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MATRIX TRANSFORMATIONS OF CLASSES OF GEOMETRIC SEQUENCES

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ABSTRACT. For any fixed t satisfying $0 < t < 1$, let G_t denote the set of all sequences which are dominated by a constant multiple of any sequence $\{r^n\}$ with $r < t$. In this paper we characterize three kinds of matrix transformations: (i) those from G_t to the convergent sequences, (ii) those from G_t to the null sequences, and (iii) those from G_t to the bounded sequences. Also, the classes of three well-known summability methods are investigated as mappings on G_t.

1. Introduction. If u is a complex number sequence and $A = [a_{n,k}]$ is an infinite matrix, then Au is the sequence whose nth term is given by

$$(Au)_n = \sum_{k=0}^{\infty} a_{nk} u_k.$$

The matrix A is called an $X - Y$ matrix if Au is in the set Y whenever u is in X. In [**4**] Selvaraj introduced the set G_t for any fixed t satisfying $0 < t < 1$ as

$$G_t = \{u : u_n = O(r^n) \text{ for some } r \in (0,t)\}$$

and gave the characterization as follows:

Theorem 1.1. *The sequence u is in G_t if and only if*

$$\limsup_k |u_k|^{1/k} < t. \tag{1}$$

In Section 2 we investigate $G_t - c$, $G_t - c_0$, and $G_t - l^\infty$ matrices. The characterizations of such matrices are established in terms of their

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rows and columns. Section 3 examines $G_t - c$, $G_t - c_0$ and $G_t - l^\infty$ mapping properties of the classical summability methods of Euler-Knopp, Nörlund, and Borel matrices.

2. Matrix transformations of G_t into c, c_0 and l^∞. First we will prove the necessary and sufficient conditions for a matrix to be a $G_t - c$ matrix. In order to characterize such a matrix, we need the following preliminary result.

Lemma 2.1. *Let x be a complex sequence such that, for any $u \in G_t$, $\sum_{n=0}^{\infty} u_n x_n$ converges. Then for each $\varepsilon > 0$ there exists a constant $B > 0$ such that, for all k, $|x_k| \le B(1/t + \varepsilon)^k$.*

Proof. Suppose the conclusion of the lemma is false. This implies that there is an $\varepsilon > 0$ so that for every $B > 0$ there exists $k = k(B)$ satisfying

$$|x_k| > B\left(\frac{1}{t} + \varepsilon\right)^k. \tag{2}$$

We now choose an increasing sequence $\{k(i)\}_{i=0}^{\infty}$ as follows. Choose $k(0)$ satisfying $|x_{k(0)}| > 0$. After selecting $k(p)$ for all $p < i$, we choose $k(i)$ as follows. For $N = k(i-1)$, there exists a constant $B = \max_{0 \le j \le N} |x_j(t/(1+\varepsilon t))^j|$ such that

$$|x_j| \le B\left(\frac{1}{t} + \varepsilon\right)^j, \qquad \text{for } j \le N. \tag{3}$$

Let $B' = B + 1$. Now we can find $k(i)$ such that

$$|x_{k(i)}| > B'\left(\frac{1}{t} + \varepsilon\right)^{k(i)}, \tag{4}$$

using (2). Thus,

$$|x_{k(i)}| > \left(\frac{1}{t} + \varepsilon\right)^{k(i)}. \tag{5}$$

This $k(i) > k(i-1)$ because, if not, $k(i) \leq N$ and hence, by (3), $|x_{k(i)}| < B'(1/t+\varepsilon)^{k(i)}$ which would contradict (4).

Now consider the sequence u given by

$$u_j = \begin{cases} \left(\dfrac{t}{1+\varepsilon t}\right)^{k(i)} \frac{|x_{k(i)}|}{x_{k(i)}}, & \text{if } j = k(i) \text{ for } i = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that $u \in G_t$. But, for each positive integer m,

$$\sum_{j=0}^{k(m)} u_j x_j > m,$$

using (5). Thus, we have a contradiction to the hypothesis. □

Theorem 2.1. *The matrix A is a $G_t - c$ matrix if and only if*

(i) *each column sequence is in c and*

(ii) *for each $\varepsilon > 0$ there exists a constant $B > 0$ such that $|a_{nk}| \leq B(1/t+\varepsilon)^k$ for all n and k.*

Proof. First assume that A satisfies both the conditions of the theorem and let u be a sequence in G_t, say $|u_k| \leq Ms^k$ for some $s \in (0,t)$. Choose $\varepsilon > 0$ such that $\varepsilon < 1/s - 1/t$. Then we have $|a_{nk}| \leq B(1/t+\varepsilon)^k$ for all n and k. Since, for each k, $\lim_n a_{nk} = L_k$, we have $|L_k| \leq B(1/t+\varepsilon)^k$ for all k. Also, we can find a positive integer l satisfying

$$2B \sum_{k=l}^{\infty} |u_k| \left(\frac{1}{t} + \varepsilon\right)^k < \frac{\varepsilon}{2}. \tag{6}$$

This is possible because the geometric series $\sum_{k=0}^{\infty} s^k(1/t+\varepsilon)^k$ converges. Also, by condition(i), we can find an N such that for $k = 0, 1, \dots, l-1$,

$$|a_{nk} - L_k|\,|u_k| < \frac{\varepsilon}{2l} \qquad \text{for } n > N. \tag{7}$$

Now, for $n > N$,

$$\left|(Au)_n - \sum_{k=0}^{\infty} u_k L_k\right| \le \sum_{k=0}^{l-1} |a_{nk} - L_k|\,|u_k| + \sum_{k=l}^{\infty} |a_{nk} - L_k|\,|u_k| < \varepsilon$$

using (6) and (7). Thus, we have proved that the sequence $\{(Au)_n\}$ converges to $\sum_{k=0}^{\infty} u_k L_k$; this series converges. Hence, $Au \in c$.

Conversely, if A is a $G_t - c$ matrix, then the basis sequences $\{\delta_n^{(k)}\}_{n=0}^{\infty}$ are mapped into c. Thus, condition (i) holds.

Suppose that condition (ii) does not hold. Then there is an $\varepsilon > 0$ so that for every $B > 0$ there exist $n = n(B)$ and $k = k(B)$ such that

$$|a_{nk}| > B\left(\frac{1}{t} + \varepsilon\right)^k. \tag{8}$$

As G_t is in the domain of the matrix A, by Lemma 2.1, for each j there exists $B(j) > 0$ such that $|a_{jk}| \le B(j)(1/t + \varepsilon)^k$ for all k. So, for $j = 0, 1, \ldots, N$, we can find $B'(N) > 0$ satisfying $|a_{jk}| \le B'(N)(1/t + \varepsilon)^k$ for all k. Since each column of the matrix A is bounded, for $k = 0, 1, \ldots, N$, there exists a constant $M'(N)$ such that $|a_{jk}| \le M'(N)(1/t + \varepsilon)^k$ for all j. Thus, given any N there exists $M = M(N) > 1$ such that

$$|a_{jk}| \le M(1/t + \varepsilon)^k, \qquad \text{for } j \le N \text{ or } k \le N. \tag{9}$$

Now we choose increasing sequences $\{u(n)\}_{n=0}^{\infty}$ and $\{v(n)\}_{n=0}^{\infty}$ as follows. Choose $u(0)$ and $v(0)$ such that $|a_{u(0),v(0)}| > 0$. After selecting $u(p)$ and $v(p)$ for all $p < i$, we choose $u(i)$ and $v(i)$ as follows. For $N = u(i-1) + v(i-1)$, there exists $M > 1$ such that for $j \le N$ or $k \le N$,

$$|a_{jk}| \le M(1/t + \varepsilon)^k \tag{10}$$

using (9). Let $H = M + i$. Now we can find $u(i)$ and $v(i)$ such that

$$|a_{u(i),v(i)}| > H(1/t + \varepsilon)^{v(i)} \tag{11}$$

using (8). Thus,

$$|a_{u(i),v(i)}| > i(1/t+\varepsilon)^{v(i)}. \tag{12}$$

This $u(i)$ and $v(i)$ each exceed $u(i-1)+v(i-1)$ because, if not, either $u(i) \le N$ or $v(i) \le N$ and hence, by (10), $|a_{u(i),v(i)}| < M(1/t+\varepsilon)^{v(i)}$ which would contradict (11).

Now consider the sequence x given by

$$x_k = \begin{cases} \left(\dfrac{t}{1+\varepsilon t}\right)^{v(i)}, & \text{if } k = v(i) \text{ for } i = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that $x \in G_t$. Define a matrix A' by $a'_{nk} = a_{nk}x_k$. For any $u \in c$, we have $xu \in G_t$. Since A is a $G_t - c$ matrix, it follows that A' is a $c - c$ matrix. But for each positive integer m, we have $|a'_{u(m),v(m)}| > m$ by (12). This contradicts that A' is a $c - c$ matrix. $\square$

We state below two theorems on the characterization of $G_t - c_0$ and $G_t - l^\infty$ matrices. The proof of Theorem 2.1 can be easily applied to these two theorems.

Theorem 2.2. *The matrix A is a $G_t - c_0$ matrix if and only if*

(i) *each column sequence is in c_0 and*

(ii) *for each $\varepsilon > 0$ there exists a constant $B > 0$ such that $|a_{nk}| \le B(1/t+\varepsilon)^k$ for all n and k.*

Theorem 2.3. *The matrix A is a $G_t - l^\infty$ matrix if and only if for each $\varepsilon > 0$ there exists a constant $B > 0$ such that $|a_{nk}| \le B(1/t+\varepsilon)^k$ for all n and k.*

In [**2**] Jacob derived similar characterizations of the above matrix transformations using the topological properties of the spaces G_t.

3. Well-known summability mappings on G_t. In this final section we shall apply the results of Section 2 to find the necessary and sufficient conditions for some well-known matrix methods to be $G_t - c$, $G_t - c_0$, and $G_t - l^\infty$ matrices.

The Euler-Knopp means [**3**, p. 54] are given by

$$E_r[n,k] = \begin{cases} \binom{n}{k}(1-r)^{n-k}r^k, & \text{if } k \le n, \\ 0, & \text{if } k > n, \end{cases}$$

where r is any complex number. In the following theorem, we shall consider the Euler matrices E_r with only real values of the parameter r.

Theorem 3.1. *The following statements are equivalent*:

(i) $r \in [0, 2/(1+t)]$;

(ii) E_r *is a* $G_t - c$ *matrix*;

(iii) E_r *is a* $G_t - l^\infty$ *matrix*.

Proof. When $r = 0$, the matrix E_r has all ones in the first column and zeros elsewhere. So, by Theorem 2.1, E_r is a $G_t - c$ matrix. If $0 < r \le 2/(1+t)$, then a simple calculation shows that $|1-r| \le 1 - rt$. Thus, for any $x \in G_t$, say $|x_k| \le Mu^k$ where $u \in (0,t)$, we have

$$|(E_r x)_n| \le M[|1-r| + ru]^n.$$

Now $|1-r| + rt \le 1$ implies that $E_r x \in c_0$ and, hence, E_r is a $G_t - c$ matrix. We have shown that (i) implies (ii).

The fact that (ii) implies (iii) is obvious from the set inclusion $c \subset l^\infty$. Next, to see that $r \in [0, 2/(1+t)]$ whenever E_r is a $G_t - l^\infty$ matrix, suppose $r < 0$. Then the first column sequence $\{(1-r)^n\}_{n=0}^\infty$ is not bounded. Consequently, by Theorem 2.3, E_r is not a $G_t - l^\infty$ matrix. Now, suppose that $r > 2/(1+t)$. If we choose u satisfying $2t/[r(1+t)] < u < t$, then $\{x_k\} = \{(-u)^k\} \in G_t$ and

$$\begin{aligned} |(E_r x)_n| &= \left|(-1)^n \sum_{k=0}^n \binom{n}{k}(r-1)^{n-k}(ru)^k\right| \\ &= |r - 1 + ru|^n. \end{aligned}$$

Since $r-1+ru>1$, it follows that E_r is not a $G_t - l^\infty$ matrix. □

It is easy to see that the following result is also true.

Corollary 3.1. *E_r is a $G_t - c_0$ matrix if and only if $r \in (0, 2/(1+t)]$.*

The Nörlund mean Np is represented by a lower triangular matrix in which

$$Np[n,k] = p_{n-k}/P_n \qquad \text{if } k \le n,$$

where $P_n = \sum_{k=0}^{n} p_k$ and p is a nonnegative sequence such that $p_0 > 0$.

Theorem 3.2. *Let Np be a Nörlund matrix.*

(i) *Np is a $G_t - c_0$ matrix if and only if $\lim_n p_n/P_n = 0$;*

(ii) *Np is a $G_t - c$ matrix if and only if each column sequence converges;*

(iii) *Np is a $G_t - l^\infty$ matrix for all p.*

Proof. (i) If $\lim_n p_n/P_n = 0$, then Np is a regular matrix and thereby maps G_t into c_0. Conversely, if Np is a $G_t - c_0$ matrix, then by Theorem 2.2, the first column is a null sequence.

(ii) Since the absolute row sums of the matrix Np are equal to 1 and $1/t > 1$, the second condition of Theorem 2.1 is always true. Hence, the result follows.

(iii) It is obvious that the condition of Theorem 2.3 is satisfied by Np matrices. □

Fricke and Fridy [**1**] introduced the extended form of Borel matrix by the following definition. For any real number δ,

$$B_\delta[n,k] = e^{-n^\delta}(n^\delta)^k/k!$$

for $k = 0, 1, \ldots,$ and $n = 0, 1, \ldots$. When $\delta = 0$, the matrix is defined by

$$B_0[n,k] = e^{-1}/k!, \qquad \text{for all } n \text{ and } k.$$

Theorem 3.3. *The matrix B_δ is a $G_t - c_0$ matrix if and only if $\delta > 0$; also, B_δ is a $G_t - c$ matrix for all δ.*

Proof. It is known [**4**, Table 3.2, Theorem 3] that if $\delta > 0$ then B_δ is a $G_t - l^1$ matrix, whence B_δ is a $G_t - c_0$ matrix. Conversely, suppose that $\delta \leq 0$. In the case of $\delta < 0$, we have $B_\delta[n, 0] = e^{-n^\delta}$ converging to 1 as $n \to \infty$. Thus, the first column of B_δ is not in c_0. Therefore, B_δ cannot be a $G_t - c_0$ matrix. Similarly, if $\delta = 0$ then the first column converges to $1/e$ so that B_δ is not a $G_t - c_0$ matrix.

Now, in order to show that B_δ is a $G_t - c$ matrix for all δ, it is enough to consider the cases $\delta \leq 0$. When $\delta \leq 0$, the preceding arugment shows that the first column of B_δ is in c. When $\delta = 0$, for each $k \geq 1$, $B_\delta[n, k]$ converges to $1/(k!e)$ as $n \to \infty$ and, when $\delta < 0$, for each $k \geq 1$, $B_\delta[n, k]$ converges to zero as $n \to \infty$. So, in both cases, $B_\delta[n, k] < (1/t + \varepsilon)^k$ for any $\varepsilon > 0$. Thus, both conditions of Theorem 2.1 are true. Hence, B_δ is a $G_t - c$ matrix. □

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