K$ is such that* $|y(\gamma)| \geq |x(\gamma)|$ *for all* $\gamma \in \Gamma$, *then* $y = x$, *and*

c) *if* $K \subseteq \Sigma([-1,1]^\Gamma)$, *then every extreme point of K is a G_δ point of K.*

Proof. We omit the easy proof of (a) and (b), and we show

c) Let x be an extreme point of K. We set for any finite subset F

of $\sigma(x)$ and $n < \omega$,

$$V(x, F, n) = \{y \in K : |y(\gamma) - x(\gamma)| < 1/n \text{ for } \gamma \in F\}.$$

Since the set $\sigma(x)$ is at most countable, the family $\{V(x, F, n)$: F is a finite subset of $\sigma(x)$ and $n < \omega\}$ is a countable family of open · neighborhoods of x in the space K. Let $y \in \cap\{V(x, F, n)$: F is a finite subset of $\sigma(x)$ and $n < \omega\}$, then clearly we have $|y(\gamma)| \geq |x(\gamma)|$ for all $\gamma \in \Gamma$, which implies by claim b) of the Lemma that $y = x$. So since K is a compact space, we get that x is a G_δ point of K.

The proof of the Lemma is complete. □

Now we get back to the proof of our implication. Let Ω be a compact and convex subset of E^* for which we assume without restriction of generality that it is contained in K. By (a) and (b) of the Lemma and Krein-Milman's theorem, we can assume that Ω has property $(*)$, so claim c) of the Lemma implies the desired result.

(6) $\to$ (5). Suppose that there is an uncountable subset Δ of Γ equivalent to the usual basis of $l^1(\Delta)$. Then the closed unit ball $\Omega = [-1, 1]^\Delta$ of the dual of $l^1(\Delta)$ is embedded as a weak* compact and convex subset in E^*, a contradiction, because Ω does not have G_δ extreme points.

It follows that the claims (1) to (6) are equivalent; since (2) $\Rightarrow$ (7) is obvious, it remains to show

(7) $\Rightarrow$ (5). Suppose for the purpose of contradiction that there exists a bounded linear one-to-one operator $T : E^* \to l_c^\infty(\Delta)$, and also that I is an uncountable subset of Γ equivalent to the usual basis of $l^1(I)$. Then clearly $l^\infty(I)$ is isomorphic to a (complemented) subspace of E^*, thus the restriction of T on the subspace $l^\infty(I)$ is a bounded linear one-to-one operator from $l^\infty(I)$ to $l_c^\infty(\Delta)$; but since I is uncountable, this contradicts the next result by Dashiell-Lindenstrauss [**7**].

Lemma. *Let I be an uncountable set and Z a Banach space with $c_0(I) \subseteq Z \subseteq l^\infty(I)$. Suppose that $T : Z \to l^\infty(\Delta)$ is a bounded linear operator such that $T|c_0(I)$ is one-to-one. Then there exists an uncountable set $A \subseteq I$, a one-to-one mapping $f : A \to \Delta$, and $\varepsilon > 0$ such that if $B \subseteq A$ with $\chi_B \in Z$ then $f(B) \subseteq \sigma_{\varepsilon/2}(T(\chi_B))$.*

Note that the above Lemma is implicitly contained in the proof of Theorem 2 of [**7**].

The proof of the theorem is complete. □

Corollary 1.8. *Let E be a Banach space with an unconditional basis. If E is a Gateaux differentiability space* (GDS) (*in particular, if E is a weak Asplund space*), *then E is weakly Lindelof.*

Proof. By a result of [**18**], every GDS Banach space E has ω^* G_δ extreme points in E^*. So Theorem 1.7 applies to E. □

Weakly Lindelof Banach spaces with an unconditional basis are useful because of the following,

Theorem 1.9. *If F is a* WLD *Banach space then* (*and only then*) *there exists a* WLD *Banach space E with an unconditional basis and a bounded linear operator $R : E \to F$ with dense range.*

Before the proof of this theorem, we shall describe some general facts concerning unconditionality, which we shall use in the proof of it.

Let Γ be a nonempty set and Ω a pointwise compact subset of the cube $[-1,1]^\Gamma$ with the following properties:

(a) if $x \in \Omega$ and $y \in [-1,1]^\Gamma$ so that $|y(\gamma)| \leq |x(\gamma)|$ for all $\gamma \in \Gamma$, then $y \in \Omega$, and

(b) for every $\gamma \in \Gamma$ there exists $x \in \Omega$ such that $x(\gamma) \neq 0$. (Clearly (a) is property $(*)$ used in the proof of Theorem 1.7).

We denote by Ω_1 the closed convex hull of Ω in $[-1,1]^\Gamma$, and by π_γ the projection at the γ-coordinate. We set E for the closed linear span of the family $\{\pi_\gamma : \gamma \in \Gamma\}$ in the Banach space $C(\Omega)$.

Under the above conditions, the following proposition has a routine proof.

Proposition 1.10. a) *The family $\{\pi_\gamma : \gamma \in \Gamma\}$ is an unconditional basis for the subspace E of $C(\Omega)$;*

b) *there exists a bounded linear one-to-one operator* $T : E^* \to l^\infty(\Gamma)$ *which is weak* to pointwise continuous, so that*

$$T(B_{E^*}) = \Omega_1.$$

Remark 1.11. The Banach space E is identified with the completion of the space Y of all real valued functions on $f : \Gamma \to R$, with finite support, endowed with the following norm,

$$||f|| = \sup\left\{\sum_{\gamma\in\Gamma} |f(\gamma)\cdot x(\gamma)| : x \in \Omega\right\}.$$

Proof of Theorem 1.9. Let $T : F^* \to l_c^\infty(\Gamma)$ be the operator of Definition 1.1. We assume that $||T|| \leq 1$, so the compact and convex $X = T(B_{F^*})$ is contained in $\Sigma([-1,1]^\Gamma)$, and also that for any $\gamma \in \Gamma$ there is $x \in X$ such that $x(\gamma) \neq 0$. We denote by Ω the compact set $\Omega = \{y \in [-1,1]^\Gamma$: there exists $x \in X$ such that $|y(\gamma)| \leq |x(\gamma)|$ for all $\gamma \in \Gamma\}$, and we notice that Ω has properties (a) and (b) above.

We consider the operator $\Phi : C(\Omega) \to C(X) : f \to \Phi(f) = f|X$ and set $R = \Phi|E$ to be the restriction of Φ to the subspace E of $C(\Omega)$ generated by the set of projections. Then it is easy to prove that the range of R is contained in the subspace F of $C(X)$ (recall that X is homeomorphic with the closed unit ball of F^*), and is dense in F. Now since E has by Proposition 1.10 an unconditional basis, the proof of the theorem is finished. (The converse of this theorem is obvious because the dual operator $R^* : F^* \to E^*$ is one-to-one and weak* to weak* continuous). □

Corollary 1.12. *Let F be a* WLD *Banach space. Then there exist* ω^* G_δ *extreme points in* F^*.

Proof. Let $R : E \to F$ be the operator of Theorem 1.9.

Then, since R has dense range, the dual operator $R^* : F^* \to E^*$ is one-to-one and of course weak* to weak* continuous, so Theorem 1.7 implies our claim. □

Conjecture. *Let E be a* WLD *Banach space. Then every nonempty weak* compact and convex subset of E^* contains a weak* exposed point. Otherwise, E is a* GDS.

Note that an affirmative answer to this conjecture would imply the existence (according to Theorem 3.6) of a GDS space which is not weak Asplund, and hence the solution of two problems by Larman and Phelps (see Problems 1 and 7 in [**18**]).

Section 2. In this section we define and investigate the properties of some special class of Banach spaces with an unconditional basis, which are useful for the construction of concrete examples.

Let Γ be a nonempty set, and $\mathcal{A}$ a family of subsets of Γ with the following properties:

(i) if $A \subseteq B$ and $B \in \mathcal{A}$, then $A \in \mathcal{A}$, and

(ii) $\{\gamma\} \in \mathcal{A}$ for all $\gamma \in \Gamma$.

We define a Banach space $E \equiv E_{0,1}(\mathcal{A})$ in the following way:

Definition 2.1. Let Y be the linear space of all real valued functions f on Γ with finite support (i.e., $\sigma(f) = \{\gamma \in \Gamma : f(\gamma) \neq 0\}$ is a finite set). For every $f \in Y$ we set,

$$||f|| = \sup\bigg\{\sum_{\gamma \in A} |f(\gamma)| : A \in \mathcal{A}\bigg\}. \tag{1}$$

It is easy to see that (1) defines a norm on the space Y. The Banach space $E \equiv E_{0,1}(\mathcal{A})$ is by definition the completion of Y with respect to the norm defined by (1), which we call the (0,1) norm of E (cf. also Proposition 1.10 and Remark 1.11).

It is simple to show that the family $\{e_\gamma : \gamma \in \Gamma\}$, where

$$e_\gamma(\delta) = \begin{cases} 1, & \gamma = \delta \\ 0, & \gamma \neq \delta \end{cases} \quad \text{for } \gamma \in \Gamma, \qquad \text{is an unconditional}$$

(normalized) basis for this space (with unconditional constant $\lambda = 1$).

Remark 2.2. We may also define an equivalent norm on Y as follows:

$$N(f) = \sup\left\{\left|\sum_{\gamma\in A} f(\gamma)\right| : A \in \mathcal{A}\right\}. \tag{2}$$

We notice that $N(f) \leq ||f|| \leq 2N(f)$ for all $f \in Y$. So the completion of the normed space (Y, N) is the space $E_{0,1}(\mathcal{A})$.

Now suppose that the family $\mathcal{A}$ is adequate ([**37**]), that is, $\mathcal{A}$ satisfies in addition the condition:

(iii) if $A \subseteq \Gamma$, and every finite subset of A belongs to $\mathcal{A}$, then $A \in \mathcal{A}$. We set,

$$\begin{aligned} K &\equiv K(\mathcal{A}) = \{x \in \{0,1\}^\Gamma : \sigma(x) \in \mathcal{A}\} \\ \Omega &\equiv \Omega(\mathcal{A}) = \{x \in \{0,\pm 1\}^\Gamma : \sigma(x) \in \mathcal{A}\} \\ D &\equiv D(\mathcal{A}) = \{x \in \{0,-1\}^\Gamma : \sigma(x) \text{ is a maximal set of the family } \mathcal{A}\}. \end{aligned}$$

It is immediate that K and Ω are compact spaces and also that $D, K \subseteq \Omega$. For every $\gamma \in \Gamma$ we define $\pi_\gamma : \Omega \to \{0, \pm 1\} : \pi_\gamma(x) = x(\gamma)$ for $x \in \Omega$. It is obvious that $\{\pi_\gamma : \gamma \in \Gamma\} \subseteq C(\Omega)$, and also that this family of functions separates the points of Ω; clearly a similar observation holds for the family $\{\pi_\gamma|K : \gamma \in \Gamma\}$ and the space $C(K)$.

The relation between the Banach space E and the spaces $C(\Omega)$ and $C(K)$ is given in the following

Lemma 2.3. (a) *The Banach space E is isometric* (*respectively, isomorphic*) *with a closed linear subspace of $C(\Omega)$* (*respectively $C(K)$*).

(b) *The compact space Ω is homeomorphically embedded as a compact subset in the unit sphere (S_{E^*}, w^*) of the dual space E^* of E, and the set of extreme points of the unit ball B_{E^*} of E^* is the image of D defined above under this homeomorphism.*

Proof. a) We define a linear isometry $T : E \to C(\Omega)$ in the obvious way, that is, $T(e_\gamma) = \pi_\gamma$ for $\gamma \in \Gamma$. It is easy to see that if $f \in E$ and $x \in \Omega$, then

$$T(f)(x) = \sum_{\gamma\in\Gamma} x(\gamma) \cdot f(\gamma).$$

Similarly, the space E is isomorphic with the closed linear span of the family $\{\pi_\gamma|K : \gamma \in \Gamma\}$ in the space $C(K)$ (cf. Remark 2.2).

b) We consider the dual operator $T^* : M(\Omega) \to E^*$ of T, and we notice that the restriction of T^* on the set $\{\delta x : x \in \Omega\}$ of Dirac measures on Ω is one-to-one. Since T^* is weak* to weak* continuous, we have the conclusion. □

Remark 2.4. A restatement of Lemma 2.3 (b) is the following: Let Ω_1 be the pointwise closure of the convex hull of the set Ω in the compact and convex set $[-1,1]^\Gamma$. Then the closed unit ball (B_{E^*}, w^*) of E^* is affinely homeomorphic with the set Ω_1; indeed, the operator $R : E^* \to l^\infty(\Gamma) : R(x^*)(\gamma) = x^*(e_\gamma)$ for $x^* \in E^*$ and $\gamma \in \Gamma$, makes the claim true (cf. the proof of Theorem 1.9).

Lemma 2.5. *Let $f \in E$ with $f \neq 0$. Then the following are equivalent*:

(a) *the $(0,1)$ norm of E is Gateaux differentiable at f;*

(b) *there exists a unique $x \in \Omega$, so that*

$$f(x) = ||f|| = \sum_{\gamma \in \sigma(x)} |f(\gamma)|;$$

in that case $x \in D$ and $\sigma(x) \subseteq \sigma(f)$;

(c) *the supremum norm of $C(\Omega)$ is Gateaux differentiable at f.*

Proof. The proof of this Lemma easily follows by the definitions and standard arguments (cf. [**27, 28**]). □

Remark 2.6. The above Lemma particularly implies that the Gateaux derivative of the norm of E at the point f (whenever it exists) belongs to the set $D \subseteq \Omega$, that is:

$$\partial(||\cdot||)(f) = x \in D.$$

Clearly in this case the point x is a weak* exposed point of the unit ball B_{E^*} of E^*.

We assume in the sequel that the Banach space $E_{0,1}(\mathcal{A})$ is always defined by an adequate family of countable sets of set Γ.

Proposition 2.7. *Let $\mathcal{A}$ be an adequate family of* (*countable*) *sets of the set Γ, then the* $(0,1)$ *norm of the Banach space $E \equiv E_{0,1}(\mathcal{A})$ is Gateaux differentiable at the points of a norm dense subset of E.*

Proof. Let $f \in E$ and $\varepsilon > 0$. We have to prove that there exists $g \in E$, so that the norm of E is Gateaux differentiable at g and, in addition, $||f - g|| \leq \varepsilon$. Suppose that $f \equiv 0$, then we consider a maximal set A of the family $\mathcal{A}$ and let $\{\gamma_n : n < \omega\}$ be a one-to-one enumeration of A. We define a function $g : \Gamma \to R$, so that $g(\gamma_n) = \varepsilon/2^n$ for $n = 1, 2, \ldots$, and $g(\gamma) = 0$ if $\gamma \in A$.

Then we have that $||0 - g|| = ||g|| = \sum_{n<\omega} g(\gamma_n)$, so Lemma 2.5 implies the claim.

Now suppose that $f \not\equiv 0$. We set $S = \sigma(f)$, and we consider the family $\mathcal{A}_s = \{A \cap S : A \in \mathcal{A}\}$ which clearly is an adequate family of subsets of the set S; so the space $F \equiv F_{0,1}(\mathcal{A}_S)$ is a complemented subspace of E. Since S is countable, the space F is separable, hence by Mazur's theorem the norm of F (and every continuous convex function on F) is Gateaux differentiable at the points of a dense G_δ subset of F. So let $h \in F$ (thus $\sigma(h) \subseteq S$) such that the norm of F is Gateaux differentiable at h, and $||f - h|| \leq \varepsilon/2$. By Lemma 2.5, there exists a unique (necessarily maximal) $A_0 \in \mathcal{A}_S$ with $A_0 \subseteq \sigma(h)$ so that $||h|| = \sum_{\gamma \in A_0} |h(\gamma)|$.

Let A be a maximal set in the family $\mathcal{A}$, containing the set A_0 (so $A \cap S = A_0$), and also let $\{\gamma_n : n < \omega\}$ be a one-to-one enumeration of the set $A \backslash A_0$. We define the function g in the following way:

$$g(\gamma) = \begin{cases} h(\gamma), & \gamma \in S \\ \varepsilon/2^n, & \gamma \in A\backslash A_0, \quad \gamma = \gamma_n \\ 0, & \gamma \in \Gamma\backslash(A \cup S), \end{cases}$$

and we notice that this is the desired g.

The proof of the Proposition is complete. □

We set G to be the set of all $f \in E$, so that the norm of E is Gateaux differentiable at $f \in E$, and also set $\Phi : G \to B_{E^*}$ to be the restriction of the subdifferentiable map of the norm of E at the set G, that is, $\Phi(f) = \partial(||\cdot||)(f)$ for $f \in G$. We conclude by the above proposition that G is a norm dense subset of E and that $\Phi(G) = D \subseteq B_{E^*}$ (cf. Remarks 2.4, 2.6).

We notice that the map $\Phi : G \to D$ is also continuous, whenever G is endowed with norm and D with the weak* topology, as a restriction of the subdifferentiable map of the norm of E, which is norm to weak* upper semicontinuous.

Our aim is to show that $\Phi : G \to D$ is moreover an open map. We shall need the next

Lemma 2.8. *Let $f \in E$ with $S = \sigma(f)$, $A \subseteq S$ be a maximal set of the family $\mathcal{A}$, and $B = S\setminus A$. Then we have*:

(a) *$f \in G$ and $\sigma(\Phi(f)) = A$ if and only if $\sum_{\gamma\in E}|f(\gamma)| < \sum_{\gamma\in A\setminus Y}|f(\gamma)|$ for any nonempty $E' \subseteq B$ and any $Y \subseteq A$ so that $Y \cup E \in \mathcal{A}$.*

(b) *For any nonempty $E' \subseteq B$ so that $E' \in \mathcal{A}$, it follows that*: $\sum_{\gamma\in E}|f(\gamma)| < \inf\{\sum_{\gamma\in A\setminus Y}|f(\gamma)| : Y \subseteq A \text{ with } Y \cup E' \in \mathcal{A}\}$.

Proof. (a) "$\to$". Let E' be a nonempty subset of B and $Y \subseteq A$ such that $Y \cup E' \in \mathcal{A}$. We set

$$y(\gamma) = \begin{cases} \left.\begin{array}{ll} 1, & f(\gamma) > 0 \\ -1, & f(\gamma) < 0 \end{array}\right\} \gamma \in E' \cup Y \\ 0, \quad \gamma \notin E' \cup Y \end{cases}.$$

It is clear that $y \in \Omega$. We have that

$$\begin{aligned} f(y) &= \sum_{E'\cup Y} f(\gamma)\cdot y(\gamma) = \sum_{E'}|f(\gamma)| + \sum_{Y}|f(\gamma)| \\ &< \sum_{A}|f(\gamma)| = ||f|| = \sum_{A\setminus Y}|f(\gamma)| + \sum_{Y}|f(\gamma)|, \end{aligned}$$

so

$$\sum_{E'} |f(\gamma)| < \sum_{A \backslash Y} |f(\gamma)|.$$

Conversely, we define

$$x : \Gamma \to R : x(\gamma) = \begin{cases} 1, & \gamma \in A \text{ and } f(\gamma) > 0 \\ -1, & \gamma \in A \text{ and } f(\gamma) < 0 \\ 0, & \gamma \notin A. \end{cases}$$

So $\sigma(x) = A$ and thus $x \in D \subseteq \Omega$. Now it is easily verified that x is the Gateaux derivative of the norm of E at f.

b) We set $X = \{\chi_Y : Y \subseteq A \text{ and } E' \cup Y \in \mathcal{A}\} \subseteq K$, and

$$\phi : X \to R : \phi(\chi_Y) = ||f|| - \sum_{Y} |f(\gamma)| \qquad \Big(= \sum_{A \backslash Y} |f(\gamma)| \Big).$$

It is clear that X is a compact subset of K (and hence of Ω) and that ϕ is a continuous map. It follows that ϕ attains its minimum at a point χ_{Y_0} of X; combining this fact together with the claim (a), we conclude (b) of the Lemma. □

Theorem 2.9. *The map* $\Phi : G \to D$ *is* (*continuous onto and*) *open.*

Proof. It is not difficult to see that it is enough to show that Φ is open at the points of the dense subset $G_1 = \{f \in G : \sigma(f) \backslash \sigma(\phi(f))$ is a finite set$\}$ of G. So let $f \in G_1$ and $\varepsilon > 0$. We set $S = \sigma(f)$, $\Phi(f) = x$, $A = \sigma(x)$ and $B = S \backslash A$. We have to show that the set $V = \Phi(B(f, \varepsilon) \cap G)$ is a neighborhood of $x = \Phi(f)$ in the space D.

We assume, without restriction of generality, that B is nonempty and A is an infinite (countable) set. We set $\mathcal{A}_B = \{X \cap B : X \in \mathcal{A}\}$. It is then clear that $\mathcal{A}_B$ is an adequate family of subsets of the (finite) set B. Let $\{E_1, E_2, \ldots, E_n\}$ be an enumeration of nonempty sets of the family $\mathcal{A}_B$. For every $k = 1, 2, \ldots, n$, we set

$$a_k = \inf \Big\{ \sum_{A \backslash Y} |f(\gamma)| : Y \subseteq A \text{ and } E_k \cup Y \in \mathcal{A} \Big\}.$$

It follows from Lemma 2.8b that

$$(1) \qquad a_k > \sum_{E_k} |f(\gamma)| \qquad \text{for all } k = 1, 2, \dots, n.$$

Now we choose a finite subset Δ of A so that

$$(2) \qquad \sum_{A \setminus \Delta} |f(\gamma)| < \min \left\{ a_k - \sum_{E_k} |f(\gamma)| : \ k = 1, 2, \dots, n \right\}.$$

Let $k \leq n$ and $Y \subseteq \Gamma$ such that $E_k \cup Y \in A$; then by using (1) and (2), we get that

$$(3) \qquad \sum_{\Delta \setminus Y} |f(\gamma)| > \sum_{E_k} |f(\gamma)|.$$

Since A is an infinite set, we may choose the finite set Δ so that (in addition to (2)) it satisfies the following condition

$$(4) \qquad \sum_{A \setminus \Delta} |f(\gamma)| < \varepsilon/2.$$

We define $U = [\{x(\gamma)\}_{\gamma \in \Delta} \times \{0\}_{\gamma \in B} \times \Pi\{0, \pm 1\}^{\Gamma \setminus (\Delta \cup B)}] \cap D$. It is obvious that U is an open and closed neighborhood of $\Phi(f) = x$ in D. We shall show that $U \subseteq \Phi(B(f, \varepsilon) \cap G)$. So let $y \in U \setminus \{x\}$, and set $A_1 = \sigma(y)$ (hence A_1 is a maximal set of $\mathcal{A}$, $A_1 \cap B = \varnothing$ and $\Delta \subseteq A_1$). We distinguish two cases:

I) $A_1 = A$. We set $g : \Gamma \to R$ such that

$$g(\gamma) = \begin{cases} f(\gamma), & \gamma \in \Delta \cup B \\ f(\gamma), & \gamma \in A \setminus \Delta \text{ and } y(\gamma) = 1 \\ -f(\gamma), & \gamma \in A \setminus \Delta \text{ and } y(\gamma) = -1. \end{cases}$$

Then it is easy to show that $||f - g|| \leq \varepsilon$, $g \in G$ and $\Phi(g) = y$.

II) $A_1 \neq A$. Since A and A_1 are maximal sets of $\mathcal{A}$, we have that $A_1 \setminus A \neq \varnothing$ and $A \setminus A_1 \neq \varnothing$. It is also obvious that $\Delta \subseteq A \cap A_1$. We consider positive numbers $\varepsilon_\gamma > 0$ for all $\gamma \in A_1 \setminus A$ so that

$$(5) \qquad \sum_{\gamma \in A_1 \setminus A} \varepsilon_\gamma = \sum_{\gamma \in A \setminus A_1} |f(\gamma)|$$

and we define $g : \Gamma \to R$ such that

$$g(\gamma) = \begin{cases} f(\gamma), & \gamma \in \Delta \cup B \\ f(\gamma), & \gamma \in (A_1 \cap A)\backslash\Delta \text{ and } y(\gamma) = 1 \\ -f(\gamma), & \gamma \in (A_1 \cap A)\backslash\Delta \text{ and } y(\gamma) = -1 \\ \varepsilon_\gamma, & \gamma \in A_1\backslash A \text{ and } y(\gamma) = 1 \\ -\varepsilon_\gamma, & \gamma \in A_1\backslash A \text{ and } y(\gamma) = -1. \end{cases}$$

Now by using (4), (5) and Lemma 2.8a we may easily show that $||f - g|| < \varepsilon$, $g \in G$ and $\Phi(g) = y$. The proof of the Theorem is complete. $\square$

Notation 2.10. Let Γ be a nonempty set and A, B disjoint finite subsets of Γ. In the sequel we shall denote by V_A^B the basic clopen set $\{1\}_A \times \{0\}_B \times \Pi\{0,1\}^{\Gamma\backslash(A\cup B)}$ of the space $\{0,1\}^\Gamma$; whenever $B = \varnothing$, we set V_A for the clopen set $\{1\}_A \times \Pi\{0,1\}^{\Gamma\backslash A}$.

As a first application of the above theorem, we show the next

Theorem 2.11. *We assume continuum hypothesis* (CH). *Then there exists on the Banach space $l^1(\omega_1)$ an equivalent norm which is "densely" but not "densely G_δ" Gateaux differentiable.*

Proof. From CH follows the existence of a compact nonmetrizable subset L of the space $\Sigma(\{0,1\}^{\omega_1})$ with a strictly positive (regular Borel probability) measure μ (i.e. L is a Corson-compact without property (M) (see Theorem 3.12 of [**3**]). Hence there exists an uncountable subset I of ω_1 such that for all $\zeta \in I$, $\mu(V_\zeta \cap L) \geq \delta$ for some $\delta > 0$. We assume (without restriction of generality) that $I = \omega_1$, and we set $X = \{x \in L$: if $y \in L$ and $\sigma(x) \subseteq \sigma(y)$ then $x = y\}$, and $\mathcal{A} = \{A \subseteq \omega_1$: there is $x \in X : A \subseteq \sigma(x)\}$. It is clear that $\mathcal{A}$ is an adequate family of (countable) subsets of ω_1 and that $X \subseteq L \subseteq K \equiv K(\mathcal{A})$. It is also obvious that $X = \{x \in \{0,1\}^{\omega_1} : \sigma(x)$ is a maximal set of $\mathcal{A}\}$. We consider the measure μ as a measure on the Corson-compact K, so the support of μ is the space L and $\mu(V_\zeta \cap K) \geq \delta$ for all $\zeta < \omega_1$.

Claim. *Let L be a compact* ccc *subset of the space $\{0,1\}^\Gamma$. Then the space $X = \{x \in L$: if $y \in L$ and $\sigma(x) \subseteq \sigma(y)$ then $x = y\}$ is also a* ccc *space.*

Proof of the claim. It suffices to show that the family of nonempty clopen subsets of X of the form $V_F \cap X$ for F a finite subset of Γ, is a basis for the topology of X. Indeed, let $U = V_{F_1}^{F_2} \cap X$ be a nonempty basic clopen subset of X, and let $x = \chi_A \in U$. By the definition of the space X, for any $\zeta \in F_2$ there exists a finite subset A_ζ of A so that $V_{A_\zeta} \cap V_\zeta \cap X = \varnothing$. We set $F' = \cup\{A_\zeta : \zeta \in F_2\} \cup F_1$, so F' is a finite subset of A, and we easily verify that $x \in V_{F'} \cap X \subseteq U$, which finishes the proof of the claim.

Since the space L is the support of the measure μ, we get that L is a ccc space; hence by the claim X is also a ccc space which easily implies that the space $D \equiv D(\mathcal{A})$ is a ccc space. As we have seen, the family $\{\pi_\zeta : \zeta < \omega_1\}$ is an unconditional basis for its closed linear span (see Definition 2.1 and Lemma 2.3); since $\mu(V_\zeta \cap K) \geq \delta$ for $\zeta < \omega_1$, we get that this family is equivalent to the usual basis of the Banach space $l^1(\omega_1)$. On the other hand, the closed linear span of this set in $C(K)$ is isomorphic with the Banach space $E \equiv E_{0,1}(\mathcal{A})$. So $E \simeq l^1(\omega_1)$. It follows by Proposition 2.7 that the set G of Gateaux differentiability points of the (0,1) norm of E is a norm dense subset of E. Now if there exists a dense G_δ subset G_1 of E on which the norm of E is Gateaux differentiable, then $G_1 \subseteq G$ and since the map $\Phi : G \to D$ is continuous onto and open, we would have from a result of Coban and Kenderov [**5**] that the space D contains a completely metrizable dense subset M. But since D is ccc, M would be ccc and so separable. It follows that D is separable, which implies that L is a metrizable space, a contradiction. □

Section 3. The aim of this section is the construction of suitable examples of WLD Banach spaces which indicate the considerable difference of this class and the class of WCD Banach spaces.

As is known, every WCD Banach space, E, admits an equivalent norm, the dual norm of which is strictly convex, so E is in particular weak Asplund and smoothable (cf. Theorem 4.8 of [**21**]). It is natural to ask if the existence of such a good norm on a WLD Banach space implies "weakly countably determiness." The answer to this question is negative as the next example (a WLD Banach space of the form $C(K)$) shows. The Corson-compact K given in this example admits a σ-distributively point-finite To-separating open cover, (otherwise K is a Gruenhage space), though it is not a Gulko-compact; and this answers a question of G. Gruenhage [**16**].

The next simple Lemma is a consequence of the "dual operation."

Lemma 3.1. *Let K be a compact Hausdorff space and S a closed subspace of K. Then the space $M(S)$ of Radon measures on S is identified with the weak* closed complemented subspace of $M(K)$ of all measures μ on K which are supported on S. Hence, any measure μ on K is written in a unique way, as $\mu = \mu_0 + \mu_1$ so that μ_0 supported on S and μ_1 vanished on that set.*

Proof. We consider the operator $T : C(K) \to C(S) : T(f) = f|S$ and simply verify that the dual operator $T^* : M(S) \to M(K)$ is a linear isometry with range the set of those measures μ on K such that $\operatorname{supp}(\mu) \subseteq S$, and in addition an isomorphism for the weak* topologies; so $M(S) \subseteq M(K)$. To see that $M(S)$ is a complemented subspace, we notice that the mapping,

$$\pi : M(K) \to M(S) : \pi(\mu)(f) = \int_S f\, d\mu,$$

is a linear projection with range $M(S)$. Since $\pi(\mu) = 0$, if and only if μ is concentrated on $K \backslash S$, we conclude that any measure μ on K has a unique decomposition, $\mu = \mu_0 + \mu_1$ so that $\operatorname{supp}(\mu_0) \subseteq S$ and $|\mu_1|(S) = 0$. □

Lemma 3.2. *Let K be a compact subset of $\{0,1\}^\Gamma$, defined by an adequate family of subsets of Γ. We denote by S the compact subset $S = \{\chi_{\{\gamma\}} : \gamma \in \Gamma\} \cup \{0\}$ of K, then we have*:

(a) *If $\mu \in M(K)$, then μ is supported by S if and only if $\mu(V_A \cap K) = 0$ for all $A \subseteq \Gamma$, such that $2 \leq |A| < \omega$ (cf.* Note 2.10*). In that case μ is an atomic measure and hence the space $M(S)$ is identified with $l^1(S)$.*

(b) *There exists an equivalent dual norm N on $M(K)$ such that $N(\mu) = |\mu_0| + ||\mu_1||$, where $\mu = \mu_0 + \mu_1$ is the unique decomposition of μ into two measures, according to the previous lemma. Furthermore, the norm $|\cdot|$ on $l^1(S)$ is strictly convex.*

Proof. The set $S = \{\chi_{\{\gamma\}} : \gamma \in \Gamma\} \cup \{0\}$ is closed with unique limit point, the point $0 \in K$, so S is a compact scattered subset of K; hence

$M(S) = l^1(S)$ is the space of all Radon measures on K supported by S.

Now we shall prove claim (a). Let $\mu \in M(K)$, such that $\mu(V_A \cap K) = 0$ for all $A \subseteq \Gamma$: $2 \leq |A| < \omega$, and also let $\gamma \in \Gamma$; $\mu(V_\gamma \cap K) \neq 0$. Since $V_\gamma = \chi_{\{\gamma\}} \cup U\{V_A : \gamma \in A,\, 2 \leq |A| < \omega\}$, and since μ is regular we find that $\mu(V_\gamma \cap K) = \mu(\chi_{\{\gamma\}})$. But $K = \{0\} \cup U\{V_\gamma \cap K : \gamma \in \Gamma\}$, so there exists a sequence of reals $r_0, r_1, \ldots, r_n, \ldots$ and a sequence $\gamma_1, \ldots, \gamma_n$ of points of Γ, so that $\mu = \sum_{n=0}^{\infty} r_n \cdot \delta_{x_n}$, where $x_0 = \{0\}$ and $x_n = \chi_{\{\gamma_n\}}$, $n = 1, 2, \ldots$. The converse of (a) is obvious.

Concerning (b), we notice that Lemma 3.1 implies that $M(K) = l^1(S) + Ł$, where $Ł = \{\mu \in M(K) : \mu$ vanished on $S\}$. Since S is an Eberlein compact, there exists an equivalent dual norm $|\cdot|$ on $M(S) = l^1(S)$, which is strictly convex. We consider such a norm on $l^1(S)$, so that $|\mu| \leq ||\mu||$ for all $\mu \in l^1(S)$.

Let $\mu \in M(K)$, then $\mu = \mu_0 + \mu_1$, where $\mu_0 \in l^1(S)$ and $\mu_1 \in Ł$, we define $N(\mu) = |\mu_0| + ||\mu_1||$. It is clear that N is an equivalent norm on $M(K)$. We shall prove that N is also a dual norm. Let $(\mu_i)_{i\in I}$ be a net in $M(K)$, with $N(\mu_i) \leq 1$ for all $i \in I$, and so that $\mu_i \xrightarrow{w^*} \mu$. It is enough to prove that $N(\mu) \leq 1$. Let $\mu_i = \mu_i^0 + \mu_i^1$, where $\mu_i^0 \in l^1(S)$ and $\mu_i^1 \in L$ for all $i \in I$, and also let λ, v be limit points of the nets $(\mu_i^0)_{i\in I}$ $(\mu_i^1)_{i\in I}$, respectively, so that $\mu_{j,j\in J}^0 \xrightarrow{w^*} \lambda$ and $\mu_{j,j\in J}^1 \xrightarrow{w^*} v$ for some cofinal subset J of I; hence, $\mu = \lambda + v$. Since the norms $||\cdot||$ and $|\cdot|$ on $M(K)$ and $l^1(S)$, respectively, are dual norms, and since $l^1(S)$ is a weak* closed subspace of $M(K)$, we get that

$$\lambda \in l^1(S), \qquad |\lambda| \leq \liminf_{j\in J} |\mu_j^0|, \tag{1}$$

and

$$||v|| \leq \liminf_{j\in J} ||\mu_j^1||. \tag{2}$$

Let $v = v_0 + v_1$, then since the measures v_0, v_1 are disjoint,

$$||v|| = ||v_0|| + ||v_1||. \tag{3}$$

Now (2) and (3) imply

$$N(v) = |v_0| + ||v_1|| \leq ||v_0|| + ||v_1|| = ||v|| \leq \liminf_{j\in J} ||\mu_j^1||. \tag{4}$$

Eventually, from (1) and (4), we conclude that

$$N(\mu) = N(\lambda + v) \leq N(\lambda) + N(v) \leq \liminf_{j\in J} |\mu_j^0| + \liminf_{j\in J} ||\mu_j^1||$$
$$\leq \liminf_{j\in J}(|\mu_j^0| + ||\mu_j^1||) = \liminf_{j\in J} N(\mu_j) \leq 1.$$

The proof of the lemma is complete. $\square$

We shall state the main result.

Theorem 3.3. *There exists a Corson and Rosenthal compact space* Ω (*and hence*) *with the property* (M) *such that*:

(a) Ω *is not a Gulko compact.*

(b) *The space* $M(\Omega)$ *of Radon measures on* Ω *admits an equivalent strictly convex dual norm.*

(c) Ω *is a Gruenhage space and hence a fragmentable space* (*cf.* [**32**]).

The space Ω is a slight modification of an earlier example which appeared in [**35**, Theorem 6.58] (see also Theorem 4.4 of [**3**]). For the proof of the above theorem, we need a description of Ω and two lemmas.

The space Ω. Let $\mathcal{N} = \{N_\zeta : \zeta < \omega_1\}$ be an almost disjoint family of subsets of ω; we define an adequate family of subsets of ω_1 with the next rule: A finite subset $F = \{\zeta_1 < \zeta_2 < \cdots < \zeta_n\}$ of ω_1 is said to be admissible if and only if

$$(1) \qquad |N_{\zeta_k} \cap N_{\zeta_l}| \geq \max\{k, l-k\} \qquad \text{for } 1 \leq k \leq l \leq n.$$

It is clear that every subset of an admissible set is also admissible so the family,

$$\mathcal{A} \equiv \mathcal{A}(\mathcal{N}) = \{A \subseteq \omega_1 : \text{ every finite subset of } A \text{ is admissible}\},$$

is adequate, and hence the space $K = \{\chi_A : A \in \mathcal{A}\}$ is a compact subset of $\{0,1\}^{\omega_1}$. It is easy to see that if $x \in K$, then $x = \chi_A$ for $A \in \mathcal{A}$, and order type $(A) \leq \omega$. So K is a Corson-compact of bounded order type and hence has the property (M) (see Proposition 4.10 of [**3**]).

Now let $\phi : \omega_1 \to [0,1]$ be a one-to-one function. We define an adequate subfamily $\mathcal{A}'$ of the family $\mathcal{A}$ in the following way: A subset A of ω_1 belongs in $\mathcal{A}$, if and only if every finite subset $F = \{\zeta_1 < \zeta_2 < \cdots < \zeta_n\}$ of A' is admissible and, in addition,

$$|\phi(\zeta_l) - \phi(\zeta_k)| \leq 1/k \qquad \text{for } 1 \leq k < l < n. \tag{2}$$

We set $\Omega = \{\chi_A : A \in \mathcal{A}'\}$. It is clear that Ω is a compact subset of K; moreover, every $A \in \mathcal{A}'$ by (2) has a unique limit point in the interval $[0,1]$, so A is a countable G_δ subset of $[0,1]$ and hence $\chi_A : [0,1] \to R$ is a Baire-1 function, implying that Ω is a Rosenthal compact.

Lemma 3.4. *There exists an almost disjoint family $\{N_\zeta : \zeta < \omega_1\}$ of subsets of ω, so that the Corson-compact Ω defined by this family is not a Gulko-compact.*

Proof. The construction of the family $\{N_\zeta : \zeta < \omega_1\}$ and the proof is the same as in the example mentioned before (see Theorem 4.4 of [**3**]). □

Lemma 3.5. *For every Corson-compact K defined by an almost disjoint family $\{N_\zeta : \zeta < \omega_1\}$ of subsets of ω as above, the space $C(K)$ admits an equivalent (necessarily) smooth norm, the dual norm of which is strictly convex.*

Proof. We set $L^n = \{\chi_{V_A \cap K} : |A| = n\}$ for $n = 1, 2, \dots$ (cf. Not. 2.10), and also set $L = (\cup_{n=1}^{\infty} L^n) \cup \{1\}$. It is clear that the linear span of the set L is a subalgebra of $C(K)$ containing the constant functions and separating the points of K, hence by Stone-Weierstrass theorem it is a norm dense subspace of $C(K)$. We consider the (bounded linear one-to-one) operator $T : M(K) \to l^\infty(L)$ defined by $T(\mu) = (\mu(f))_{f \in L}$ which actually is weak* to pointwise continuous, and we set $D = \cup_{n \geq 2} L^n$.

Claim. *There exists a partition $\{D_m : m = 1, 2, \dots\}$ of D, so that if we define the (bounded linear) operator $S : l^\infty(L) \to l^\infty(D)$ such that*

$$S(f) = \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot (f | D_m),$$

then $S \circ T[M(K)]$ is a subspace of the Banach space $c_0(D)$.

Proof of the claim. We define the following partition of D,

$$D_m = \{\chi_{V_A \cap K} \in D : \max\{|N_\zeta \cap N_\xi| : \zeta \neq \xi, \zeta, \xi \in A\} = m\} \quad \text{for } m = 1, 2, \dots .$$

We shall show that for every $x \in K$ and $m = 1, 2, \dots$, the set $x(m) = \{\chi_{V_A \cap K} \in D_m\colon\ \delta_x(V_A \cap K) \neq 0\}$ is finite. Indeed, since the order type of $(x, <)$ is ω, we write the support of x as $\sigma(x) = \{\zeta_1 < \zeta_2 < \dots, \zeta_n < \dots\}$, also $\delta_x(V_A \cap K) \neq 0$ is the same as to say $A \subseteq \sigma(x)$. Now let $\chi_{V_A \cap K} \in x(m)$, and $A = \{\zeta_{n_1} < \zeta_{n_2} < \dots < \zeta_{n_q}\} \subseteq \sigma(x)$; since A is an admissible set, we get that

$$|N_{\zeta_{n_k}} \cap N_{n_l}| \geq \max\{n_k, n_l - n_k\} \qquad \text{for } 1 \leq k < l \leq q,$$

since $\chi_{V_A \cap K} \in x(m)$, we conclude that

$$m \geq \max\{\max\{n_k, n - n_k\} :\ 1 \leq k \leq q\},$$

which clearly implies that $n_q < 2m$. So $x(m)$ is a finite set.

Since the operator $S \circ T$ is weak* to pointwise continuous, it follows from the above that the set $S \circ T(K)$ is pointwise and hence weakly compact subset of $c_0(D)$ (recall that the weak and the pointwise topology coincide on bounded subsets of $c_0(D)$). We show further that the range of the operator $S \circ T$ is contained in $c_0(D)$.

Set $X = \{\pm\delta_t : t \in K\} \subseteq M(K) \equiv C(K)^*$; then by Krein-Milman's theorem, the closed unit ball B of $M(K)$ is equal to the weak* closure of the convex hull $\langle X \rangle$ of X, namely $B = \overline{\langle X \rangle}^*$. We notice that $S \circ T(\langle X \rangle) = \langle S \circ T(X) \rangle$, so by Krein's theorem the convex hull $\langle S \circ T(X) \rangle$ of the weakly compact subset $S \circ T(X)$ of $c_0(D)$ is weakly relatively compact, hence pointwise relatively compact. It follows from the weak*-pointwise continuity of the operator $S \circ T$ that $S \circ T(B) = S \circ T(\overline{\langle X \rangle}^*) \subseteq \overline{S \circ T(\langle X \rangle)}^p = \overline{\langle S \circ T(X) \rangle}^p \subseteq c_0(D)$; therefore, $S \circ T(M(K)) \subseteq c_0(D)$.

We notice that if $\mu, v \in M(K)$ and $\mu = \mu_0 + \mu_1$, $v = v_0 + v_1$ are the unique decompositions of μ and v according to Lemma 3.1, then Lemma 3.2a easily implies that $S \circ T(\mu) = S \circ T(v)$, if and only if $\mu_1 = v_1$.

To finish our proof, we define an equivalent norm $||\cdot||$ on $M(K)$ in the following way,

$$|||\mu||| = ((N(\mu))^2 + (||S \circ T(\mu)||_D))^{1/2},$$

where N is the (dual) norm defined by Lemma 3.2b and $||S \circ T(\mu)||_D$ denotes $\cdot$ the Day's norm of the space $c_0(D)$. It is clear that $|||\cdot|||$ is a dual norm on $M(K)$. It remains to prove that $|||\cdot|||$ is a strictly convex norm. Let $\mu = \mu_0 + \mu_1$, $v = v_0 + v_1$ are measures on K so that $|||\mu||| = |||v||| = |||(\mu + v)/2||| = 1$, then by standard arguments we have that

$$N(\mu) = N(v) = N((\mu + v)/2), \tag{1}$$

and

$$||S \circ T(\mu)||_D = ||S \circ T(v)||_D = ||S \circ T((\mu + v)/2)||_D. \tag{2}$$

By the strict convexity of Day's norm on $c_0(D)$, we get from (2) that $S \circ T(\mu) = S \circ T(v)$, namely $\mu_1 = v_1$, so (1) implies that $|\mu_0| = |v_0| = |(\mu_0 + v_0)/2|$, and by the strict convexity of $|\cdot|$, we get that $\mu_0 = v_0$, which finishes the proof of the Lemma. $\square$

Proof of Theorem 3.3. We consider an almost disjoint family $\mathcal{N} = \{N_\zeta : \zeta < \omega_1\}$ of subsets of ω, as in Lemma 3.4, and we set Ω to be the space defined by the adequate family $\mathcal{A}' = \mathcal{A}'(\mathcal{N})$ so Ω is not a Gulko compact. Since Ω is a closed subspace of the Corson-compact K defined by the adequate family $\mathcal{A} = \mathcal{A}(\mathcal{N})$, we conclude from Lemmas 3.1 and 3.5 that $M(\Omega)$ admits a dual strictly convex norm. Hence, claims (a) and (b) have been proved.

Now we prove claim (c). We recall that a topological space Ω is said to be a Gruenhage space (see [**16**] and Definition 2.1 of [**32**]), if there exists a family $U = \cup_{n<\omega} U_n$ of open subsets of Ω so that for all $x, y \in \Omega$ with $x \neq y$ there exist $n \in \omega$ and $V \in U_n$ which separate x, y and $\operatorname{ord}(x, U_n) < \omega$ or $\operatorname{ord}(y, U_n) < \omega$ (where $\operatorname{ord}(x, U_n)$ denotes the cardinality of the set $\{V \in U_n : x \in V\}$).

We set for $\zeta < \omega_1$ and $m = 1, 2, \ldots$, $V_{\zeta,m} = \cup\{V_{\{\zeta,\eta\}} \cap \Omega : \zeta < \eta < \omega_1$ and $|\phi(\zeta) - \phi(\eta)| > 1/m$, and $U_m = \{V_{\zeta,m} : \zeta < \omega_1\}$. We also set $U_0 = \{V_\zeta \cap \Omega : \zeta < \omega_1\}$.

It is easy to verify that the family $U = \cup_{m<\omega} U_m$ makes Ω a Gruenhage space. It then follows by Proposition 2.2 of [**32**] that Ω is a fragmentable space, that is, there is a metric d on Ω so that, for every $\varepsilon > 0$ and each nonempty subset Y of Ω there is a nonempty relatively open subset U of Y such that d-diam $(U) \leq \varepsilon$. The proof of the theorem is complete. $\square$

The WLD Banach space $C(K)$ described in Theorem 3.3 is "dually" strictly convexifiable and thus (by using standard results) weak Asplund and smoothable. Now we give an example of a WLD Banach space E with an unconditional basis, whose dual space is strictly convexifiable, which has neither of these properties. This example provides a solution of a problem of Larman and Phelps (see Problem 6 of [**18**]):

"If the Banach space E is weakly Lindelof in its weak topology, is it a weak Asplund space?"

Theorem 3.6. *There exists a Corson-compact space K with property* (M)*, such that*:

a) *K does not contain a dense G_δ metrizable subset and* (*hence*)

b) *the Banach space $C(K)$ is neither weak-Asplund nor smoothable, and*

c) *$C(K)$ contains a closed linear subspace E* (*so E is weakly Lindelof*) *with an unconditional basis, which is not weak Asplund nor smoothable.*

We first give a description of the space K.

A finite subset $A = \{t_1 < t_2 < \cdots < t_n\}$ of the unit interval $I = [0, 1]$ is called admissible, if and only if

$$t_n - t_m < 1/m \qquad \text{for } m < n.$$

It is clear that if A is admissible then every subset of A is also admissible; so the family

$$\mathcal{A} = \{A \subseteq I : \text{ every finite subset of } A \text{ is admissible}\},$$

is adequate. It follows that the spaces

$$\Omega = \{x \in \{0, \pm 1\}^I : \sigma(x) \in \mathcal{A}\}, \quad \text{and} \quad K = \{\chi_A : A \in \mathcal{A}\},$$

are compact.

We notice that if A is an infinite set belonging in $\mathcal{A}$, then A in its natural order is well ordered with order type A which is equal either to ω (so $A = \{t_1 < t_2 < \cdots < t_n < \cdots\}$, $n < \omega$) or to $\omega + 1$ (and so $A = \{t_1 < t_2 < \cdots < t_n < \cdots < t\}$); it is obvious that, in the second case, we have $t = \lim_{n\to\infty} t_n$. So K and Ω are Corson compact spaces. Moreover, it follows by the above remarks that every $A \in \mathcal{A}$ is a scattered subset of I, so $\chi_A : I \to R$ is a Baire-1 function, which implies that K and Ω are Rosenthal-compacts; thus these spaces have the property (M).

For the proof of (a) of the above Theorem, we shall need two simple lemmas.

Lemma 1. *Let A be a nonempty admissible set. If $a = \max A < 1$, then there exists $b \in (a, 1)$ so that the set $A \cup \{t\}$ is also admissible for any $t \in (a, b)$.*

Proof. Let $A = \{t_1 < t_2 < \cdots < t_n\}$, so $a = t_n$. We set $t'_m = t_m + 1/m$ for $m \leq n$. It is easy to see that $t'_m > a$ for all $m \leq n$, so if $b = \min\{1, t'_1, \ldots, t'_n\}$ then we have $a < b$. Now it is easy to verify that this b satisfies the desired property. $\square$

Lemma 2. *Let A be a nonempty admissible set with $a = \max A < 1$, and also let $b \in (a, 1)$. Then there exists an admissible set $B \supseteq A$ with $\phi \neq B \backslash A \subseteq (a, b)$, so that if $\Gamma \in \mathcal{A}$ and $B \subseteq \Gamma$, then $\sup \Gamma < b$.*

Proof. Let $A = \{t_1 < t_2 < \cdots < t_n = a\}$. Then it is easy to show that there exists $\varepsilon > 0$ so that for every $t \in (a, a + \varepsilon)$ there is a strictly increasing sequence $(t_\lambda)_{\lambda \geq n+1} \subseteq (a, t)$ converging to t, so that $\{t_n : n \in \omega\} \in \mathcal{A}$. Let $t \in (a, a + \varepsilon)$ such that $t - a < (b - a)/2$, and also let $\lambda \geq n$ with $1/\lambda < (b - a)/2$. We set

$$B = \{t_1 < t_2 < \cdots < t_\lambda\},$$

and we easily verify that this is the desired B. $\square$

The above Lemmas imply the following facts:

1) The set $D = \{x \in \Omega : \sigma(x) \text{ is a maximal set of } \mathcal{A}\}$ is a dense subset of Ω (and the set $D \cap K$ is a dense subset of K).

2) If $V = V_A^B \cap K$ is a nonempty basic clopen subset of K, with A nonempty and $a = \max A < 1$, then there exists $b \in (a, 1)$ such that $B \cap (a, b) = \varnothing$ and $A \cup \{t\} \in \mathcal{A}$ for every $t \in (a, b)$.

It follows that we may associate to every basic clopen set V of K an open interval $(a, b) \subseteq I$ as in (2) above.

We are now ready to complete the

Proof of Theorem 3.6. a) Let $G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq \cdots$, $n < \omega$ a sequence of dense open subsets of K, so that the (dense G_δ) $G = \cap_{n\in\omega} G_n$ is metrizable, and also let $d : G \times G \to R$ be a metric on G, with $d(x, y) \leq 1$ for all $x, y \in G$. We set $V_0 = V_{\{0\}} \cap K$, and we construct a decreasing sequence $V_1 \supseteq V_2 \supseteq \cdots \supseteq V_n \supseteq \cdots$, $n < \omega$ of nonempty clopen subsets $V_n = V_{A_n}^{B_n} \cap K$ of K so that $A_{n-1} \subseteq A_n$, $0 < a_n < a_{n+1} < b_{n+1} < b_n < 1$, and also

$$d\text{-diam}\,(G \cap V_n) \leq 1/n,$$

and

$$G_1 \supseteq V_1, G_2 \supseteq V_2, \ldots, G_n \supseteq V_n, \ldots \qquad \text{for } n < \omega.$$

Suppose that the set V_{n-1} has been chosen for some $n \geq 1$, and let $(a_{n-1}, b_{n-1}) \subseteq I$ be the corresponding interval (according to fact (2)).

From Lemma 2, there exists an admissible set $A_n' \supseteq A_{n-1}$ with $\phi \neq A_{n-1}' \backslash A_{n-1} \subseteq (a_{n-1}, b_{n-1})$ such that if $\Gamma \in \mathcal{A}$ with $A_{n-1}' \subseteq \Gamma$, then we have $\sup \Gamma < b_{n-1}$. We set $W = V_{A_{n-1}'}^{B_{n-1}} \cap K$. Then W is a nonempty clopen subset of V_{n-1}. Since $(G_n \cap W) \cap G$ is a nonempty relatively open subset of G, there exists a nonempty basic clopen subset V_n of K such that

$$\phi \neq V_n \cap G \subseteq (G_n \cap W) \cap G,$$

and

$$d\text{-diam}\,(V_n \cap G) \leq 1/n.$$

It is clear that we may assume that $V_n \subseteq G_n \cap W$. So we have $V_n \subseteq G_n$ and $V_n \subseteq W \subseteq V_{n-1}$; the second inclusion implies $A_n \supseteq A_{n-1}' \neq A_{n-1}$, so $a_{n-1} < a_n < b_{n-1}$. Now let $b_n \in (a_n, b_{n-1})$ with $(a_n, b_n) \cap B_n = \varnothing$, and so that the set $A \cup \{t\}$ is admissible for all $t \in (a_n, b_n)$. The induction is complete.

Set $A = \cup_{n<\omega} A_n$. Since $(A_n)_{n<\omega}$ is an increasing sequence, $(V_n)_{n<\omega}$ is a decreasing sequence of clopen sets, and $\chi_{A_n} \in V_n \subseteq G_n$ for $n < \omega$, we conclude that $\chi_A \in \cap_{n<\omega} V_n \subseteq \cap_{n<\omega} G_n = G \subseteq K$. Our construction implies that the set A has the form $A = \{t_1 < t_2 < \cdots < t_n < \cdots\} \subseteq [0,1)$. Let $t = \lim t_n$; since $A_n \subseteq A$ for $n < \omega$, we find that $\sup A = t \in (a_n, b_n)$ for all $n < \omega$, so $\chi_{A \cup \{t\}} \in V_n$ for $n < \omega$, and thus $\chi_{A\cup\{t\}} \in \cap_{n<\omega} V_n$. But this is absurd because d-diam$\,(G \cap V_n) \leq 1/n$ for all $n < \omega$, and so

$$d\text{-diam}\left(\bigcap_{n<\omega} V_n\right) = 0.$$

We notice that an easy modification of the previous proof shows that the space Ω does not contain a dense G_δ metrizable subset.

b) Concerning this claim, we notice that if the Banach space $C(K)$ was weak-Asplund, then by a result of Coban and Kenderov [**5**], the compact space K would contain a dense G_δ metrizable subset, which contradicts claim (a). Now to show that $C(K)$ is not smoothable, we may use a recent deep result by Preiss and Phelps [**30**] according to which every smoothable Banach space is weak-Asplund. However, we shall give (after the proof of claim (c)) a direct proof of this fact.

c) We set $E \equiv E_{0,1}(\mathcal{A})$. Then E has an unconditional basis and $E \subseteq C(K)$ (cf. Definition 2.1 and Lemma 2.3). We also set $G = \{f \in E$: the (0,1) norm of E is Gateaux differentiable$\}$, and $\Phi : G \to D$ to be the restriction of the subdifferentiable map of the norm of E on G. Assume that E is weak Asplund. Then the set G contains a dense G_δ set of points of E, so since Φ is a (continuous) open and onto map (according to Theorem 2.9), by the above mentioned result of [**5**] the set D also contains a dense completely metrizable subset. But D is dense in Ω (see the fact (1)), therefore Ω contains a dense completely metrizable (hence dense G_δ and metrizable) subset, which contradicts claim (a). □

Remarks. 1) Let K be a nonmetrizable ccc Corson-compact space with property (M) (if we assume CH, there exist several examples of this type according to Theorem 2.3 of [**3**]). Then in a similar way we may show that the space $C(K)$ satisfies Theorem 3.6.

2) We notice that: (a) Theorem 3.6 answers in the negative question Q_1 in [**3**, p. 222], and (b) the above mentioned result of [**5**] gives an affirmative answer to question Q_2 (p. 222 of [**3**]).

Now we give a direct proof of the fact that $C(K)$ is not smoothable. It is enough for this to show that the Banach space $E \equiv E_{0,1}(\mathcal{A})$ is not smoothable. We shall use a Lemma due to M. Day, and which is implicitly contained in the proof of Theorem 9 of [**6**].

Lemma. *Let $(E, ||\cdot||)$ be a Banach space and also let $|\cdot|$ be an equivalent norm on E, so that $||x|| \geq |x| \geq m||x||$. For every x with $||x|| = 1$, we define $E_x = \{y \in E : ||y|| = 1$ and $|y| \leq |x| - m||x-y||/4\}$, then*:

a) *E_x is a norm-closed subset of E, and $x \in E_x$;*

b) *if $y \in E_x$, then $E_y \subseteq E_x$;*

c) *if (x_n) is a sequence of norm-one points of E, so that for each n, $x_{n+1} \in E_{x_n}$, then (x_n) is norm converging to some $x \in E$, and x is in all E_{x_n}.*

If, moreover, we suppose that $|\cdot|$ is a smooth norm, then we have:

d) *if x is a norm-one point of E so that there exists $z \in E$ with $||\lambda x + \mu z|| = |\lambda| + |\mu|$ for all $\lambda, \mu \in R$, then there exists $y \in E_x$ with $y = tz + (1-t)x$ for some $t \in (0,1)$, such that $|y| < |x|$.*

Proposition 3.7. *The Banach space E is not smoothable.*

Proof. We assume, for the purpose of contradiction, that there is an equivalent smooth norm $|\cdot|$ on E such that $||x|| \geq |x| \geq m||x||$.

Let $x \in E$ with $||x|| = 1$, so that the support of x, $\sigma(x) = \{t \in I : x(t) \neq 0\}$ is an admissible set, and also let $a = \max(\sigma(x))$. We consider, using Lemma 1, $b \in R$, so that $a < b$ and with the property that $\sigma(x) \cup \{t\}$ is admissible for all $t \in (a,b)$. We set

$$\begin{aligned}\tilde{E}_x &\equiv \tilde{E}(x,(a,b)) \\ &= \{y \in E_x : \sigma(y) \in \mathcal{A}, \sigma(x) \subseteq \sigma(y), \text{ and } \sigma(y)\backslash\sigma(x) \subseteq (a,b)\}.\end{aligned}$$

We notice that:

(i) each $\tilde{E}_x$ is a nonempty norm closed subset of E.

(ii) for any $\tilde{E}_x$ there exists $y \in \tilde{E}_x$ such that the support of y is an admissible set and $|y| < |x|$: (Indeed, let $t \in (a,b)$, and also let $z = e_t$, then $||\lambda x + \mu y|| = |\lambda| + |\mu|$ for all $\lambda, \mu \in R$. So by (d) of Day's lemma, there exists $y = tz + (1-t)x \in E_x$ for some $t \in (0,1)$ such that $|y| < |x|$. Now it is obvious that this is the desired y).

(iii) Let $y \in \tilde{E}(x,(a,b))$ such that the support of y is an admissible set. If $a' = \max(\sigma(y))$ and $b' \in (a',b)$, then $\tilde{E}(y,(a',b')) \subseteq \tilde{E}(x,(a,b))$.

We set $x_0 = e_0$ and $\tilde{E}_{x_0} = \tilde{E}(x_0,(0,1))$, and then we construct inductively a decreasing sequence

$$\tilde{E}_{x_1} \supseteq \tilde{E}_{x_2} \supseteq \cdots \tilde{E}_{x_n} \supseteq \cdots, \qquad n < \omega,$$

where $\tilde{E}_{x_n} = \tilde{E}(x_n,(a_n,b_n))$, so that for any $n = 1,2,\ldots,$ we have

$$\sigma(x_{n-1}) \subsetneqq \sigma(x_n),$$
$$0 < a_n < a_{n+1} < b_{n+1} < b_n < 1,$$
$$x_n \in \tilde{E}_{x_{n-1}},$$

and

$$|x_n| - \inf\{|x| : x \in \tilde{E}_{x_{n-1}}\} < 1/2^n.$$

Indeed, suppose that the set $\tilde{E}_{x_{n-1}}$ has been chosen for some $n \geq 1$. From remark (ii) above there exists $x_n \in \tilde{E}_{x_{n-1}}$, the support of which is an admissible set, with $\sigma(x_{n-1}) \subsetneqq \sigma(x_n)$ such that

$$|x_n| - \inf\{|x| : x \in \tilde{E}_{x_{n-1}}\} < 1/2^n. \tag{1}$$

Set $a_n = \max(\sigma(x_n))$ so $a_n \in (a_{n-1}, b_{n-1})$. By using Lemma 1, there is $b_n \in (a_n, b_{n-1})$ such that $\sigma(x_n) \cup \{t\} \in \mathcal{A}$ for any $t \in (a_n, b_n)$. We define $\tilde{E}_{x_n} = \tilde{E}(x_n,(a_n,b_n))$, and the induction is complete.

From Day's lemma, the sequence (x_n) is norm converging to some $x \in E$; it is clear from the above construction that $\sigma(x) = \cup_{n<\omega}\sigma(x_n) \in \mathcal{A}$ and also that this set has the form $\sigma(x) = \{t_1 < t_2 < \cdots < t_n < \cdots\} \subseteq [0,1)$. We set $t = \lim t_n$, and we notice that $||x|| = 1$ and that $\sigma(x) \cup \{t\} \in \mathcal{A}$, hence $||\lambda x + \mu e_t|| = |\lambda| + |\mu|$ for all $\lambda, \mu \in R$. So Day's lemma implies that there exists $y \in E_x$ (of the form $y = t'x + (1-t')e_t$ for some $t' \in (0,1)$) such that $|y| < |x|$ (2). Since $a_n < t < b_n$ for $n < \omega$, and since $\sigma(y) = \sigma(x) \cup \{t\} \in \mathcal{A}$, we have that $y \in \tilde{E}_{x_n}$, $n < \omega$; so from

(1) we conclude that $|y| = \lim |x_n| = |x|$, which obviously contradicts (2). $\square$

Remark. Since every WCD Banach space admits an equivalent smooth norm (Theorem 4.8 of [**21**]) the above result particularly implies that the space K is not a Gulko-compact. On the other hand, it is easily seen that K is a pointwise compact subset of the Banach space $C_2(I)$ endowed with supremum-norm of all bounded functions $f : I \to R$ such that for every $\varepsilon > 0$, $\sigma_\varepsilon''(f) = \varnothing$, where $\sigma_\varepsilon(\phi) = \{t \in I : |f(t)| \geq \varepsilon\}$, and $\sigma_\varepsilon''(f)$ denotes the second derived set of the set $\sigma_\varepsilon(f)$ (see [**7**]). It should be noticed that the pointwise compact subsets of the space $C_2(I)$ are quite near to the class of Gulko-compact (see [**22**]), and also that this space is strictly convexifiable (see [**7**]). By using strict convexity of the Banach space $C_2(I)$, we may show the next

Theorem 3.8. *The dual space E^* of the Banach space E is strictly convexifiable.*

Proof. It is enough by standard results to prove that there is a bounded linear one-to-one operator $T : E^* \to C_2(I)$.

For this purpose, we define $T : E^* \to l^\infty(I)$: $T(x^*) = (x^*(e_t))_{t \in I}$. It is clear that T is bounded ($||T|| \leq 1$) linear one-to-one, and weak* to pointwise continuous. Let B be the closed unit ball of E^*; we have to prove that $T(B) \subseteq C_2(I)$ (1). We notice that the ball B of E^* endowed with weak* topology is affinely homeomorphic with the pointwise compact and convex subset $T(B)$ of the ball of $l^\infty(I)$.

By Lemma 2.3 and Remark 2.4, the set of extreme points of $T(B)$ is the set $D(= \{x \in \Omega : \sigma(x)$ is a maximal set of $\mathcal{A}\})$. It follows that, to prove (1), it is enough to show the following

Claim. *The pointwise closure X in the space $[0,1]^I$ of the convex hull of the set $K = \{\chi_A : A \in \mathcal{A}\}$ is contained in the space $C_2(I)$.*

Proof of the claim. We shall prove that for any $f \in X$ and $n < \omega$, the set $\sigma_{1/n}(f)$ has at most n limit points in I; so we assume that for some $f \in X$ and $n < \omega$, the set $\sigma_{1/n}(f)$ has $n+1$ limit points. In this case we may clearly find a natural number m and $n+1$-many disjoint sets $F_1, \dots, F_{n+1}$, so that:

(i) $|F_k| = m$, $F_k \subseteq \sigma_{1/n}(f)$ for $k = 1, 2, \dots, n+1$, and

(ii) if $1 \leq k < l \leq n+1$ and $t \in F_k$, $t' \in F$, then $|t - t'| \geq 1/m$. We set $F = \cup_{k=1}^{n+1} F_k$ and $\mu = \sum_{t \in F} \delta_t$ (namely, $\mu(g) = \sum_{t \in F} g(t)$ for every $g \in l^\infty(I)$), and then we get that

$$\mu(f) > (1/(n+1)) \cdot (n+1) \cdot m = m. \tag{2}$$

Now let $\mathcal{A} = \{t_1 < t_2 < \cdots < t_q\}$ be an admissible set with $q > m$. If $m \leq k < l \leq q$, then $t_l - t_k < 1/m$, so (ii) implies that the set $\{t_m < t_{m+1} < \cdots < t_q\} \cap F_k$ is nonempty for at most one index $k = 1, 2, \dots, n+1$. Therefore, the set $F \cap A$ has at most $(m-1)+1 = m$ elements. But since $\mu : l^\infty(I) \to R$ is a continuous linear functional, when $l^\infty(I)$ is endowed with pointwise topology we conclude that

$$\mu(g) \leq m \qquad \text{for all } g \in \Omega,$$

which contradicts (2). $\square$

Remark. Though the space $C_2(I)$ is strictly convexifiable, it does not admit an equivalent strictly convex norm $|\cdot|$ that is pointwise lower semicontinuous (namely, with the property that for each $r > 0$ the r-closed ball of $(C_2(I), |\cdot|)$ is a pointwise closed set). Since in that case (by using the properties of the operator T of the previous theorem), the space E^* would admit an equivalent dual strictly convex norm, and thus E would admit an equivalent smooth norm, a contradiction according to Theorem 3.6(b) (cf. Theorem 4.5 of [**21**]).

Using the above results, we give another proof to a result of [**7**, Theorem 2 (i)].

Corollary 3.9. *There is no bounded linear one-to-one operator $\Phi : C_2(I) \to c_0(\Gamma)$, for any set Γ.*

Proof. Suppose that for some set Γ there exists such an operator $\Phi : C_2(I) \to c_0(\Gamma)$. Set $R = \Phi \circ T : E^* \to c_0(\Gamma)$ where T is the operator of Theorem 3.8. Then, clearly, R is a bounded linear one-to-one operator. However, since E has an unconditional basis, such an operator cannot exist; because in that case by a result by W. Johnson (Proposition 1.3 of [**32**]) the space E would be WCG, so the closed unit ball B of E^* in its weak* topology would be an Eberlein compact (see Proposition 34 of [**32**]); which implies that $\Omega \subseteq B$ is an Eberlein compact, a contradiction. $\square$

Note. We may prove in a similar way a stronger result, namely: there is no bounded linear one-to-one operator $T : C_2(I) \to C_1(X)$ for any countably determined topological space X (the space $C_1(X)$ is defined in Section 1).

The next example proved by Haydon (see the introduction) is a WLD Banach space E (with an unconditional basis) so that the dual E^* of E is not even strictly convexifiable. For this purpose, we shall use a combinatorial construction due to Todorcevic [**37**]. We begin by recalling the definition of a tree.

A tree is a partially ordered set $(T, \leq)$ such that for $s \in T$ the set $\{t \in T : t < s\}$ is well ordered. A chain in T is a set $C \subseteq T$ which is totally ordered by $\leq$. An antichain in T is a set $A \subseteq T$, consisting of pairwise incomparable elements. A branch of a tree T is a maximal chain of T. A path of T is any chain of T which is also an initial segment of T. If $t \in T$, then t^+ denotes the set of all immediate successors of t in T. In any tree, we use normal interval notation so that, for instance, $(s, u] = \{t \in T : s < t \leq u\}$; also for convenience we introduce two "imaginary" elements, not in Y, denoted 0 and ∞, with the property that $0 < t < \infty$.

For a tree $(T, \leq)$ we set:

$$A_T = \{C \subseteq T : C \text{ is a chain}\},$$

and

$$B_T = \{C \in A_T : s < t \in C \to s \in C\}.$$

It is clear that A_T is an adequate family of subsets of T and that B_T is the set of all parts of T. We also set: $E \equiv E_{01}(\mathcal{A}_T)$ and

$P \equiv P(T) = \{x \in \{0, \pm 1)^T : \sigma(x) \in B_T\}$. Note that P is compact and that E embeds isometrically in $C(P)$.

Proposition 3.10. *If every branch of T is countable, then P is a Corson-compact with property* (M).

Proof. It is obvious that P is Corson, so let μ be a positive measure on P, with support S. Define $T_\mu = \{t \in T$: there is $x \in S$ with $x(t) \neq 0\}$. Then T_μ is a sub-tree of T (that is, if $s < t \in T_\mu$ then $s \in T_\mu$).

For $t \in T$, set $\phi(t) = \mu(\{x \in P : x(t) \neq 0\})$. Then $T_\mu = \{t \in T : \phi(t) \neq 0\}$ and $\phi(t) \geq \Sigma\{\phi(u) : u \in t^+\}$ for all t. We shall show that T_μ is countable. To do this, it is enough to show that for no $\varepsilon > 0$ is the set $\{t \in T : \phi(t) \geq \varepsilon\}$ uncountable. By a well-known combinatorial principle (the theorem by Dushnik and Miller [**9**, Theorem 44, p. 475], see also [**23**]) any uncountable subset of T contains either an uncountable chain or an infinite antichain. The first possibility is impossible because all branches of T are countable. The second possibility cannot also hold here since $\Sigma\{\phi(t) : t \in A\} \leq ||\mu||$, for any antichain A in T.

The proof is now complete since S is obviously homeomorphic to a subset of $\{0, \pm 1\}^{T_\mu}$. $\square$

We say that a tree $(T, \leq)$ has property $(*)$, if there is no sequence $(A_n)_{n \in \omega}$ of antichains in T such that

$$T = \cup_{n \in \omega} \cup_{\alpha \in A_n} (0, \alpha].$$

Lemma 3.11. *There exists a tree $(T, \leq)$ with property $(*)$.*

Proof. The existence of a tree with property $(*)$ has been shown by Todorcevic. We shall give an outline of its construction and refer the reader to [**37**, Theorems 9.13–9.14 and Lemma 9.12] for more detailed proofs.

So, let A be a subset of ω_1 such that A and $\omega_1 \backslash A$ are both stationary (α subset of ω_1 is called stationary if it intersects every closed and unbounded subset of ω_1). We define T to be the set of all subsets t of

A that are closed in ω_1. We write $t \leq s$ if t is an initial segment of s, and note that $(T, \leq)$ is an uncountable branching tree (that is, t^+ is uncountable for any $t \in T$). The tree T has no uncountable branch since, if it did, A would contain a closed unbounded subset. $\square$

Note that Todorcevic has proved among other things that the compact space $P \equiv P(T)$ does not contain a dense metrizable subset (cf. Theorem 3.6 (a)).

Theorem 3.12. *Let $(T, \leq)$ be a tree with property $(*)$, then the Banach space $E^* \equiv E^*(\mathcal{A}_T)$ has no equivalent strictly convex norm.*

For the proof of this theorem, it is convenient to refine the tree T so that a property stronger than $(*)$ holds. We set

$$T' = \left\{ \begin{array}{ll} t \in T : & \text{there is no sequence } (A_n) \text{ of antichains in } [t,\infty] \\ & \text{such that } [t,\infty) = \cup_{n\in\omega} \cup_{a\in A_n} [t,a) \end{array} \right\}.$$

Then T' (is a nonempty sub-tree of T which) has the following property:

$$(**) \qquad \begin{array}{l} \text{If } (A_n) \text{ is a sequence of antichains in } T' \text{ then} \\ [t,\infty)\backslash\Big(\bigcup_{n\in\omega}\bigcup_{a\in A_n}(0,a]\Big) \neq \varnothing, \text{ for every } t \in T'. \end{array}$$

We now assume $T = T'$ and note the following equivalent formulation of $(**)$.

$$(**) \qquad \begin{array}{l} \text{If } \theta : T \to R^+ \text{ is a decreasing function, then there is} \\ \text{a maximal antichain } A \text{ in } T \text{ such that } \theta \text{ is constant on} \\ \text{each set } [a,\infty) \text{ with } a \in A. \end{array}$$

Lemma 1. *If T satisfies $(**)$ and $\phi \in E^{**}$, then there is a maximal antichain A in T such that $\langle \phi, x \rangle = 0$ whenever $x \in R^T$ and $|x| \leq \chi_{(a,b]}$ for some $a \in A$ and $b \in [a,\infty)$.*

Proof. Define $\theta : T \to R^+$ by

$$\theta(t) = \sup\{\langle \phi, y \rangle : y \in R^T \quad \text{and} \quad |y| < \chi_{(t,u]} \quad \text{for some } u \in [t,\infty\}.$$

Evidently θ is decreasing, so that by $(**)$ there exists a maximal antichain A with θ constant on each $[a,\infty)$ $(a \in A)$. We claim that the constant value must in each case be 0. For suppose otherwise, and let N be a natural number with $N > 2||\phi||/_{\theta(a)}$. Set $t_0 = a$ and, given t_n, choose $t_{n+1} > t_n$ together with x_{n+1} with $|x_{n+1}| \leq \chi_{(t_n, t_{n+1}]}$ and $\langle \phi, x_{n+1}\rangle(1/2)\theta(a)$ (recall that $\theta(t_n) = \theta(a)$). Note that $||\sum_{n=1}^N y_n|| \leq 1$ so that $\langle \phi, \sum_{n=1}^N y_n\rangle \leq ||\phi||$, a contradiction. □

Now let $|||\cdot|||$ be any equivalent norm on E^* when $t \in T$ and $y \in \{0,\pm 1\}^T$ satisfies $|y| \leq \chi_{(0,t]}$, define

$$\omega(t,y) = \sup\left\{ \begin{array}{ll} |||z||| : & z \in \{0,\pm1\}^T, z|(0,t] = y|(0,t] \text{ and there is} \\ & u \geq t \text{ with } |z| \leq \chi_{(0,u]} \end{array} \right\}.$$

Lemma 2. *Given $s \in T$ and $x \in \{0,\pm1\}^T$ with $|x| \leq \chi_{(0,s]}$, there exists a maximal antichain A in $[s,\infty)$ such that $|||y||| \geq |||x|||$ whenever $y \in \{0,\pm1\}^T$, and there exist $a \in A$, $t \in [a,\infty)$ with $y|(0,a] = x|(0,a]$, $|y| \leq \chi_{(0,t]}$.*

Proof. We choose $\phi \in E^{**}$ such that $|||\phi||| = 1$ and $\langle \phi, x\rangle = |||x|||$. We then choose A so that $\langle \phi, z\rangle = 0$ whenever $|z| \leq \chi_{(a,t]}$ for some $a \in A$, $t \in (a,\infty)$. □

Lemma 3. *Given $s \in T$, $\varepsilon > 0$ and $x \in \{0,\pm1\}^T$ with $|x| \leq \chi_{(0,s]}$, there exists a maximal antichain $\mathcal{B}$ in $[s,\infty)$ together with elements y_b $(b \in \mathcal{B})$ of $\{0,\pm1\}^T$ such that:*

(i) $y_b|_{(0,s]} = x|_{(0,s]}$,

(ii) $|y_b| \leq \chi_{(0,b]}$,

(iii) $|||y_b||| \geq \omega(b, y_b) - \varepsilon$,

for all $b \in \mathcal{B}$.

Proof. For any $t \in (s,\infty)$, there exists $u \in [t,\infty)$ and z with $z|_{(0,t]} = x|_{(0,t]}$, $|z| \leq \chi_{(0,u]}$, $|||z||| \geq \omega(t,x) - \varepsilon$. Thus, if we take an antichain $\mathcal{B}$ in (s,∞) maximal subject to the existence of elements

y_b ($b \in \mathcal{B}$) satisfying (i), (ii), (iii), the antichain $\mathcal{B}$ will actually be maximal. $\square$

Finally we prove Theorem 3.12.

Proof of Theorem 3.12. We consider the following: A sequence $(A_n)_{n\in\omega}$ of maximal antichains in T and elements y_a, $a \in \cup_{n\in\omega} A_n$ of $\{0, \pm 1\}^T$ satisfying

(i) for each $a \in A_n$ there is $b \in A_{n+1}$ with $a < b$,

(ii) if $a < b$, then $y_b|_{(0,a]} = y_a|_{(0,a]}$,

(iii) if $a \in A_n$, $b \in A_{n+1}$ and $a < b$, then $|||y_a||| \leq |||z||| \leq 2^{-n} + |||y_b|||$ for any $z \in \{0, \pm 1\}^T$ satisfying $z|_{(0,b]} = y_b$ and $|z| \leq \chi_{(0,u]}$ for some $u \in [b, \infty)$.

We start by letting A_0 be the set of minimal elements of T, with $y_a = 0$ for all $a \in A_0$. If A_n has been defined, together with y_a ($a \in A_n$) we work repeatedly with each $a \in A_n$. For such an a we start by applying Lemma 2, with $s = a$, $x = y_a$, to find a maximal antichain A' in $[s, \infty)$ such that $|||z||| \geq |||y_a|||$ whenever $z \in \{0, \pm 1\}^T$ and $z|_{(0,a']} = y_a|_{(0,a']}$, $|z| \leq \chi_{(0,t]}$ for some $a' \in A'$, $t \in [a', \infty)$. Next, for each $a' \in A'$ we apply Lemma 2, getting a maximal antichain A'' in (a', ∞) together with elements $y_{a''}$ ($a'' \in A''$) such that the conclusion of Lemma 3 holds. It is clear that if we take the union of these sets A' for all a' and a, we obtain a maximal antichain which has the properties we require for A_{n+1}.

By our hypothesis about T, there exists a $u \in T \setminus \cup_{n\in\omega} \cup_{a\in A_n} (0, a]$. For each $n \in \omega$ there is a unique $a_n \in A_n$ with $a_n < u$, and we can define an element z of $\{0, \pm 1\}^T$ by $z|_{(0,a_n]} = y_{a_n}|_{(0,a_n]}$, $n \in \omega$, $z|_{Y \setminus \cup_{n\in\omega}(0,a_n]} = 0$. This element z satisfies $|z| \leq \chi_{(0,u]}$ and $|||y_{a_{n+1}}||| \leq |||z||| \leq |||y_{a_{n+1}}||| + 2^{-n}$ for all n, so that $|||z||| = \lim_{n\to\infty} |||y_{a_n}|||$. Moreover, any z' satisfying $|z'| \leq \chi_{(0,u]}$, $z'|_{(0,u)} = z|_{(0,u)}$ has $|||z'||| = |||z|||$. In particular, if we define $z^{\pm}$ by

$$\{z^{\pm}|_{(0,u)} = z|_{(0,u)}, z^{\pm}(u) = \pm 1, z^{\pm}|Y \setminus (0, u) = 0\},$$

we have $|||z^{\pm}||| = |||z||| = |||\frac{z^+ + z^-}{2}|||$, contradicting strict convexity of $|||\cdot|||$. $\square$

In the sequel we define a class of dual WLD Banach spaces by using a quite general method of construction of Banach spaces with an unconditional boundedly complete basis (and so dual Banach spaces) mentioned in a paper by D.N. Kutzarova and S.L. Troyanski [**17**]. Roughly speaking, we associate to each (totally disconnected) Corson compact space K with property (M), a WLD Banach space E with an unconditional boundedly complete basis in such a way that K is embedded in the unit ball of E^* as a weak* compact subset. This result particularly implies, by using standard arguments, that if K is Talagrand compact, which is not an Eberlein compact, then E is a weakly $\mathcal{K}$-analytic Banach space which is not a subspace of a WCG Banach space; and this solves a problem posed by Talagrand [**35**, Problem 4.6 c], "Is a dual weakly $\mathcal{K}$-analytic Banach space a subspace of some WCG Banach space?".

Definition 3.13 [**17**]. Let Γ be a nonempty set and $\mathcal{A}$ a family of subsets of Γ as in Definition 2.1.

We set
$$E \equiv E_{1,2}(\mathcal{A}) = \{f : \Gamma \to R : ||f|| < +\infty\},$$
where
$$||f|| = \sup\left\{\left[\sum_{i\in I}\left(\sum_{\gamma\in A_i}|f(\gamma)|\right)^2\right]^{1/2} : I \text{ finite, } A_i \text{ finite}\right.$$
$$\left.\text{for all } i\in I, A_i\cap A_j=\varnothing, i\neq j, \text{ and } A_i\in\mathcal{A}, i\in I\right\}.$$
It is rather easy to see that $(E, ||\cdot||)$ is a Banach space, having the set $\{e_\gamma : \gamma\in\Gamma\}$ as an unconditional boundedly complete (normalized) basis. So $(E, ||\cdot||)$ is isomorphic with the dual of the subspace F of E^*, which is generated by the family of biorthogonal functionals of the family $\{e_\gamma : \gamma\in\Gamma\}$ (see Proposition 1.b.4 of [**19**]).

Now suppose that the family $\mathcal{A}$ is adequate, and set $K \equiv K(\mathcal{A})$ for the compact space defined by $\mathcal{A}$.

Then we have the following

Lemma 3.14. a) *There is a bounded linear one-to-one operator* $T : E \to C(K)$ *such that* $T(e_\gamma) = \pi_\gamma$, $\gamma\in\Gamma$.

b) *The compact space K is homeomorphic with a weak* compact subset of the closed unit ball B_{E^*} of E^*.*

c) *If Δ is a nonempty subset of Γ such that the family $\{e_\gamma : \gamma \in \Delta\}$ is equivalent to the usual basis of $l^1(\Delta)$, then (and only then) the family $\{\pi_\gamma : \gamma \in \Delta\}$ is equivalent to the usual basis of $l^1(\Delta)$.*

Proof. a) We define $T(e_\gamma) = \pi_\gamma$ for $\gamma \in \Gamma$. Then it is easy to see that this correspondence can be extended to a bounded linear one-to-one operator to the whole E such that $||T|| = 1$.

b) This is similar to the proof of claim (b) of Lemma 2.3.

c) We recall that a family $\{e_\gamma : \gamma \in \Gamma\}$ of vectors of a Banach space E is said to be equivalent to the usual basis of $l^1(\Gamma)$ if there is a constant $\delta > 0$ such that

$$\left\| \sum_{\gamma \in A} \lambda_\gamma \cdot e_\gamma \right\| \geq \delta \cdot \sum_{\gamma \in A} |\lambda_\gamma|, \tag{1}$$

for every finite subset A of Γ and every choice of scalars $\{\lambda_\gamma : \gamma \in A\}$.

Now suppose (without restrictions of generality) that $\Delta = \Gamma$, and let $\delta > 0$ be a constant such that the family $\{e_\gamma : \gamma \in \Gamma\}$ satisfies (1). We shall show that

$$\left\| \sum_{\gamma \in A} \lambda_\gamma \cdot \pi_\gamma \right\|_{0,1} \geq \delta^2 \sum_{\gamma \in A} |\lambda_\gamma|. \tag{2}$$

Set $f = \sum_{\gamma \in A} \lambda_\gamma \cdot e_\gamma$ for some finite nonempty subset $A \subseteq \Gamma$ and some choice of (nonzero) scalars $\{\lambda_\gamma : \gamma \in A\}$. By the definition of the norm of E, there is a family $\{A_1, \dots, A_n\}$ of pairwise disjoint nonempty subsets of A, each of them belonging in the family $\mathcal{A}$ such that

$$||f||_{1,2} = \left(\left(\sum_{\gamma \in A_1} |\lambda_\gamma| \right)^2 + \cdots + \left(\sum_{\gamma \in A_n} |\lambda_\gamma| \right)^2 \right)^{1/2}. \tag{3}$$

We set $a_i = \sum_{\gamma \in A_i} |\lambda_\gamma| / \sum_{\gamma \in A} |\lambda_\gamma|$ for $i = 1, 2, \dots, n$. Then from (1) and (3), we get that

$$\delta^2 \leq \alpha_1^2 + \cdots + \alpha_n^2. \tag{4}$$

We assume (without restriction of generality) that α_n is the largest of numbers $\alpha_1, \dots, \alpha_n$, then since $\alpha_i > 0$ for $i = 1, 2, \dots, n$ and $\sum_{i=1}^n \alpha_i \leq 1$, we get from (4)

$$\delta^2 \leq \alpha_1^2 + \cdots + \alpha_n^2 \leq \alpha_n(\alpha_1 + \cdots + \alpha_n) \leq \alpha_n.$$

It follows that

$$\sum_{\gamma \in A_n} |\lambda_\gamma| \geq \delta^2 \sum_{\gamma \in A} |\lambda_\gamma|, \tag{5}$$

therefore, $||f||_{0,1} = \sup\{\sum_{\gamma \in B} |\lambda_\gamma| : B \in \mathcal{A}\} \geq \sum_{\gamma \in A_n} |\lambda_\gamma| \geq \delta^2 \sum_{\gamma \in A} |\lambda_\gamma|$, which is the desired inequality (2).

For the converse of the claim (c), we notice that if A is a nonempty finite subset of Γ and $\{\lambda_\gamma : \gamma \in A\}$ is a choice of scalars, then

$$\begin{aligned} \delta \cdot \sum_{\gamma \in A} |\lambda_\gamma| &\leq \left\| \sum_{\gamma \in A} \lambda_\gamma \cdot \pi_\gamma \right\| = \left\| T\left(\sum_{\gamma \in A} \lambda_\gamma \cdot e_\gamma \right) \right\| \\ &\leq ||T|| \cdot \left\| \sum_{\gamma \in A} \lambda_\gamma \cdot e_\gamma \right\| = \left\| \sum_{\gamma \in A} \lambda_\gamma \cdot e_\gamma \right\|, \end{aligned}$$

which implies the conclusion.

The proof of the Lemma is complete. □

Corollary 3.15. *If Δ is an infinite subset of Γ such that $\{e_\gamma : \gamma \in \Delta\}$ is equivalent to the usual basis of $l^1(\Delta)$, then there is an infinite subset Δ_1 of Δ with $\Delta_1 \in \mathcal{A}$.*

Proof. Set $K_\Delta = \{\chi_{A \cap \Delta} : A \in \mathcal{A}\}K$. It is obvious that K_Δ is a compact subset (in fact, a retract) of K. From the claim (c) of Lemma 3.14, we have that the family $\{\pi_\gamma : \gamma \in \Delta\} \subseteq C(K_\Delta)$ is equivalent to the usual basis of $l^1(\Delta)$. Hence there is $A \in \mathcal{A}$ so that $A \cap \Delta$ is an infinite set, otherwise the space K_Δ would be a scattered compact space, and thus $l^1(\omega)$ is not isomorphic with a closed linear subspace of the space $C(K_\Delta)$. It follows that the set $\Delta_1 = A \cap \Delta$ is the desired set. □

Note. As Professor S. Negrepontis noticed (it is easy to verify that) Corollary 3.15 is in fact a restatement of a fundamental combinatorial result on the existence of convex means due to Pták [**31**] (see also Section 1 of [**4**] and [**24**]).

Theorem (Pták). *Let Γ be an infinite set, $\mathcal{A}$ a family of subsets of Γ and $\delta > 0$. We assume that for every nonnegative real-valued function f on Γ with finite support and such that $\sum_{\gamma\in\Gamma} f(\gamma) = 1$, there exists $B \subseteq \Gamma$ with $B \in \mathcal{A}$ such that $\sum_{\gamma\in B} f(\gamma) \geq \delta$. Then there are a sequence $\{\gamma_n : n < \omega\}$ of distinct elements of Γ, and a sequence $\{B_n : n < \omega\}$ of members of $\mathcal{A}$ such that $\{\gamma_1, \dots, \gamma_n\} \subseteq B_n$ for all $n \in \omega$.*

Note that Pták proved his result by a direct argument. We can now state the main result of this last unity.

Theorem 3.16. *Let Γ be an infinite set and let $\mathcal{A}$ be an adequate family of subsets of Γ. We set $K \equiv K(\mathcal{A})$, and $E \equiv E_{1,2}(\mathcal{A})$, then we have*:

a) *The space $C(K)$ is* WCG *if and only if the space E is* WCG;

b) *the space $C(K)$ is weakly $\mathcal{K}$-analytic if and only if the space E is weakly $\mathcal{K}$-analytic*;

c) *the space $C(K)$ is* WCD *if and only if the space E is* WCD.

d) *the space $C(K)$ is* WLD *if and only if the space E is* WLD (*if and only if $C(K)$ is weakly Lindelof*).

Proof. We set $L = \{\pi_\gamma : \gamma \in \Gamma\} \cup \{0\}$. Since the family $\mathcal{A}$ is adequate, the set L is a pointwise closed subset of $C(K)$ with only (possible) limit point the point $0 \in C(K)$ (cf. [**37**, Theorem 4.1]).

a) Suppose that $C(K)$ is a WCG Banach space. It follows then from a result of Talagrand [**35**, Theorem 4.2 (b)] that L is a countable union of pointwise (and hence weakly) compact sets, say $L = \cup_{n<\omega} L_n$. Let $\Gamma_n = \{\gamma \in \Gamma : \pi_\gamma \in L_n\}$, so we have $\Gamma = \cup_{n<\omega}\Gamma_n$. Now using Lemma 3.14 c), we may easily show that each set $\{e_\gamma : \gamma \in \Gamma_n\}$ $(n \in \omega)$ is a weakly relatively compact subset of E, which clearly implies that E is WCG.

Conversely, if we assume that E is WCG, then by a result of Johnson (see Proposition 1.3 of [**33**]) the subset $\{e_\gamma : \gamma \in \Gamma\} \cup \{0\}$ of E is a countable union of weakly compact sets, hence by Lemma 3.14 a) the subset $L = \{\pi_\gamma : \gamma \in \Gamma\} \cup \{0\}$ of $C(K)$ is a countable union of weakly compact sets, and so $C(K)$ is WCG (cf. also Theorem 4.2 of [**35**]).

b) We denote by Σ the Baire space ω^ω of irrationals, and we set for $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \leq \sigma_2$ if $\sigma_1(n) \leq \sigma_2(n)$ for $n < \omega$. It is clear that $\sigma_1 \leq \sigma_2$ if and only if $\Sigma(\sigma_1) \subseteq \Sigma(\sigma_2)$, where for $\sigma \in \Sigma$ the set $\Sigma(\sigma)$ denotes the compact subset $\Pi_{K<\omega}\{1, 2, \ldots , \sigma_{(K)}\}$ of Σ. Now if we assume that the space $C(K)$ is weakly $\mathcal{K}$-analytic, then the weakly closed subset L of $C(K)$ can be written in the form: $L = \cup\{L_\sigma : \sigma \in \Sigma\}$ where L_σ is weakly compact for all $\sigma \in \Sigma$, and for $\sigma_1, \sigma_2 \in \Sigma$, $L_{\sigma_1} \subseteq L_{\sigma_2}$ if $\sigma_1 \leq \sigma_2$. We may show as above that the set, $L_\sigma = \{e_\gamma : \pi_\gamma \in L_\sigma\}$ is a weakly relatively compact subset of E for $\sigma \in \Sigma$. We have clearly $\{e_\gamma : \gamma \in \Gamma\} = \cup\{L_\sigma : \sigma \in \Sigma\}$, and also $L_{\sigma_1} \subseteq L_{\sigma_2}$ if $\sigma_1 \leq \sigma_2$. Since the set $\{e_\gamma : \gamma \in \Gamma\}$ is a basis for E, a result of Talagrand (see Proposition 6.13 of [**35**]) implies that E is weakly $\mathcal{K}$-analytic.

For the converse, we notice that Lemma 3.14 a) implies that the subset L of $C(K)$ is weakly $\mathcal{K}$-analytic, hence by Theorem 4.2 of [**35**], $C(K)$ is weakly $\mathcal{K}$-analytic.

c) This implication is proved as the previous one, using the following (unpublished) result, obtained in [**20**]; For a Banach space E the following are equivalent:

1) E is weakly countably determined, and

2) there is a separable metric space M, and a total subset L of E, that can be written in the form, $L = \cup\{L_K : K \in \mathcal{K}(M)\}$, where L_K is weakly compact for all compact subsets K of M, and $L_{K_1} \subseteq L_{K_2}$ if $K_1 \subseteq K_2$ for $K_1, K_t \in \mathcal{K}(M)$; (where $K(M)$ denotes the set of compact subsets of M).

It is clear that this result generalizes Proposition 6.13 of [**35**].

d) We define a bounded linear one-to-one operator $\Phi : E^* \to l^\infty(\Gamma)$, so that $\Phi(x^*) = (x^*(e_\gamma))_{\gamma\in\Gamma}$. It is obvious that Φ is weak* to pointwise continuous.

Assume that the set $\Delta = \{\gamma \in \Gamma : |x^*(e_\gamma)| \geq \varepsilon\}$ is uncountable for some $x^* \in E^*$ and some $\varepsilon > 0$. It is then followed by unconditionality and Lemma 3.14 c) that the space $l^1(\Delta)$ embeds into $C(K)$; since Δ is uncountable, the space $C(K)$ cannot be weakly Lindelof.

Now we assume that E is weakly Lindelof; then the above defined operator has as a range the space $l_c^\infty(\Gamma)$ (cf. the proof of Theorem 1.7), hence the closed unit ball B of E^* in its weak* topology is affinely homeomorphic to a pointwise compact and convex subset of $l_c^\infty(\Gamma)$, and therefore has the property (M) (see Lemma 3.3 of [**3**]). Now claim b) of Lemma 3.14 finishes the proof of d).

The proof of the theorem is complete. $\square$

Theorem 3.17. *There exists a dual weakly $\mathcal{K}$-analytic Banach space E, which is not isomorphic with a closed linear subspace of any* WCG *Banach space.*

Proof. We consider a totally disconnected Talagrand compact space K which is not an Eberlein compact (as it is known M. Talagrand [**35**] was the first who constructed such an example), and we assume without restriction of generality that K is defined by an adequate family of sets. Indeed, since K is totally disconnected is homeomorphic to a pointwise compact subset of $C_1(Y)$, consisting of characteristic functions, for some $\mathcal{K}$-analytic topological space Y (see Proposition 3.4 of [**21**]). We set $\Omega = \{\chi_A\colon$ there is $B \subseteq Y$ with $A \subseteq B$ and $\chi_B \in K\}$. It is easily seen that Ω is a pointwise compact subset of $C_1(Y)$ (and hence Talagrand compact) defined by an adequate family $\mathcal{A}$ of sets and which contains space K. By Theorem 3.16 b) the space $E \equiv E_{1,2}(\mathcal{A})$ is a dual weakly $\mathcal{K}$-analytic Banach space, and by Lemma 3.14 b) the space Ω is homeomorphic with a weak* compact subset of the dual ball B of E^*. Now since K is not an Eberlein compact, and since a continuous image of an Eberlein compact is again an Eberlein compact (Theorem 6.34 of [**25**]), we conclude that E is not isomorphic with a closed linear subspace of a WCG Banach space. $\square$

Remark. Note that Fabian and Troyanski [**11**], and more recently Fabian [**12**], have proved that if E is a Banach space so that E^* is WCD, then:

a) the space E admits an equivalent locally uniformly convex norm [**11**], and

b) the space E^* admits an equivalent dual locally uniformly convex norm [**12**].

The above authors used Theorem 3.17 to show that (a) and (b) are real extensions of previous corresponding results by Godefroy, Troyanski, Whitfield and Zizler (see [**13**] and [**14**], see also [**8**]).

We conclude with two related questions. Suppose that the dual E^* of a Banach space E is WLD.

Q_1: Does E admit an equivalent locally uniformly convex norm?

Q_2: Does E^* admit an equivalent dual locally uniformly convex norm?

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