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VALUATED VECTOR SPACES, KUREPA'S HYPOTHESIS AND ABELIAN p-GROUPS

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Dedicated to Professor Djuro Kurepa on the occasion of his 86th birthday.

ABSTRACT. The key result states that, for a regular cardinal $\varkappa$, the $\varkappa$-Kurepa hypothesis (the existence of a tree of height $\varkappa$ with levels of cardinality $< \varkappa$ and at least $\varkappa^+$ branches) is equivalent to the existence of a valuated vector space V of cardinality $\varkappa$ with the following properties: (a) its $\varkappa$-topology is Hausdorff; (b) for every $i < \varkappa$, $|V/V(i)| < \varkappa$; and (c) the completion $\tilde{V}$ of V in the $\varkappa$-topology has cardinality greater than $\varkappa$. Another equivalence to the $\varkappa$-Kurepa hypothesis is obtained by replacing (b) by the following condition (b$'$): For every $i < \varkappa$ and every subspace $W \leq V/V(i)$, with $|W| < \varkappa$, its closure $\overline{W}$, in the i-topology, also satisfies $|\overline{W}| < \varkappa$.

This is used to prove in a short and elegant way some results previously established by P. Keef; namely, Kurepa's hypothesis is equivalent to the existence of a C_{ω_1}-group G of length ω_1 and cardinality at least $\aleph_2$ with a p^{ω_1}-pure subgroup A of cardinality $\aleph_1$ whose closure in the ω_1-topology of G has cardinality at least $\aleph_2$. This is also equivalent to the existence of a C_{ω_1}-group of length ω_1 and balanced projective dimension 2.

Let V be a valuated vector space over a field F with valuation $v : V \to \mathrm{Ord} \cup \{\infty\}$, i.e., a function satisfying $v(a) = \infty$ if and only if $a = 0$, $v(ta) = v(a)$ for all scalars $t \neq 0$, and $v(a+b) \geq \min\{v(a), v(b)\}$. Then by $V(\alpha)$ we mean the subspace $V(\alpha) = \{x \in V : v(x) \geq \alpha\}$. If λ is a limit ordinal, then by the λ-*topology* on V we mean the linear topology having as a base for the neighborhoods of 0 the set $\{V(\alpha) : \alpha < \lambda\}$. All the topologies in this paper will be of this kind. It is easy to see that if $a, b \in V$ with $v(a) \neq v(b)$ then $v(a + b) = \min\{v(a), v(b)\}$.

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If U and W are subspaces of V, then by $V = U\check{\oplus}W$ we mean the valuated direct sum, i.e., $V = U \oplus W$ and, for all $a \in U$ and $b \in W$, $v(a+b) = \min\{v(a), v(b)\}$. All valuated vector spaces will be subject to some cardinality restrictions in relation to the cardinality of the ground field F; these conditions will always be ensured for countable F. For more details on valuated vector spaces, see [**3**].

Recall that a strict partially ordered set $(T, \leq)$ is a *tree* if for every $x \in T$, the set $\{y \in T : y < x\}$ is well ordered in the induced ordering. The *height of* $x \in T$, denoted by $ht(x, T)$, is the ordinal that is order equivalent to the well ordered set $\{y \in T : y < x\}$. If α is an ordinal, then $\mathrm{Lev}_{\alpha}(T) = \{x \in T : h(x, T) = \alpha\}$ is the α-th level of T. The *height of* $(T, \leq)$, denoted $ht(T)$, is the least ordinal τ such that $\mathrm{Lev}_{\tau}(T) = \varnothing$. A *branch of* T is a maximal linearly ordered subset of T. If b is a branch of T, then b is well ordered by the ordering of T. If $b \cap \mathrm{Lev}_{\alpha}(T) \neq \varnothing$, then it is a singleton and $b \cap \mathrm{Lev}_{\beta}(T) \neq \varnothing$ for all $\beta < \alpha$. If b is a branch, then the *length of* b is the least ordinal λ such that $b \cap \mathrm{Lev}_{\lambda}(T) = \varnothing$. Note that this is the same as the ordinal that is order equivalent to b. If the length of a branch b is λ we shall refer to b as a λ-branch. In this paper we shall be concerned with $\varkappa$-trees and families of $\varkappa$-branches, for a regular cardinal $\varkappa$.

A tree $(T, \leq)$ is a *$\varkappa$-Kurepa tree* if:

(1) $ht(T) = \varkappa$;

(2) for every $\alpha < \varkappa$, $|\mathrm{Lev}_{\alpha}(T)| < \varkappa$;

(3) T has at least $\varkappa^+$ $\varkappa$-branches.

The *$\varkappa$-Kurepa hypothesis* ($\varkappa$-KH) is the statement "there is a $\varkappa$-Kurepa tree." When $\varkappa = \omega_1$ we have Kurepa's hypothesis (KH).

Let X be a set of cardinality $\varkappa$ and $\{X_\alpha\}_{\alpha<\varkappa}$ be a chain of subsets of X of cardinality $< \varkappa$ such that $X = \cup_{\alpha<\varkappa} X_\alpha$. A *$\varkappa$-Kurepa family* is a family $\mathcal{F} \subseteq \mathcal{P}(X)$ of cardinality at least $\varkappa^+$ such that, for each $\alpha < \varkappa$, $|\{X_\alpha \cap Y : Y \in \mathcal{F}\}| < \varkappa$. By a Kurepa tree or Kurepa family we shall mean an ω_1-Kurepa tree or ω_1-Kurepa family, respectively. The existence of a $\varkappa$-Kurepa family is equivalent to the existence of a $\varkappa$-Kurepa tree. ZFC is equiconsistent with $ZFC + KH$ and with $ZFC + GCH + KH$. On the other hand, if I represents the statement that there exists a strongly inaccessible cardinal, then $ZFC + I$ is equiconsistent with $ZFC + \neg KH$ and with $ZFC + GCH + \neg KH$ (see

[**11**] or [**7**]).

Let us prove a statement about the existence of $\varkappa$-Kurepa trees in terms of valuated vector spaces:

Theorem 1. *Let $\varkappa$ be an uncountable regular cardinal and $\aleph$ a cardinal greater than $\varkappa$. Then there is a $\varkappa$-Kurepa tree with at least $\aleph$ $\varkappa$-branches if and only if, for every field F of cardinality $< \varkappa$, there exists a valuated F-vector space V with the following properties*:

(a) $|V| = \varkappa$,

(b) $V(\varkappa) = 0$,

(c) *for every* $i < \varkappa$, $|V/V(i)| < \varkappa$,

(d) *the completion* $\hat{V}$ *of* V *in the* $\varkappa$*-topology has cardinality* $\geq \aleph$.

Proof. We first assume the existence of a $\varkappa$-Kurepa tree $(T, \leq)$ with $\aleph$ $\varkappa$-branches. Consider any field F of cardinality $\leq \varkappa$. If $x \in \operatorname{Lev}_\alpha(T)$, we will let $\langle x \rangle$ denote the one-dimensional F-vector space generated by x. Define an F-vector space by

$$P = \prod_{\alpha < \varkappa} (\oplus_{x \in \operatorname{Lev}_\alpha(T)} \langle x \rangle),$$

and define

$$v : P \to \mathrm{Ord} \cup \{\infty\} \qquad \text{by } v(g) = \min\{\alpha \in \varkappa : g(\alpha) \neq 0\}.$$

Then (P, v) is a valuated vector space over F. Let $\mathcal{F}$ denote the set of $\varkappa$-branches in $(T, \leq)$. Each $f \in \mathcal{F}$ may be thought of as an element of P, i.e., for each $\alpha \in \varkappa$, $f(\alpha) = \{f \cap \operatorname{Lev}_\alpha(T)\}$. If $f \in \mathcal{F}$ and $\alpha < \varkappa$ we define f_α to be the element of P defined by $f_\alpha(\beta) = f(\beta)$ if $\beta < \alpha$ and $f_\alpha(\beta) = 0$ if $\alpha \leq \beta < \varkappa$. Note that $v(f_\alpha) < \alpha$, for nonzero f_α, and that the same is true for every linear combination $\sum_{f \in \mathcal{F}} s_\alpha f_\alpha \neq 0$. Define

$$V = \langle \{f_\alpha : f \in \mathcal{F} \text{ and } \alpha \in \varkappa\} \rangle.$$

We prove that V is the desired vector space with the completion $\hat{V}$, in the $\varkappa$-topology, of cardinality $\geq |\mathcal{F}|$. Note that if $x \in \operatorname{Lev}_\alpha(T)$ and f is a branch containing x, then in P, $x = f_{\alpha^+} - f_\alpha$. Every level $\operatorname{Lev}_\alpha(T)$ is of cardinality $< \varkappa$ as T is a $\varkappa$-Kurepa tree. $\langle\{f_\alpha : f \in \mathcal{F} \text{ and } \alpha \in \varkappa\}\rangle$ has cardinality $\varkappa$ as at every level α, there are less than $\varkappa$ f_α's. Thus, V is a vector space of cardinality $\varkappa$. It is a valuated vector space as (P, v) is such. The $\varkappa$-topology on V is Hausdorff since the height of the tree is $\varkappa$. We next establish (c) and (d). For every $i < \varkappa$ define

$$V_i = \langle\{f_\alpha : f \in \mathcal{F} \text{ and } \alpha \leq i\}\rangle.$$

Then $V = V_i \check{\oplus} V(i)$ for the following reasons. First of all, this is a valuated direct sum, for if $a \in V_i$, $v(a) < i$, and for every $b \in V(i)$, $v(b) \geq i$. Secondly, we show that $V = V_i + V(i)$ as follows. Let $i < \varkappa$ be fixed, and let

$$y \in V = \langle\{f_\alpha : f \in \mathcal{F} \text{ and } \alpha \in \varkappa\}\rangle.$$

If $y = \sum_{\alpha, f} s_\alpha, f_\alpha$, then $y = \sum_{\alpha \leq i} s_\alpha f_\alpha + \sum_{\alpha > i} s_\alpha f_\alpha$. For every $\alpha > i$, $f_\alpha = f_{1\alpha} + f_{2\alpha}$, where $f_{1\alpha} = f_i$ and $f_{2\alpha}$ is an element that has zero projections outside the ordinal interval (i, α). Thus, $y = \sum_{\alpha \leq i} s_\alpha f_\alpha + \sum_{\alpha > i} s_\alpha f_{1\alpha} + \sum_{\alpha > i} s_\alpha f_{2\alpha}$; the first two sums give an element in V_i, the third, an element in $V(i)$. Thus, indeed we have the desired decomposition.

Since $(T, \leq)$ is a $\varkappa$-Kurepa tree, $|\mathrm{Lev}_\alpha(T)| < \varkappa$ for each $\alpha < \varkappa$ and $\{f_\alpha : f \in \mathcal{F} \text{ and } \alpha \leq i\}$ is of cardinality less than $\varkappa$. Therefore $|V_i| < \varkappa$, for all i and so $|V/V(i)| < \varkappa$ as these spaces are isomorphic; this establishes (c).

Next note that for each $f \in \mathcal{F}$, $\{f_i\}_{i<\varkappa}$ is a Cauchy net in V converging to f in the $\varkappa$-topology on V. Thus the completion $\hat{V}$ of V in the $\varkappa$-topology on V has cardinality at least $|\mathcal{F}|$ and this establishes (d).

For the reverse implication we can use Theorem 5, but we also give an alternative proof. Thus, given a valuated F-vector space V with the noted properties, we think of the completion $\hat{V}$ as an inverse limit $\varprojlim V/V(i) \hookrightarrow \prod_{i<\varkappa} V/V(i)$ over the universal cone $q_{ij} : V/V(j) \to V/V(i)$, $i < j$, $\hat{q}_i : \hat{V} \to V/V(i)$, where $\hat{q}_i$ is continuously extended from $q_i : V \to V/V(i)$. Define $T = \cup_{i<\varkappa} V/V(i)$ and, for $a \in V/V(i)$ and $b \in V/V(j)$ define $a < b$ in T if $i < j$ and $q_{ij}(b) = a$. Note that, for any x and α, $h(x, T) = \alpha \Leftrightarrow x \in V/V(\alpha)$, thus $(T, \leq)$ is the desired $\varkappa$-Kurepa tree whose branches may be identified with the elements of

$\hat{V}$, via the bijection $x = (\dots, a_i, \dots, a_j, \dots) \in \hat{V} \leftrightarrow b = (\dots < a_i < \dots < a_j < \dots) \subset T$. $\square$

Remark. If, in the theorem above, $|\hat{V}| = \aleph$, then the constructed $\varkappa$-Kurepa tree T has exactly $\aleph$ $\varkappa$-branches.

We will prove another theorem on the existence of $\varkappa$-Kurepa trees, and to that end we first establish several results we will use in the proof. The following is an easy but useful lemma in which we employ the notion of a high subspace.

Lemma 2. *Let V be a valuated vector space and α be an ordinal. If U is a valuated subspace of V, then U is maximal with respect to the property $U \cap V(\alpha) = 0$ if and only if $V = U \oplus V(\alpha)$. Moreover, such a U may be chosen to contain any subspace W of V trivially intersecting $V(\alpha)$.*

Proof. Assume first that U is maximal with respect to the stated property. Let $x \in V \backslash U$. By the maximality of U, $(U + \langle x \rangle) \cap V(\alpha) \neq 0$. Thus, there exists a $y \in U$ such that $y + x = z \in V(\alpha)$ and therefore $x = -y + z$. Hence, we have $V = U \check{\oplus} V(\alpha)$ as $v(-y) < v(z)$. The statement concerning W follows from Zorn's lemma.

Conversely, let $V = U \check{\oplus} V(\alpha)$. Then $U' \geq U$ implies $U' = U \check{\oplus} U'(\alpha)$. If U' trivially intersects $V(\alpha)$, $U'(\alpha) = 0$ and $U' = U$, showing that U is maximal. $\square$

For a limit ordinal λ and an uncountable cardinal $\varkappa$, we will say that the valuated vector space V has the $(\lambda, \varkappa)$-closure property if it is Hausdorff in the λ-topology and if, for every subspace W of V of cardinality $< \varkappa$, its closure $\overline{W}$ in the λ-topology is also of cardinality $< \varkappa$. If V is Hausdorff in the λ-topology and is of cardinality $< \varkappa$, then V has the $(\lambda, \varkappa)$-closure property. We will use a consequence of the following result in the sequel, namely, that free valuated vector spaces have the $(\lambda, \varkappa)$-closure property.

Lemma 3. *Let λ be a limit ordinal, and let $\varkappa$ be an uncountable regular cardinal. If $V = \check{\oplus}_{i \in I} W_i$ is a valuated vector space over a* *field of cardinality* $< \varkappa$, *which is Hausdorff in the* λ*-topology, and if* $|W_i| < \varkappa$, *for all* $i \in I$, *then* V *has the* $(\lambda, \varkappa)$*-closure property.*

Proof. Let W be a subspace of V of cardinality $< \varkappa$. Then there is a $J \subseteq I$, such that $|J| < \varkappa$ and $W \subseteq \check{\oplus}_{i\in J} W_i$. Then also $\overline{W} \subseteq \check{\oplus}_{i\in J} W_i$, thus $|\check{\oplus}_{i\in J} W_i| < \varkappa$, because $\varkappa$ is regular, hence $|\overline{W}| < \varkappa$. □

If $\varkappa$ is a regular cardinal and V a module, then any ascending chain $\{U_\alpha\}_{\alpha<\varkappa}$ of submodules of V with the property that $V = \cup_{\alpha<\varkappa} U_\alpha$ and $U_\gamma = \cup_{\alpha<\gamma} U_\alpha$ for every limit ordinal $\gamma < \varkappa$ is called a *filtration of* V. If, in addition, V is generated by $\varkappa$ elements and, for all $\alpha < \varkappa$, U_α is generated by fewer than $\varkappa$ elements, then $\{U_\alpha\}_{\alpha<\varkappa}$ is called a $\varkappa$-filtration of V. For example, by well ordering elements of a generating set of V, say $\{a_\gamma\}_{\gamma<\varkappa}$ and letting $U_\alpha = \langle \{a_\gamma\}_{\gamma<\alpha} \rangle$, $\{U_\alpha\}_{\alpha<\varkappa}$ is a $\varkappa$-filtration of V.

The following technical lemma will be important in the proof of Theorem 5.

Lemma 4. *Let* $\varkappa$ *be an uncountable regular cardinal and let* F *be a base field of cardinality* $< \varkappa$. *Consider a* $\varkappa$*-filtration* $\{U_\alpha\}_{\alpha<\varkappa}$ *of a valuated vector space* V *over* F *of dimension* $\varkappa$. *If the* $\varkappa$*-topology on* V *is Hausdorff, then there is a two-dimensional array* $\{U_\alpha^i\}_{\alpha<\varkappa}^{i<\varkappa}$ *of valuated subspaces of* V *with the following properties*:

(1) $U_0^0 = 0$;

(2) *for a fixed* $\alpha < \varkappa$, $\{U_\alpha^i\}^{i<\varkappa}$ *is a filtration of* U_α, *and, for a fixed* $i < \varkappa$, $\{U_\alpha^i\}_{\alpha<\varkappa}$ *is an ascending chain*;

(3) *for all* α, $i < \varkappa$, $U_\alpha = U_\alpha^{i+1} \check{\oplus} U_\alpha(i)$; *and if* $V^i = \cup_{\alpha<\varkappa} U_\alpha^i$, *for all* $i < \varkappa$, *and* $\hat{V}$ *denotes the completion of* V *in the* $\varkappa$*-topology, then* $\{V^i\}^{i<\varkappa}$ *is a filtration of* V *such that, for all* $i < \varkappa$,

(4) $V = V^{i+1} \check{\oplus} V(i)$ *and*

(5) $\hat{V} = V^{i+1} \check{\oplus} \hat{V}(i)$.

Proof. We will construct subspaces $U_\alpha^i (i, \alpha < \varkappa)$ inductively in two dimensions. Thus, assume that we have constructed all U_β^m, $\beta < \alpha$, $m < \varkappa$ and all U_α^j, $j < i < \varkappa$, satisfying conditions (1)–(3). We show

how to construct U_α^i. By (1), we know how to start; to satisfy the first part of (2), we define $U_\alpha^i = \cup_{j<i} U_\alpha^j$, if i is a limit ordinal. Choose U_α^{i+1} to be a valuated subspace of U_α containing $U_\alpha^i + \cup_{\beta<\alpha} U_\beta^{i+1}$ and maximal with respect to the property $X \cap U_\alpha(i) = 0$. By Lemma 2, $U_\alpha = U_\alpha^{i+1} \check{\oplus} U_\alpha(i)$. We must justify that this choice is possible by showing that $(*)$ $(U_\alpha^i + \cup_{\beta<\alpha} U_\beta^{i+1}) \cap U_\alpha(i) = 0$. To the contrary, assume that there is an $x \in U_\alpha^i$ and $y \in \cup_{\beta<\alpha} U_\beta^{i+1}$ such that $x + y \neq 0$ and $v(x+y) \geq i$. Then there is a $k < i$ such that $x \in U_\alpha^{k+1}$ (by (2)) and $y \in U_\beta^{i+1}$ for some $\beta < \alpha$. By (3) and the inductive hypothesis, we have the decomposition $U_\beta = U_\beta^{k+1} \check{\oplus} U_\beta(k)$. Hence, $y = z + w$, where $z \in U_\beta^{k+1}$, $w \in U_\beta(k)$, and $v(w) = v(y-z) \geq k$. Also, $x+z = (x+y)-(y-z)$, where $v(x+y) \geq i$ and $v(y-z) \geq k$. Therefore, $v(x+z) \geq k$. On the other hand, $x+z \in U_\alpha^{k+1} + U_\beta^{k+1} \leq U_\alpha^{k+1}$ (by (2)), and by (3) either $v(x+y) < k$ or $x+z = 0$, the latter remaining as the only possibility; however $x+y = x+z+w = w \in U_\beta^{i+1} \cap U_\alpha(i) = 0$, which is a contradiction. Thus, $(*)$ holds.

By the construction, the chains in (2) are ascending. In addition, the first chain in (2) is smooth and $U_\alpha = \cup_{i<\varkappa} U_\alpha^i$. The reason for the latter is that $|U_\alpha| < \varkappa$. Thus, for every $\alpha < \varkappa$ there is an $i < \varkappa$ such that $U_\alpha(i) = 0$ since $\varkappa$ is regular. Hence, by (3), $U_\alpha^j = U_\alpha$ for all $j \geq i+1$.

If $i < \varkappa$ is a limit ordinal, then $\cup_{j<i} V^j = \cup_{j<i} \cup_{\alpha<\varkappa} U_\alpha^j = \cup_{\alpha<\varkappa} \cup_{j<i} U_\alpha^j = \cup_{\alpha<\varkappa} U_\alpha^i = V^i$, by (2). Also $V = \cup_{\alpha<\varkappa} U_\alpha = \cup_{\alpha<\varkappa} \cup_{i<\varkappa} U_\alpha^i = \cup_{i<\varkappa} \cup_{\alpha<\varkappa} U_\alpha^i = \cup_{i<\varkappa} V^i$. Thus $\{V^i\}^{i<\varkappa}$ is indeed a filtration of V. Since $V^{i+1} = \cup_{\alpha<\varkappa} U_\alpha^{i+1}$, $v(x) < i$ for every $x \in V^{i+1} \setminus 0$ as $x \in U_\alpha^{i+1}$, for some $\alpha < \varkappa$. Thus $V^{i+1} \check{\oplus} V(i) \leq V$. For the reverse inequality, let $x \in V \setminus 0$. Then $x \in U_\alpha = U_\alpha^{i+1} \check{\oplus} U_\alpha(i)$ for $\alpha < \varkappa$ by (3). Hence, $x = a + b$, where $a \in U_\alpha^{i+1} \leq V^{i+1}$, $b \in U_\alpha(i) \leq V(i)$, and $x \in V^{i+1} \check{\oplus} V(i)$ (in fact $v(x) = v(a)$). This settles (4).

For (5), notice that the elements of $\hat{V}$ may be thought of as limits of Cauchy nets in V. Let $(f_\gamma)_{\gamma<\varepsilon}$ be a Cauchy net in V, i.e., for all $i < \varkappa$, $f_\gamma - f_\delta \in V(i)$ for all sufficiently large γ, δ. Using the decomposition $V = V^{i+1} \check{\oplus} V(i)$, we have $f_\gamma = f_{\gamma i} + f'_{\gamma i}$ and $f_\delta = f_{\delta i} + f'_{\delta i}$ where $f_{\gamma i}$, $f_{\delta i} \in V^{i+1}$ and $f'_{\gamma i}, f'_{\delta i} \in V(i)$. Thus, $f_\gamma - f_\delta = (f_{\gamma i} - f_{\delta i}) + (f'_{\gamma i} - f'_{\delta i}) \in V(i)$, and this is possible only if $f_{\gamma i} = f_{\delta i}$, as the nonzero elements in V^{i+1} have values $< i$. Thus, (f_γ) is a Cauchy net in V, if and only if $(f'_{\gamma i})$ is a Cauchy net in V (or $V(i)$) and the limit of the net (f_γ) is determined by the limit of the Cauchy net of its components $(f'_{\gamma i})$. Thus, we have (5): $\hat{V} = V^{i+1}\check{\oplus}\hat{V}(i)$. □

If we weaken condition (c) of Theorem 1, we will still get the following equivalence:

Theorem 5. *Let $\varkappa$ be an uncountable regular cardinal and $\aleph$ a cardinal greater than $\varkappa$. Then there is a $\varkappa$-Kurepa family of cardinality $\geq \aleph$, if and only if there exists a valuated vector space V of cardinality $\varkappa$, over a field of cardinality $< \varkappa$, with the following properties*:

(a) $V(\varkappa) = 0$,

(b) *for every* (*limit*) $i < \varkappa$, $V/V(i)$ *has the* $(i, \varkappa)$-*closure property.*

(c) *The completion* $\hat{V}$ *of* V *in the* $\varkappa$-*topology has cardinality* $\geq \aleph$.

Proof. Let $\{U_\alpha\}_{\alpha<\varkappa}$ be a $\varkappa$-filtration of V. Our hypotheses ensure the existence of a two-dimensional array $\{U_\alpha^i\}_{\alpha<\varkappa}^{i<\varkappa}$ and a filtration $\{V^i\}_{i<\varkappa}$ satisfying the conditions of Lemma 4. For each $f \in \hat{V}$ and $i < \varkappa$, let $f = f_i + f'_i$ be the unique decomposition given by condition 5 in Lemma 4, with $f_i \in V^{i+1}$ and $f'_i \in \hat{V}(i)$. Let

$$\mathcal{F} = \{\{f_i : i < \varkappa\} : f \in \hat{V}\backslash V\}.$$

Note that for each $f \in \hat{V}\backslash V$, $\{f_i : i < \varkappa\}$ is a Cauchy net converging to f. Also observe that $|\mathcal{F}| = \aleph$ and that $\mathcal{F}$ is a family of subsets of V. We will show that for each $\alpha < \varkappa$, $\{\{f_i : i < \varkappa\} \cap U_\alpha : f \in \hat{V}\backslash V\}$ is of cardinality $< \varkappa$. Note that if $f_i \in U_\alpha$, then $f_j \in U_\alpha$ for all $j < i$. This follows since $f_i = a + b$ with $a \in U_\alpha^{j+1} \leq V^{j+1}$ and $b \in U_\alpha(j) \leq V(j)$ and, by the uniqueness of the decomposition, $a = f_j$, as $f_i = f_j + (f_i - f_j)$ is another decomposition in $V^{j+1}\check{\oplus}V(j)$. Since U_α is of cardinality $< \varkappa$, there must exist an $l < \varkappa$ such that $\hat{V}(l) \cap U_\alpha = \varnothing$. For α, $j < \varkappa$, denote by $\mathcal{F}_{\alpha,j}$ the set of those $\{f_i : i < \varkappa\} \cap U_\alpha$ such that $\{f_i : i < \varkappa\} \in \mathcal{F}$ and $\{f_i : i < \varkappa\} \cap U_\alpha = \{f_i : i < j\}$. If j is a limit ordinal, then the sets in $\mathcal{F}_{\alpha,j}$ represent an element of the closure of U_α^j in V^{j+1} (which is isometric to $V/V(j)$) in the j-topology, hence by the $(j, \varkappa)$-closure property, there are $< \varkappa$ sets of this form. If j is an isolated ordinal, then the sets in $\mathcal{F}_{\alpha,j}$ correspond to $f_{j-1} \in V^j \cap U_\alpha$. There are likewise $< \varkappa$ sets of this form, because $|U_\alpha| < \varkappa$. The regularity of $\varkappa$ implies that $\cup_{j<l}\mathcal{F}_{\alpha,j}$ is also of cardinality $< \varkappa$. Since

$\cup_{j<l}\mathcal{F}_{\alpha,j} = \{\{f_i : i < \varkappa\} \cap U_\alpha : f \in \hat{V}\backslash V\}$, $\mathcal{F}$ is a $\varkappa$-Kurepa family of cardinality $\aleph$.

The proof of the converse follows from Theorem 1. $\square$

Applications to abelian p-groups. We will use the above theorems to give proofs of two results of Keef's, utilizing some of the techniques similar to those he used. Our approach is however to deal only with the socles of the groups in question whenever possible. The notation and terminology will be the same as that in [**2**]. Recall that an abelian p-group G is a C_{ω_1}-group if $G/p^\alpha G$ is a direct sum of countable groups for all $\alpha < \omega_1$. A subgroup A of G is p^α-pure in G if the short exact sequence $0 \to A \to G \to G/A \to 0$ represents an element of $p^\alpha \mathrm{Ext}\,(G/A, A)$. If λ is an ordinal, H_λ is the generalized Prüfer group of length λ [**5**, p. 59]. For $\lambda < \omega_1$, H_λ is countable, and $H_{\omega_1} = \oplus_{\lambda<\omega_1} H_\lambda$. By [**12**, Proposition 3], if G is a C_{ω_1}-group, then $\mathrm{Tor}\,(G, H_{\omega_1})$ is a direct sum of countable groups. We will need the following [**8**, Lemma 1]: Let A be a pure subgroup of G and $\lambda \le \omega_1$. Then $\mathrm{Tor}\,(A, H_\lambda)$ is a summand of $\mathrm{Tor}\,(G, H_\lambda)$ if and only if A is a p^λ-pure subgroup of G. If, in addition, G is a C_λ-group, then so is G/A. We will also use the following result from [**10**, Lemma 1]: If $\lambda \le \omega_1$ is a limit ordinal and A is a subgroup of a C_λ-group G, then A is a p^λ-pure subgroup of G if and only if, for each $\alpha < \lambda$, the natural embedding of $A/p^\alpha A$ in $G/p^\alpha G$ is a summand of $G/p^\alpha G$.

Proposition 6 [**10**, Theorem 5]**.** *Kurepa's hypothesis is equivalent to the existence of a C_{ω_1}-group G of length ω_1 and cardinality at least $\aleph_2$ with a p^{ω_1}-pure subgroup A of cardinality $\aleph_1$ such that the closure of A in G in the ω_1-topology has cardinality at least $\aleph_2$.*

Proof. Assuming Kurepa's hypothesis, there exists a valuated vector space V satisfying conditions of Theorem 1 (we take $\varkappa = \omega_1$). Note that the completion $\hat{V}$ of V in the ω_1-topology contains $\tilde{V}_{\mathcal{F}} = V + \langle\{f : f \in \mathcal{F}\}\rangle$ ($\mathcal{F}$ is the set of ω_1-branches of the Kurepa tree). By [**14**, Theorem 1], there exists an abelian p-group G such that

(1) $\hat{V} \subseteq G[p]$,

(2) $\hat{V}$ is a nice subgroup of G,

(3) for all $x \in \hat{V}$ the height of x in G is the same as the valuation of x in $\hat{V}$, and

(4) $G/\hat{V}$ is a direct sum of countable groups.

We will show that G is a C_{ω_1}-group. We need only show that $G/p^\alpha G$ is a direct sum of countable groups for all $\alpha \in \omega_1$. Note that $p^\alpha(G/\hat{V}) = (p^\alpha G + \hat{V})/\hat{V}$ by [**2**, Lemma 79.2]. Therefore

$$\begin{aligned}(G/\hat{V})/p^\alpha(G/\hat{V}) &= [G/\hat{V}]/[(p^\alpha G + \hat{V})/\hat{V}] \\ &\cong G/(p^\alpha G + \hat{V}) \\ &\cong [G/p^\alpha G]/[(\hat{V} + p^\alpha G)/p^\alpha G].\end{aligned}$$

By [**2**, Exercise 10, p. 75], $(\hat{V}+p^\alpha G)/p^\alpha G$ is a nice subgroup of $G/p^\alpha G$. Since $\hat{V} = V_\alpha \check{\oplus} \hat{V}(\alpha)$, V_α is countable, with countably many values, hence V_α must be free as a valuated vector space, by [**3**, Theorem 1]. Thus $(\hat{V} + p^\alpha G)/p^\alpha G$ is a summable subsocle of $G/p^\alpha G$. Let C be a direct sum of countable groups of length α having a subsocle S isomorphic to V_α as a valuated vector space and such that, for all β, the relative Ulm invariants $f_\beta(G/p^\alpha G, (\hat{V}+p^\alpha G)/p^\alpha G) \le f_\beta(C, S)$. By [**2**, Theorem 81.2], there is a height preserving isomorphism from $G/p^\alpha G$ into C. Thus $G/p^\alpha G$ is isomorphic to an isotype subgroup of C. By [**5**, Theorem 104], isotype subgroups of direct sums of countable groups of countable length are direct sums of countable groups. Thus, $G/p^\alpha G$ is a direct sum of countable groups. Therefore, G is a C_{ω_1}-group.

Next we will show that there exists a p^{ω_1}-pure subgroup A of G such that $V \subseteq A$ and $|A| = \aleph_1$. Let H_{ω_1} be the generalized Prüfer group of length ω_1. Note that $\mathrm{Tor}\,(G, H_{\omega_1})$ is a direct sum of countable groups since H_{ω_1} is a direct sum of countables [**12**, Theorem 6]. Fix a decomposition $\mathrm{Tor}\,(G, H_{\omega_1}) = \oplus_{\nu\in X} J_\nu$ with the J_ν's countable. We will construct a sequence $\{A_i\}_{i\in\omega}$ of subgroups of G and a sequence $\{I_{2n+1}\}_{n\in\omega}$ of subsets of X recursively as follows. Let $A_0 = V$. Assume that A_n has been constructed and has cardinality $\aleph_1$. If n is even, let A_{n+1} be a pure subgroup of G containing A_n and of cardinality $\aleph_1$. If n is odd, we proceed as follows. Let I_n be the least subset of X such that $\mathrm{Tor}\,(A_n, H_{\omega_1}) \subseteq \oplus_{\nu\in I_n} J_\nu$. Note that, since $|A_n| = \aleph_1$, $|I_n| = \aleph_1$. Let $B_n = \oplus_{\nu\in I_n} J_\nu$. For each $x \in B_n$, choose a $g_x \in G$ such that there exists a positive integer m and an $h \in H_{\omega_1}$ with (g_x, m, h) a representative of x in the representation of $\mathrm{Tor}\,(G, H_{\omega_1})$ as equivalence classes of ordered triples. Let $A_{n+1} = \langle \{g_x : x \in B_n\} \rangle$. Then $B_n \subseteq \mathrm{Tor}\,(A_{n+1}, H_{\omega_1})$ and $|A_{n+1}| = \aleph_1$. Let $A = \cup_{i \in \omega} A_i$ and $I = \cup_{n \in \omega} I_{2n+1}$. Then A is a pure subgroup of G and $\mathrm{Tor}\,(A, H_{\omega_1}) = \oplus_{\nu \in I} J_\nu$. Thus, by [**8**, Lemma 1], A is a p^{ω_1}-pure subgroup of G of cardinality $\aleph_1$.

Since A is p^{ω_1}-pure in G, A is isotype in G [**13**, p. 196]. Since $V \subseteq A$ and the closure of V in the ω_1-topology on G contains $\hat{V}$, the closure of A in G has cardinality at least $\aleph_2$. This completes one direction of the equivalence.

For the converse, assume that there exists a C_{ω_1}-group G of length ω_1, of cardinality at least $\aleph_2$, with a p^{ω_1}-pure subgroup A of cardinality $\aleph_1$, whose closure $\overline{A}$ in the ω_1-topology is of cardinality at least $\aleph_2$. We will prove that $V = A[p]$ satisfies the conditions of Theorem 5. The ω_1-topology is clearly Hausdorff. By [**10**, Lemma 1], $A/p^\alpha A$ is a direct summand of $G/p^\alpha G$, for all $\alpha < \omega_1$, and hence $A/p^\alpha A$ is also a direct sum of countables. If B is a $p^\alpha A$-high subgroup of A, then by [**1**, Lemma 1.1] $A[p] = B[p] \dot{\oplus} p^\alpha A[p]$ and consequently $V/V(\alpha) \cong B[p]$, for all $\alpha < \omega_1$. As B is an isotype subgroup of $A/p^\alpha A$, B is a direct sum of countables by [**5**, Theorem 104], and hence $B[p]$ is free as a valuated vector space by [**5**, Proposition 111]; by Lemma 3, it has the (α, ω_1)-closure property. This establishes (b) in Theorem 5. In addition, since the closure of $A[p]$ in $G[p]$ in the relative ω_1-topology is $\overline{A}[p]$, we have that the cardinality of $\hat{V}$ exceeds $\aleph_1$. This concludes the proof. □

For the proof of the following theorem we recall several definitions and results from [**4**]. Let $\varkappa$ be a cardinal. A subgroup A of G is $\varkappa$-separable in G if, for all $g \in G$, $\sup\{h(g+a) : a \in A\} = \sup\{h(g+s) : s \in S\}$ for some subset S of G of cardinality $\leq \varkappa$ ($h(g)$ is the height of $g \in G$). A family $\mathcal{C}$ of subgroups of G is an $H(\varkappa)$-family in G if $\{0\} \in \mathcal{C}$, $\mathcal{C}$ is closed under group unions, and for $B \in \mathcal{C}$ and any subgroup A of G of cardinality $\leq \varkappa$, there is a $D \in \mathcal{C}$ that contains both B and A such that D/B has cardinality at most $\varkappa$. An abelian group G satisfies Axiom 3: $\varkappa$ if G has an $H(\varkappa)$-family of $\varkappa$-separable subgroups. By [**4**, Theorem 4.5], for each $n \geq 0$, the balanced projective dimension of G is $\leq n$ if and only if G satisfies Axiom 3: $\aleph_{n-1}$. By [**4**, Lemma 5.1], if

$$0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots \subseteq G_\lambda = G, \qquad \alpha < \lambda$$

is a well ordered continuous chain of subgroups of G, where each G_α, $\alpha < \lambda$, is balanced in G and the balanced projective dimension (*bpd*) of

$G_{\alpha+1}/G_\alpha$ is $\leq n$ for all $\alpha < \lambda$, then the $bpd\, G \leq n$. We will also use [**10**, Theorem 3]: Suppose that G is a C_{ω_1}-group, λ is a limit ordinal, and $\{A_i\}_{i<\lambda}$ is a filtration of p^{ω_1}-pure subgroups of G. Then the closure of the union of the A_i's is the union of their closures in the ω_1-topology on G.

Theorem 7 [**9**, Theorem 6]. *Kurepa's hypothesis is equivalent to the existence of a C_{ω_1}-group of length ω_1 and balanced projective dimension 2.*

Proof. Assuming Kurepa's hypothesis, let G be a C_{ω_1}-group of length ω_1 and cardinality at least $\aleph_2$ having a p^{ω_1}-pure subgroup A of cardinality $\aleph_1$ such that the closure of A in G in the ω_1-topology has cardinality at least $\aleph_2$ as constructed in Proposition 6. If G has cardinality greater than $\aleph_2$, replace G by a p^{ω_1}-pure subgroup of G which contains A and a subgroup of $\overline{A}$ of cardinality $\aleph_2$. This new G is a C_{ω_1}-group by [**10**, Lemma 1]. Note that in a group of length ω_1, every subgroup is $\aleph_1$-separable. This implies that the family $\mathcal{C}$ of all subgroups of the group G is an $H(\aleph_1)$-family of that group and that G satisfies Axiom 3: $\aleph_1$. By [**4**, Theorem 4.5], this means that $bpd\, G \leq 2$. The dimension must be exactly 2, for if $bpd\, G \leq 1$, then, by [**4**, Theorem 4.5] G would have an $H(\omega_0)$-family $\mathcal{C}$ of $\aleph_0$-separable submodules. Thus, for every $a \in A$, there is a countable $B(a) \in \mathcal{C}$ containing a. Let $C = \sum_{a \in A} B(a)$. Then $C \in \mathcal{C}$ and C has cardinality $\aleph_1$. Hence, there exists $g \in \overline{A} \backslash C$ since A has cardinality $\aleph_2$. Thus, $\sup\{g + c : c \in C\} = \omega_1$. This is a contradiction as $\aleph_0$-separable means that, for all $x \in G$, $\sup\{h(x+c) : c \in C\} = \sup\{h(x+s) : s \in S\}$ for some countable subset S of G whereas $\{h(x+s) : s \in S\}$ is a countable set of countable ordinals.

Conversely, assume the negation of Kurepa's hypothesis. Then, by Proposition 6, for every C_{ω_1}-group G of length ω_1 and cardinality at least $\aleph_2$, if A is a p^{ω_1}-pure subgroup of G of cardinality $\aleph_1$, then the closure of A in the ω_1-topology has cardinality $\aleph_1$. Let G be a C_{ω_1}-group of length ω_1. We will show that $bpd\, G \leq 1$. Let $\varkappa$ be the cardinality of G. If $\varkappa \leq \aleph_1$, then G has an ω_1-filtration which is an $H(\varkappa)$-family of $\aleph_0$-separable subgroups. Hence, by [**4**, Theorem 4.5], $bpd\, G \leq 1$. Thus, we may assume that $|G| \geq \aleph_2$.

We first consider the cases $|G| = \aleph_2$. Let $\mathrm{Tor}\,(G, H_{\omega_1}) = \oplus_{i\in I} K_i$ be a fixed decomposition with $|K_i| = \aleph_0$ for each $i \in I$. We will construct a filtration $\{A_\alpha\}_{\alpha<\omega_2}$ of G such that, for each $\beta \in \lim \omega_2$ ($\lim \omega_2$ is the set of limit ordinals less than ω_2) and each $n < \omega$,

(1) $A_0 = \{0\}$;

(2) $A_{\beta+2n+1}$ is the closure of $A_{\beta+2n}$ in the ω_1-topology of G;

(3) $A_{\beta+2n+2}$ is a p^{ω_1}-pure subgroup of G containing $A_{\beta+2n+1}$ such that

$$\mathrm{Tor}\,(A_{\beta+2n+2}, H_{\omega_1}) = \bigoplus_{i\in J(\beta+n+1)} K_i$$

for some $J(\beta+n+1) \subseteq I$ and $|A_{\beta+2n+2}/A_{\beta+2n+1}| = \aleph_1$; and

(4) $A_\beta = \cup_{\alpha<\beta} A_\alpha$.

The construction is clear except perhaps for (3). In this case, we choose $A_{\beta+2n+2}$ as in the third paragraph of the proof of Proposition 6. Note that, for each limit ordinal $\beta < \omega_1$, $\mathrm{Tor}\,(A_\beta, H_{\omega_1}) = \oplus_{i\in X_\beta} K_i$ where $X_\beta = \cup_{\alpha<\beta} J_{\alpha+1}$. Thus, by [**8**, Lemma 1], A_β is p^{ω_1}-pure in G. Hence, by [**5**, Theorem 91], the image under the natural map of $p^\alpha G[p]$ is equal to $p^\alpha(G/A_\beta)[p]$ for each $\alpha < \omega_1$. By [**10**, Theorem 3], A_β is closed in G for each limit ordinal $\beta < \omega_1$. Thus, G/A_β has length $\leq \omega_1$. Hence, by [**2**, Proposition 80.2], A_β is balanced in G. Therefore, $\{A_\beta\}_{\beta\in\lim \omega_1}$ is a filtration of G consisting of balanced subgroups of G. For each $\beta \in \lim \omega_1$, $A_{\beta+\omega}/A_\beta$ has cardinality $\aleph_1$ and length $\leq \omega_1$. Hence, any ω_1-filtration of this group is an $H(\aleph_0)$-family of $\aleph_0$-separable subgroups. Therefore, $\mathit{bpd}\, A_{\beta+\omega}/A_\beta \leq 1$. Thus $\{A_\beta\}_{\beta\in\lim \omega_1}$ is a filtration of G satisfying the hypotheses of [**4**, Lemma 5.1], and $\mathit{bpd}\, G \leq 1$.

In the general case when the cardinality of G is greater than $\aleph_2$, we apply the same process as in the previous case to get a chain of balanced p^{ω_1}-pure subgroups of G closed in the ω_1-topology whose union is a closed, balanced, p^{ω_1}-pure subgroup H of G of cardinality $\aleph_2$. The group G/H is a C_{ω_1}-group by [**8**, Lemma 1] and it is of length ω_1 since G is of length ω_1 and H is closed in the ω_1-topology. Thus, we will apply the same process to the group G/H to get a chain $\{B_\alpha\}_{\alpha<\omega_1}$ with the same properties as the one just described. Let $\phi : G \to G/H$ be the natural map. Then $\{\phi^{-1}B_\alpha\}_{\alpha\leq\omega_1}$ is the continuation of the previous chain with every quotient of consecutive terms of cardinality at most $\aleph_1$. Each $\phi^{-1}(B_\alpha)$ is closed since ϕ is continuous in the ω_1-topology,

p^{ω_1}-pure by [**6**, Theorem 18], and balanced by [**2**, (d) p. 78]. We apply the process again to $G/\phi^{-1}(B_{\omega_1})$ to obtain the desirable continuation of the existing chain, and so on, until the group G is exhausted. In this way, we obtain a smooth chain of balanced subgroups of G such that every quotient of consecutive terms is of cardinality at most $\aleph_1$. As above, we may conclude that each consecutive quotient has *bpd* 1. Thus, by [**4**, Lemma 5.1], $bpd\, G \leq 1$. This completes the proof. $\square$

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