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THE GREEN'S MATRIX FUNCTION AND RELATED EIGENVALUE RESULTS FOR A VECTOR DIFFERENCE EQUATION

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Dedicated to Paul Waltman on the occasion of his 60th birthday

We will derive a special form of a Green's matrix function for a second-order, self-adjoint vector difference equation, and then take advantage of this form in the application of cone theory in a Banach space to prove results concerning eigenvalues for a corresponding boundary value problem. The self-adjoint equation has been considered by Ahlbrandt and Hooker [**1**], and Peil and Peterson [**11**], among others. A discussion of the corresponding scalar self-adjoint equation appears in Kelley and Peterson [**9**]. Background on cone theory to difference equations can be found in Krasnosel'skiĭ [**10**] and Diaz [**2**]. Similar applications of cone theory to difference equations can be found in Hankerson and Henderson [**4**] and Hankerson and Peterson [**5, 6**]. Applications of cone theory to differential equations can be found in Eloe, Hankerson and Henderson [**3**].

We initially consider the second-order, self-adjoint vector difference equation

$$Ly(t) = -\Delta[P(t-1)\Delta y(t-1)] + Q(t)y(t) = 0 \tag{1}$$

on the discrete interval $[a+1, b+1] \equiv \{a+1, \dots, b+1\}$, where here $P(t)$ and $Q(t)$ are $n \times n$ matrix functions with $P(t)$ positive definite on $[a, b+1]$ and $Q(t)$ Hermitian on $[a+1, b+1]$. Solutions of (1) are defined on $[a, b+2]$.

If $y(t)$ is a complex solution of $Ly(t) = 0$, then on $[a+1, b+2]$,

$$y^*(t-1)P(t-1)y(t) - y^*(t)P(t-1)y(t-1) = c,$$

where c is complex constant and $*$ denotes the conjugate transpose. If $c = 0$, then $y(t)$ is called a prepared solution of (1), in which case $y^*(t-1)P(t-1)y(t)$ is real-valued on $[a+1, b+2]$.

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We say that a nontrivial prepared solution $y(t)$ of (1) has a generalized zero at a if $y(a) = 0$, and a generalized zero at $t_0 \in [a+1, b+2]$ if $y^*(t_0-1)P(t_0-1)y(t_0) \leq 0$ and $y(t_0-1) \neq 0$. If no nontrivial prepared solution of (1) has two generalized zeros in $[a, b+2]$, then (1) is said to be disconjugate on $[a, b+2]$.

If (1) is disconjugate on $[a, b+2]$, we are guaranteed the existence (see [**12**] and [**13**]) of a unique matrix function $G(t,s)$ such that:

i) $G(t,s)$ is defined on $[a, b+2] \times [a+1, b+1]$,

ii) $LG(t,s) = \delta_{ts}I$ on $[a, b+2] \times [a+1, b+1]$, where δ_{ts} is the Kronecker delta and I is the $n \times n$ identity matrix,

iii) $G(a,s) = 0 = G(b+2, s)$ for $s \in [a+1, b+1]$.

We call $G(t,s)$ the Green's matrix function for the boundary value problem

$$\begin{gathered} Ly(t) = 0, \qquad t \in [a+1, b+1] \\ y(a) = 0 = y(b+2). \end{gathered} \tag{2}$$

Note that while the assumption of disconjugacy is sufficient for the existence of the Green's matrix function, it is not necessary.

Similar to the differential equations case, we get an inversion formula, offered without proof.

Proposition 1. *The solution of the boundary value problem*

$$\begin{gathered} Ly(t) = h(t), \qquad t \in [a+1, b+1] \\ y(a) = 0 = y(b+2) \end{gathered}$$

is given by

$$y(t) = \sum_{s=a+1}^{b+1} G(t,s)h(s),$$

where $G(t,s)$ is the above Green's matrix function.

Throughout the remainder of this paper, we will consider the case when $Q(t) \equiv 0$. In this case, the Green's matrix function for (2) has a particularly accessible form. A derivation of this Green's matrix function using the Cauchy matrix function can be found in [**12**] and [**13**].

Proposition 2. *The Green's matrix function for the boundary value problem* (2) *is given by*

$$G(t,s) = \begin{cases} \sum_{\tau=a}^{t-1} P^{-1}(\tau) M^{-1} \sum_{\tau=s}^{b+1} P^{-1}(\tau), & t \le s \\ \sum_{\tau=t}^{b+1} P^{-1}(\tau) M^{-1} \sum_{\tau=a}^{s-1} P^{-1}(\tau), & s \le t, \end{cases} \tag{3}$$

where $M = \sum_{\tau=a}^{b+1} P^{-1}(\tau)$.

This form can also be arrived at by starting with the difference equation

$$-\Delta[P(s-1)\Delta y(s-1)] = h(s),$$

taking the sum from $\tau+1$ to $b+1$, then from a to $t-1$ and applying the boundary conditions $y(a) = 0$ and $y(b+2) = 0$. This yields

(4)

$$\begin{aligned} y(t) &= \sum_{\tau=a}^{t-1} \left[P^{-1}(\tau) \sum_{s=\tau+1}^{b+1} h(s) \right] \\ &\quad - \sum_{\tau=a}^{t-1} P^{-1}(\tau) \left[M^{-1} \sum_{k=a}^{b+1} P^{-1}(k) \sum_{s=k+1}^{b+1} h(s) \right] \\ &= \sum_{s=a+1}^{t} \sum_{\tau=a}^{s-1} P^{-1}(\tau) h(s) + \sum_{s=t+1}^{b+1} \sum_{\tau=a}^{t-1} P^{-1}(\tau) h(s) \\ &\quad - \sum_{s=a+1}^{b+1} \left[\sum_{\tau=a}^{t-1} P^{-1}(\tau) M^{-1} \sum_{k=a}^{s-1} P^{-1}(k) \right] h(s) \\ &= \sum_{s=a+1}^{t} \left[-\sum_{\tau=a}^{t-1} P^{-1}(\tau) M^{-1} \sum_{k=a}^{s-1} P^{-1}(k) + \sum_{\tau=a}^{s-1} P^{-1}(\tau) \right] h(s) \\ &\quad + \sum_{s=t+1}^{b+1} \left[-\sum_{\tau=a}^{t-1} P^{-1}(\tau) M^{-1} \sum_{k=a}^{s-1} P^{-1}(k) + \sum_{\tau=a}^{t-1} [P^{-1}(\tau)] \right] h(s) \end{aligned}$$

$$\begin{aligned}
&= \sum_{s=a+1}^{t} \left[-\sum_{\tau=a}^{t-1} P^{-1}(\tau) M^{-1} \sum_{k=a}^{s-1} P^{-1}(k) + M M^{-1} \sum_{\tau=a}^{s-1} P^{-1}(\tau) \right] h(s) \\
&\quad + \sum_{s=t+1}^{b+1} \left[-\sum_{\tau=a}^{t-1} P^{-1}(\tau) M^{-1} \sum_{k=a}^{s-1} P^{-1}(k) + \sum_{\tau=a}^{t-1} P^{-1}(\tau) M^{-1} M \right] h(s) \\
&= \sum_{s=a+1}^{t} \left[\sum_{\tau=t}^{b+1} P^{-1}(\tau) M^{-1} \sum_{k=a}^{s-1} P^{-1}(k) \right] h(s) \\
&\quad + \sum_{s=t+1}^{b+1} \left[\sum_{\tau=a}^{t-1} P^{-1}(\tau) M^{-1} \sum_{k=s}^{b+1} P^{-1}(k) \right] h(s) \\
&= \sum_{s=a+1}^{b+1} G(t,s) h(s).
\end{aligned}$$

Note the agreement of the pieces of the Green's matrix function for $t = s$, which can be seen in (4). A further special case which yields a Green's matrix function of a fairly nice form occurs when we, in addition to $Q(t) \equiv 0$, assume that $P(t)$ is of the form

$$P(t) = D(t)B,$$

where B is a constant nonsingular $n \times n$ matrix and $D(t) =$ diag $(d_1(t), \ldots, d_n(t))$, with $d_i(t) > 0$ for $1 \le i \le n$, $t \in [a, b+1]$. In this case, we will not assume that $P(t)$ is positive definite, but it is easy to see that the Green's matrix function for (2) exists and is still given by (3).

Here we have

$$P^{-1}(t) = B^{-1}\text{diag}\left(\frac{1}{d_1(t)}, \ldots, \frac{1}{d_n(t)}\right)$$

and

$$\begin{aligned}
M^{-1} &= \left[B^{-1} \sum_{\tau=a}^{b+1} \text{diag}\left(\frac{1}{d_1(\tau)}, \ldots, \frac{1}{d_n(\tau)}\right)\right]^{-1} \\
&= \text{diag}\left(\left(\sum_{\tau=a}^{b+1} \frac{1}{d_1(\tau)}\right)^{-1}, \ldots, \left(\sum_{\tau=a}^{b+1} \frac{1}{d_n(\tau)}\right)^{-1} \right) B.
\end{aligned}$$

Using Proposition 2, we get

$$G(t,s) = \begin{cases} B^{-1}\operatorname{diag}\left(\dfrac{\sum_{\tau=a}^{t-1}\frac{1}{d_1(\tau)}\sum_{\tau=s}^{b+1}\frac{1}{d_1(\tau)}}{\sum_{\tau=a}^{b+1}\frac{1}{d_1(\tau)}}, \dots, \right. \\ \qquad\qquad \left. \dfrac{\sum_{\tau=a}^{t-1}\frac{1}{d_n(\tau)}\sum_{\tau=s}^{b+1}\frac{1}{d_n(\tau)}}{\sum_{\tau=a}^{b+1}\frac{1}{d_n(\tau)}}\right), & t \le s \\ B^{-1}\operatorname{diag}\left(\dfrac{\sum_{\tau=t}^{b+1}\frac{1}{d_1(\tau)}\sum_{\tau=a}^{s-1}\frac{1}{d_1(\tau)}}{\sum_{\tau=a}^{b+1}\frac{1}{d_1(\tau)}}, \dots, \right. \\ \qquad\qquad \left. \dfrac{\sum_{\tau=t}^{b+1}\frac{1}{d_n(\tau)}\sum_{\tau=a}^{s-1}\frac{1}{d_n(\tau)}}{\sum_{\tau=a}^{b+1}\frac{1}{d_n(\tau)}}\right), & s \le t \end{cases}$$
$$= B^{-1}\operatorname{diag}(g_1(t,s), \dots, g_n(t,s)),$$

where, for $1 \le i \le n$, $g_i(t,s)$ is the Green's function for the scalar boundary value problem

$$\begin{gathered} -\Delta[d_i(t-1)\Delta u(t-1)] = 0, \qquad t \in [a+1, b+1] \\ u(a) = 0 = u(b+2). \end{gathered}$$

Note that $g_i(t,s) > 0$ on $[a+1, b+1] \times [a+1, b+1]$ for $1 \le i \le n$. Throughout, we will write

$$G^D(t,s) \equiv \operatorname{diag}(g_1(t,s), \dots, g_n(t,s)).$$

The remainder of this paper will consider the eigenvalue problem

$$\begin{gathered} -\Delta[D(t-1)B\Delta y(t-1)] = \lambda R(t)y(t), \qquad t \in [a+1, b+1] \\ y(a) = 0 = y(b+2). \end{gathered} \tag{5}$$

First, define the Banach space

$$\mathcal{B}_0 = \{y : [a, b+2] \to \mathbf{R}^n \mid y(a) = 0 = y(b+2)\}$$

equipped with the norm

$$||y||_\infty = \max_{t \in [a+1, b+1]} ||y(t)||_1$$

where $||\cdot||_1$ is the l_1 vector norm.

Define an operator N on $\mathcal{B}_0$ by

$$Ny(t) = \sum_{s=a+1}^{b+1} B^{-1}G^D(t,s)R(s)y(s).$$

Now note that $\lambda = 0$ is not an eigenvalue of (5). Using Proposition 1, we have that (λ_0, y^0) is an eigenpair for (5) if and only if $(1/\lambda_0, y^0)$ is an eigenpair for N.

We will use this relationship, along with results from cone theory, to obtain eigenvalue results for (5). The development of most of the cone theory used can be found in [**10**].

Given a Banach space $\mathcal{B}$, a nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is a cone if:

i) $\mathcal{P}$ is a closed set

ii) If $u, v \in \mathcal{P}$, then $\alpha u + \beta v \in \mathcal{P}$ for any scalars $\alpha, \beta > 0$

iii) If $u \in \mathcal{P}$ and $-u \in \mathcal{P}$, then $u = 0$.

A cone $\mathcal{P}$ is solid if $\mathcal{P}^0 \neq \varnothing$, where $\mathcal{P}^0$ is the interior of $\mathcal{P}$. We say $\mathcal{P}$ is a reproducing cone if $\mathcal{B} = \mathcal{P} - \mathcal{P}$, the difference set of $\mathcal{P}$. A basic result in [**10**] is that if $\mathcal{P}$ is a solid cone, then it is a reproducing cone. In $\mathbf{R}^n$, the converse is also true.

For $u, v \in \mathcal{B}$, we write $u \leq v$ if $v - u \in \mathcal{P}$. A linear operator $M : \mathcal{B} \to \mathcal{B}$ is said to be positive with respect to the cone $\mathcal{P}$ if $M : \mathcal{P} \to \mathcal{P}$. We say M is strongly positive if $M : \mathcal{P}\backslash\{0\} \to \mathcal{P}^0$. For a nonzero element u_0 of $\mathcal{P}$, M is u_0-positive if, for each nonzero element x of $\mathcal{P}$, there exist $\alpha_x, \beta_x > 0$ such that $\alpha_x u_0 \leq Mx \leq \beta_x u_0$. Another result that will be of use is that if M is a strongly positive linear operator with respect to the solid cone $\mathcal{P}$, then M is u_0-positive for any $u_0 \in \mathcal{P}^0$.

Two of the main results we will use are due to Krasnosel'skií [**10**].

Theorem 1. *Let M be a compact linear operator on the Banach space $\mathcal{B}$, positive with respect to a reproducing cone $\mathcal{P}$ in $\mathcal{B}$. Suppose there exist a nonzero element u of $\mathcal{B}$, with $-u \notin \mathcal{P}$, and an $\varepsilon > 0$ such that $Mu \geq \varepsilon u$. Then M has an eigenvector in $\mathcal{P}$, whose corresponding eigenvalue η is larger than the modulus of any other eigenvalue of M and satisfies $\eta \geq \varepsilon$.*

Theorem 2. *Let $\mathcal{P}$ be a reproducing cone in a Banach space $\mathcal{B}$ and M a compact u_0-positive linear operator on $\mathcal{B}$. Then M has an essentially unique eigenvector in $\mathcal{P}$, and its corresponding eigenvalue is simple, positive, and larger than the modulus of any other eigenvalue of M.*

We now return to our previously mentioned Banach space $\mathcal{B}_0$. Let $\mathcal{K}$ be a solid cone in $\mathbf{R}^n$. Then

$$\mathcal{P} \equiv \{y \in \mathcal{B}_0 \mid y(t) \in \mathcal{K}, t \in [a+1, b+1]\}$$

is a solid cone in $\mathcal{B}_0$, and

$$\mathcal{P}^0 = \{y \in \mathcal{B}_0 \mid y(t) \in \mathcal{K}^0, t \in [a+1, b+1]\}.$$

Throughout the following, (a_{ij}) will denote the $n \times n$ matrix with (i, j) entry a_{ij}, and $\langle b_i \rangle$ will denote the $n \times 1$ vector with ith component b_i. Also, write $B^{-1} = (c_{ij})$, and let $R(t) = (r_{ij}(t))$ be an $n \times n$ real matrix function on $[a+1, b+1]$.

We will also assume that the entries of B^{-1} and $R(t)$ satisfy the sign conditions

1) $\sigma_{ij} c_{ij} \geq 0$, $1 \leq i, j \leq n$
2) $\sigma_{ij} r_{ij}(t) \geq 0$, $1 \leq i, j \leq n$, $t \in [a+1, b+1]$

where $\sigma_{ij} \in \{-1, 1\}$ and $\sigma_{ij} = \sigma_{1i}\sigma_{ij}$, $1 \leq i, j \leq n$.

Remark. We will consider quadrants in $\mathbf{R}^n$, and the condition $\sigma_{ij} = \sigma_{1i}\sigma_{ij}$, $1 \leq i, j \leq n$, above will ensure that the matrices B^{-1} and $R(t)$, for each $t \in [a+1, b+1]$, will act as positive operators on an appropriate quadrant.

Theorem 3. *Suppose there exist $1 \leq i_0, j_0 \leq n$ and $s_0 \in [a+1, b+1]$ such that $c_{i_0 j_0} \neq 0$ and $r_{j_0 i_0}(s_0) \neq 0$. Then (5) has a least positive eigenvalue λ_0 which is smaller than the modulus of any other eigenvalue of (5). Also, there is an eigenfunction y^0 corresponding to λ_0 such that $\sigma_{1i} y_i^0(t) \geq 0$ for $1 \leq i \leq n$, $t \in [a+1, b+1]$.*

Proof. Consider the solid cone

$$\mathcal{K}_0 \equiv \{u \in \mathbf{R}^n \mid \sigma_{1i} u_i \geq 0, 1 \leq i \leq n\}$$

and the corresponding cone of functions

$$\mathcal{P}_0 \equiv \{y \in \mathcal{B}_0 \mid y(t) \in \mathcal{K}_0, t \in [a+1, b+1]\}.$$

We wish to find $y \in \mathcal{B}_0$ and $\varepsilon > 0$ such that $-y \notin \mathcal{P}_0$ and $Ny \geq \varepsilon y$. The result will then follow using Theorem 1.

Let $t_0 \in [a+1, b+1]$ and $1 \leq k \leq n$, and define $y^k(t)$ on $[a, b+2]$ by

$$y^k(t) = \begin{cases} 0, & t \neq t_0 \\ \sigma_{1k} e_k, & t = t_0, \end{cases}$$

where e_k is the kth unit basis vector in $\mathbf{R}^n$. Note that $y^k \in \mathcal{P}_0 \backslash \{0\}$, and so $-y^k \notin \mathcal{P}_0$.

For $t = t_0$, we have

$$\begin{aligned} Ny^k(t_0) &= \sum_{s=a+1}^{b+1} B^{-1} G^D(t_0, s) R(s) y^k(s) \\ &= B^{-1} G^D(t_0, t_0) R(t_0) \sigma_{1k} e_k \\ &= \left(\sum_{l=1}^{n} c_{il} g_l(t_0, t_0) r_{lj}(t_0) \right) \sigma_{1k} e_k \\ &= \sigma_{1k} \left\langle \sum_{l=1}^{n} c_{il} g_l(t_0, t_0) r_{lk}(t_0) \right\rangle. \end{aligned}$$

Now consider the vector

$$\begin{aligned} z(t_0) = \sigma_{1k} &\left\langle \sum_{l=1}^{n} c_{il} g_l(t_0, t_0) r_{lk}(t_0) \right\rangle \\ &- \sum_{l=1}^{n} c_{kl} g_l(t_0, t_0) r_{lk}(t_0) \sigma_{lk} e_k. \end{aligned}$$

For $1 \leq i \leq n$, $i \neq k$, the ith component of $z(t_0)$ satisfies

$$\begin{aligned} \sigma_{1i} z_i(t_0) &= \sigma_{1i} \left[\sigma_{1k} \sum_{l=1}^{n} c_{il} g_l(t_0, t_0) r_{lk}(t_0) - 0 \right] \\ &= \sigma_{1i} \sigma_{1k} \sum_{l=1}^{n} \sigma_{1i} \sigma_{1l} |c_{il}| \sigma_{1l} \sigma_{1k} |r_{lk}(t_0)| g_l(t_0, t_0) \\ &= \sum_{l=1}^{n} g_l(t_0, t_0) |c_{il} r_{lk}(t_0)| \geq 0. \end{aligned}$$

For $i = k$,

$$\sigma_{1k} z_k(t_0) = \sigma_{1k}\left[\sigma_{1k}\sum_{l=1}^{n} c_{kl} g_l(t_0,t_0) r_{lk}(t_0) - \sum_{l=1}^{n} c_{kl} g_l(t_0,t_0) r_{lk}(t_0)\sigma_{1k}\right]$$
$$= 0.$$

Hence, $z(t_0) \in \mathcal{K}_0$, and so

$$\sigma_{1k}\left\langle \sum_{l=1}^{n} c_{il} g_l(t_0,t_0) r_{lk}(t_0)\right\rangle \geq \sum_{l=1}^{n} c_{kl} g_l(t_0,t_0) r_{lk}(t_0)\sigma_{1k} e_k,$$

where the inequality is with respect to $\mathcal{K}_0$.

From above, we get

$$\begin{aligned} Ny^k(t_0) &\geq \sum_{l=1}^{n} c_{kl} g_l(t_0,t_0) r_{lk}(t_0)\sigma_{1k} e_k \\ &= \sum_{l=1}^{n} c_{kl} g_l(t_0,t_0) r_{lk}(t_0) y^k(t_0). \end{aligned}$$

For $t \in [a+1, b+1]$, $t \neq t_0$, we have

$$\begin{aligned} Ny^k(t) &= \sum_{s=a+1}^{b+1} B^{-1} G^D(t,s) R(s) y^k(s) \\ &\geq B^{-1} G^D(t,t) R(t) y^k(t) \\ &= 0 \\ &= \sum_{l=1}^{n} c_{kl} g_l(t_0,t_0) r_{lk}(t_0) y^k(t). \end{aligned}$$

Hence,

$$Ny^k \geq \sum_{l=1}^{n} c_{kl} g_l(t_0,t_0) r_{lk}(t_0) y^k,$$

where the inequality is now with respect to $\mathcal{P}_0$.

Recall that $t_0 \in [a+1, b+1]$ and $1 \le k \le n$ were chosen arbitrarily, and so, in particular,

$$Ny^{i_0} \ge \sum_{l=1}^{n} c_{i_0 l} g_l(s_0, s_0) r_{l i_0}(s_0) y^{i_0}.$$

Also,

$$\sum_{l=1}^{n} c_{i_0 l} g_l(s_0, s_0) r_{l i_0}(s_0) \ge c_{i_0 j_0} g_{j_0}(s_0, s_0) r_{j_0 i_0}(s_0) > 0,$$

and so Theorem 1 gives that the compact operator N has a positive eigenvalue η_0 that is larger than the modulus of any other eigenvalue of N. Further, there is an eigenfunction y^0, corresponding to η_0, with $y^0 \in \mathcal{P}_0$.

The result follows by noting that $\lambda_0 = 1/\eta_0$, and by the definition of $\mathcal{P}_0$, since $y^0 \in \mathcal{P}_0$ implies $\sigma_{1i} y_i^0(t) \ge 0$ for $1 \le i \le n$, $t \in [a+1, b+1]$. □

In the next theorem, we use the maximum column sum matrix norm, defined by

$$|||A|||_1 \equiv \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|,$$

where $A = (a_{ij})$.

Theorem 4. *Assume the hypothesis in Theorem* 3*, and let* λ_0 *be the least positive eigenvalue of* (5)*. Then for any* $t_0 \in [a+1, b+1]$ *and* $1 \le k \le n$*, we have*

$$\sum_{l=1}^{n} g_l(t_0, t_0) c_{kl} r_{lk}(t_0) \le \lambda_0^{-1} \le M |||B^{-1}|||_1 \sum_{s=a+1}^{b+1} |||R(s)|||_1, \tag{6}$$

where $M = \max\{g_i(t,t) \mid 1 \le i \le n, t \in [a+1, b+1]\}$.

Proof. From the proof of Theorem 3, we have

$$Ny^k \ge \sum_{l=1}^{n} g_l(t_0, t_0) c_{kl} r_{lk}(t_0) y^k$$

for any $1 \leq k \leq n$ and $t_0 \in [a+1, b+1]$, where $y^k \in \mathcal{P}_0 \backslash \{0\}$, and so the left inequality in (6) follows from Theorem 1.

For the right inequality in (6), let y^0 be an eigenfunction of N in $\mathcal{P}_0$, corresponding to λ_0. Then, for $t \in [a+1, b+1]$,

$$\begin{aligned} \lambda_0^{-1} ||y^0(t)||_1 &= ||Ny^0(t)||_1 \\ &= \left\| \sum_{s=a+1}^{b+1} B^{-1} G^D(t,s) R(s) y^0(s) \right\|_1 \\ &\leq \sum_{s=a+1}^{b+1} |||B^{-1} G^D(t,s) R(s)|||_1 ||y^0(s)||_1 \\ &\leq \sum_{s=a+1}^{b+1} |||B^{-1}|||_1 |||G^D(t,s)|||_1 |||R(s)|||_1 ||y^0(s)||_1 \\ &\leq \left(M |||B^{-1}|||_1 \sum_{s=a+1}^{b+1} |||R(s)|||_1 \right) ||y^0||_\infty. \end{aligned} \tag{7}$$

Inequality (7) follows from a result which can be found in [**7**]. Since the above holds for all $t \in [a+1, b+1]$, we get

$$\lambda_0^{-1} ||y^0||_\infty \leq \left(M |||B^{-1}|||_1 \sum_{s=a+1}^{b+1} |||R(s)|||_1 \right) ||y^0||_\infty,$$

which implies

$$\lambda_0^{-1} \leq M |||B^{-1}|||_1 \sum_{s=a+1}^{b+1} |||R(s)|||_1. \qquad \square$$

Corollary. *Assume the hypothesis in Theorem* 3, *and let* λ_0 *be the least positive eigenvalue of* (5). *Then*

$$\lambda_0^{-1} \leq M_1 |||B^{-1}|||_1 \sum_{s=a+1}^{b+1} |||R(s)|||_1,$$

where

$$M_1 = \frac{1}{4} \max_{1 \le i \le n} \sum_{\tau=a}^{b+1} \frac{1}{d_i(\tau)}.$$

Proof. Let $i \in \{1, \dots, n\}$. For $t \in [a+1, b+1]$, we have

$$g_i(t,t) = \frac{\sum_{\tau=a}^{t-1} \frac{1}{d_i(\tau)} \sum_{\tau=t}^{b+1} \frac{1}{d_i(\tau)}}{\sum_{\tau=a}^{b+1} \frac{1}{d_i(\tau)}}.$$

We will use the inequality

$$4(a+b)^{-1} \le a^{-1} + b^{-1} \tag{8}$$

for $a, b > 0$. Letting $a = \sum_{\tau=a}^{t-1} 1/d_i(\tau)$ and $b = \sum_{\tau=t}^{b+1} 1/d_i(\tau)$ we have

$$\begin{aligned} g_i(t,t) &= ab(a+b)^{-1} \\ &\le \frac{1}{4} ab(a^{-1} + b^{-1}) \\ &= \frac{1}{4}(a+b) = \frac{1}{4} \sum_{\tau=a}^{b+1} \frac{1}{d_i(\tau)}. \end{aligned}$$

Hence,

$$g_i(t,t) \le \frac{1}{4} \sum_{\tau=a}^{b+1} \frac{1}{d_i(\tau)}$$

for all $1 \le i \le n$ and $t \in [a+1, b+1]$. The result follows from the right inequality in (6). $\square$

Example. We will find bounds for the least positive eigenvalue λ_0 of the eigenvalue problem

$$-\Delta \left\{ \begin{bmatrix} (t-1)^{-1} & 0 \\ 0 & 1 \end{bmatrix} \Delta y(t-1) \right\} = \lambda \begin{bmatrix} t & -1 \\ 0 & 1 \end{bmatrix} y(t), \qquad t \in [2,5]$$
$$y(1) = 0 = y(6).$$

Here,

$$D(t) = \begin{bmatrix} t^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \qquad B^{-1} = I \quad \text{and} \quad R(t) = \begin{bmatrix} t & -1 \\ 0 & 1 \end{bmatrix}.$$

The Green's matrix function is given by $G(t,s) = B^{-1}\mathrm{diag}\,(g_1(t,s), g_2(t,s))$, where

$$g_1(t,s) = \begin{cases} \dfrac{t(t-1)(6-s)(5+s)}{60}, & t \le s \\ \dfrac{s(s-1)(6-t)(5+t)}{60}, & s \le t \end{cases}$$

and

$$g_2(t,s) = \begin{cases} \dfrac{(6-s)(t-1)}{5}, & t \le s \\ \dfrac{(6-t)(s-1)}{5}, & s \le t \end{cases}.$$

Now

$$\max\left\{ \sum_{l=1}^{n} g_l(t_0,t_0) c_{kl} r_{lk}(t_0) \mid t_0 \in [2,5], k \in \{1,2\} \right\} = 50/3$$

and

$$M_1 \sum_{s=2}^{5} \left|\left|\left| \begin{bmatrix} s & -1 \\ 0 & 1 \end{bmatrix} \right|\right|\right|_1 = \frac{1}{4}(15)(14) = \frac{105}{2}.$$

Hence,

$$2/105 \le \lambda_0 \le 3/50.$$

Theorem 5. *Suppose that for each $1 \le i \le n$, the following condition holds*:

Let $c_{ij_1}, \dots, c_{ij_m}$, $m = m(i)$ be the nonzero entries of the ith row of B^{-1}. Given $k_0 \in \{1,\dots,n\}$ and $s_0 \in [a+1, b+1]$, there exists some $l_i \in \{j_1,\dots,j_m\}$ such that $r_{l_i k_0}(s_0) \ne 0$.

Then (5) *has a least positive eigenvalue λ_0 that is smaller than the modulus of any other eigenvalue of* (5)*. Also, there exists an essentially unique eigenfunction y^0, corresponding to λ_0, which satisfies $\sigma_{1i} y_i^0(t) > 0$ for $1 \le i \le n$, $t \in [a+1, b+1]$.*

Proof. We will show that N is u_0-positive with respect to $\mathcal{P}_0$, and the result will then follow using Theorem 2. Let $y \in \mathcal{P}_0 \backslash \{0\}$. Then there exist $k_0 \in \{1,\dots,n\}$ and $s_0 \in [a+1,b+1]$ such that $y_{k_0}(s_0) \ne 0$.

For $t \in [a+1, b+1]$,

$$\begin{aligned} Ny(t) &= \sum_{s=a+1}^{b+1} B^{-1}G^D(t,s)R(s)y(s) \\ &= \Big\langle \sum_{s=a+1}^{b+1} \sum_{l=1}^{n} c_{il} \sum_{k=1}^{n} g_l(t,s) r_{lk}(s) y_k(s) \Big\rangle, \end{aligned}$$

and for $1 \le i \le n$, we have

$$\begin{aligned} &\sigma_{1i} \sum_{s=a+1}^{b+1} \sum_{l=1}^{n} c_{il} \sum_{k=1}^{n} g_l(t,s) r_{lk}(s) y_k(s) \\ &\qquad = \sigma_{1i} \sum_{s=a+1}^{b+1} \sum_{l=1}^{n} \sigma_{il}|c_{il}| \sum_{k=1}^{n} g_l(t,s)\sigma_{lk}|r_{lk}(s)|\sigma_{1k}|y_k(s)| \\ &\qquad = \sum_{s=a+1}^{b+1} \sum_{l=1}^{n} |c_{il}| \sum_{k=1}^{n} g_l(t,s)|r_{lk}(s)y_k(s)| \\ &\qquad \ge \sum_{l=1}^{n} |c_{il}| \sum_{k=1}^{n} g_l(t,s_0)|r_{lk}(s_0)y_k(s_0)| \\ &\qquad \ge g_{l_i}(t,s_0)|c_{il_i} r_{l_i k_0}(s_0) y_{k_0}(s_0)| > 0. \end{aligned}$$

Hence, $Ny(t) \in \mathcal{K}_0^0$. It follows that $Ny \in \mathcal{P}_0^0$.

So we have that N is strongly positive with respect to the solid cone $\mathcal{P}_0$ and so is u_0-positive. Using Theorem 2, N has an essentially unique eigenvector y^0 in $\mathcal{P}_0$, and its corresponding eigenvalue η_0 is simple, positive, and larger than the modulus of any other eigenvalue of N.

The result follows by noting again that $\lambda_0 = 1/\eta_0$, and that $y^0 = \lambda_0 N y^0 \in \mathcal{P}_0^0$. □

We conclude with a comparison theorem, also arrived at via cone theory.

Again, let $\mathcal{P}$ be a cone in a Banach space $\mathcal{B}$. For linear operators $M, N : \mathcal{B} \to \mathcal{B}$, we write $M \le N$ (with respect to $\mathcal{P}$) if $Mu \le Nu$ for all $u \in \mathcal{P}$. The following rsult is due to Keener and Travis [**8**].

Theorem 6. *Let $\mathcal{P}$ be a cone in a Banach space $\mathcal{B}$, and $M, N : \mathcal{B} \to \mathcal{B}$ be boudned linear operators, one of which is u_0-positive. If $M \leq N$ and there exist nontrivial elements u_1, u_2 of $\mathcal{P}$ and $\eta_1, \eta_2 > 0$ such that*

$$\eta_1 u_1 \leq M u_1 \quad \text{and} \quad N u_2 \leq \eta_2 u_2,$$

then $\eta_1 \leq \eta_2$. Further, if $\eta_1 = \eta_2$, then u_1 is a scalar multiple of u_2.

Now let B_1 and B_2 be constant, nonsingular $n \times n$ matrices, with $B_1^{-1} = (c_{ij}^1)$ and $B_2^{-1} = (c_{ij}^2)$. Also, let $D_i(t) = \operatorname{diag}(d_1^i(t), \dots, d_n^i(t))$, $i = 1, 2$ satisfy the assumptions on $D(t)$, with respective Green's matrix functions $G^{D_i}(t,s) = \operatorname{diag}(g_1^i(t,s), \dots, g_n^i(t,s))$, $i = 1, 2$. Finally, let $R_1(t) = (r_{ij}^1(t))$ satisfy the assumptions on $R(t)$ in Theorem 3 with respect to B_1, and let $R_2(t) = (r_{ij}^2(t))$ satisfy the assumptions on $R(t)$ in Theorem 5 with respect to B_2.

Theorem 7. *Suppose that for $1 \leq i,\ j \leq n$ we have*

$$\begin{aligned} |r_{ij}^1(t)| &\leq |r_{ij}^2(t)|, \qquad t \in [a+1, b+1] \\ |c_{ij}^1| &\leq |c_{ij}^2| \end{aligned}$$

and

(9)
$$\sum_{\tau=a}^{s_1-1} \frac{1}{d_i^1(\tau)} \sum_{\tau=s_2}^{b+1} \frac{1}{d_i^1(\tau)} \sum_{\tau=a}^{b+1} \frac{1}{d_i^2(\tau)} \leq \sum_{\tau=a}^{s_1-1} \frac{1}{d_i^2(\tau)} \sum_{\tau=s_2}^{b+1} \frac{1}{d_i^2(\tau)} \sum_{\tau=a}^{b+1} \frac{1}{d_i^1(\tau)}$$

for $a+1 \leq s_1 \leq s_2 \leq b+1$. Then the least positive eigenvalues λ_1 and λ_2 of

$$\begin{aligned} -\Delta[D_1(t-1)B_1\Delta y(t-1)] &= \lambda R_1(t) y(t), \qquad t \in [a+1, b+1] \\ y(a) &= 0 = y(b+2) \end{aligned} \tag{10}$$

and

$$\begin{aligned} -\Delta[D_2(t-1)B_2\Delta y(t-1)] &= \lambda R_2(t) y(t), \qquad t \in [a+1, b+1] \\ y(a) &= 0 = y(b+2), \end{aligned} \tag{11}$$

respectively, satisfy $\lambda_2 \leq \lambda_1$. Further, if $\lambda_1 = \lambda_2$, then $R_1(t) = R_2(t)$ on $[a+1, b+1]$; in addition, if $R_2(t)$ is symmetric, we also have that $B_1 = B_2$ and $G^{D_1}(t,s) = G^{D_2}(t,s)$ on $[a, b+2] \times [a+1, b+1]$.

Proof. Define operators N_1 and N_2 on $\mathcal{B}_0$ by

$$N_1y(t) = \sum_{s=a+1}^{b+1} B_1^{-1}G^{D_1}(t,s)R_1(s)y(s)$$

and

$$N_2y(t) = \sum_{s=a+1}^{b+1} B_2^{-1}G^{D_2}(t,s)R_2(s)y(s).$$

Note that N_2 is u_0-positive with respect to $\mathcal{P}_0$.

Let $y \in \mathcal{P}_0$. Then

$$\begin{aligned} &N_2y(t) - N_1y(t) \\ &\quad = \sum_{s=a+1}^{b+1} [B_2^{-1}G^{D_2}(t,s)R_2(s) - B_1^{-1}G^{D_1}(t,s)R_1(s)]y(s) \\ &\quad = \left\langle \sum_{s=a+1}^{b+1} \sum_{k=1}^{n}\sum_{l=1}^{n} [c_{il}^2 g_l^2(t,s) r_{lk}^2(s) - c_{il}^1 g_l^1(t,s) r_{lk}^1(s)] y_k(s) \right\rangle. \end{aligned}$$

Also note that inequality (9) implies that $g_i^1(t,s) \le g_i^2(t,s)$ for $1 \le i \le n$ and $t,s \in [a+1,b+1]$. Now let $t \in [a+1,b+1]$. For $1 \le i \le n$, we have

$$\begin{aligned} &\sigma_{1i} \sum_{s=a+1}^{b+1} \sum_{k=1}^{n}\sum_{l=1}^{n} [c_{il}^2 g_l^2(t,s) r_{lk}^2(s) - c_{il}^1 g_l^1(t,s) r_{lk}^1(s)] y_k(s) \\ &= \sigma_{1i} \sum_{s=a+1}^{b+1} \sum_{k=1}^{n}\sum_{l=1}^{n} \sigma_{il}\sigma_{lk}\sigma_{1k} [g_l^2(t,s)|c_{il}^2 r_{lk}^2(s)| - g_l^1(t,s)|c_{il}^1 r_{lk}^1(s)|]|y_k(s)| \\ &= \sum_{s=a+1}^{b+1} \sum_{k=1}^{n}\sum_{l=1}^{n} [g_l^2(t,s)|c_{il}^2 r_{lk}^2(s)| - g_l^1(t,s)|c_{il}^1 r_{lk}^1(s)|]|y_k(s)| \ge 0. \end{aligned}$$

Hence, for $t \in [a+1,b+1]$, $N_2y(t) - N_1y(t) \in \mathcal{K}_0$, and so

$$N_2y - N_1y \in \mathcal{P}_0.$$

By definition, $N_1y \le N_2y$ with respect to $\mathcal{P}_0$. Since y was an arbitrary element of $\mathcal{P}_0$, we have $N_1 \le N_2$.

Now, letting y^1 and y^2 be eigenfunctions in $\mathcal{P}_0$ for λ_1 and λ_2, respectively, we get

$$\frac{1}{\lambda_1}y^1 = N_1 y^1 \quad \text{and} \quad \frac{1}{\lambda_2}y^2 = N_2 y^2.$$

Theorem 6 then gives that $1/\lambda_1 \le 1/\lambda_2$, i.e., $\lambda_2 \le \lambda_1$.

Now assume $\lambda_1 = \lambda_2$. Then $y^1 = \alpha y^2$ for some nonzero scalar α, and so

$$\begin{aligned} N_2 y^2 - N_1 y^2 &= N_2 y^2 - \frac{1}{\alpha} N_1 y^1 \\ &= \frac{1}{\lambda_2} y^2 - \frac{1}{\alpha}\frac{1}{\lambda_1} y^1 \\ &= \frac{1}{\lambda_2} y^2 - \frac{1}{\lambda_2} y^2 = 0. \end{aligned}$$

Hence, for $t \in [a+1, b+1]$ and $1 \le i \le n$, we have

$$\sigma_{1i} \sum_{s=a+1}^{b+1} \sum_{k=1}^{n} \sum_{l=1}^{n} [c_{il}^2 g_l^2(t,s) r_{lk}^2(s) - c_{il}^1 g_l^1(t,s) r_{lk}^1(s)] y_k^2(s) = 0,$$

or

$$\sum_{s=a+1}^{b+1} \sum_{k=1}^{n} \sum_{l=1}^{n} [g_l^2(t,s)|c_{il}^2||r_{lk}^2(s)| - g_l^1(t,s)|c_{il}^1||r_{lk}^1(s)|]|y_k^2(s)| = 0. \tag{12}$$

Note that for $s \in [a+1, b+1]$ and $1 \le k,\ l \le n$,

$$g_l^2(t,s)|c_{il}^2||r_{lk}^2(s)| - g_l^1(t,s)|c_{il}^1||r_{lk}^1(s)| \ge 0.$$

Also, Theorem 5 gives that $y^2 \in \mathcal{P}_0^0$ and so $y_k^2(s) \ne 0$ for $1 \le k \le n$, $s \in [a+1, b+1]$. So (12) implies that

$$g_l^2(t,s)|c_{il}^2||r_{lk}^2(s)| - g_l^1(t,s)|c_{il}^1||r_{lk}^1(s)| = 0$$

for all $s \in [a+1, b+1]$ and $1 \le k,\ l \le n$. We then get

$$\begin{aligned} 0 &= g_l^2(t,s)|c_{il}^2||r_{lk}^2(s)| - g_l^1(t,s)|c_{il}^1||r_{lk}^1(s)| \\ &\ge g_l^1(t,s)|c_{il}^1|[|r_{lk}^2(s)| - |r_{lk}^1(s)|] \ge 0 \end{aligned}$$

for all $s \in [a+1, b+1]$ and $1 \leq k,\ l \leq n$.

Since $t \in [a+1, b+1]$ and $1 \leq i \leq n$ were arbitrary, we have

$$g_l^1(t,s)|c_{il}^1|[|r_{lk}^2(s)| - |r_{lk}^1(s)|] = 0 \tag{13}$$

for all $t, s \in [a+1, b+1]$ and $1 \leq i, k, l \leq n$. Now B_1^{-1} is nonsingular, and so for each $1 \leq l \leq n$, there exists $1 \leq i_l \leq n$ such that $c_{i_l l} \neq 0$. Hence, (13) gives that

$$r_{lk}^2(s) = r_{lk}^1(s)$$

for all $s \in [a+1, b+1]$ and $1 \leq k,\ l \leq n$, and so $R_1(t) = R_2(t)$ on $[a+1, b+1]$.

Now suppose that $R_2(t)$ is symmetric. Starting with (12), and proceeding in a manner similar to that above, we get

$$g_l^1(t,s)|r_{lk}^2(s)|[|c_{il}^2| - |c_{il}^1|] = 0 \tag{14}$$

for all $t, s \in [a+1, b+1]$ and $1 \leq i, k, l \leq n$. Now let $s \in [a+1, b+1]$ and $1 \leq i,\ l \leq n$. From the hypotheses of Theorem 5, there exists $1 \leq k_i \leq n$ such that

$$r_{lk_i}^2(s) = r_{k_i l}^2(s) \neq 0.$$

Hence, (14) gives that $c_{il}^2 = c_{il}^1$ for $1 \leq i,\ l \leq n$, and so $B_1^{-1} = B_2^{-1}$; i.e., $B_1 = B_2$.

Finally, again starting with (12), we get

$$|r_{lk}^2(s)|\,|c_{il}^2|[g_l^2(t,s) - g_l^1(t,s)] = 0 \tag{15}$$

for all $t, s \in [a+1, b+1]$ and $1 \leq i, k, l \leq n$. Letting $t, s \in [a+1, b+1]$ and $1 \leq l \leq n$, there exist $1 \leq i \leq n$ and $1 \leq k_i \leq n$ such that

$$c_{il}^2 \neq 0 \quad \text{and} \quad r_{lk_i}^2(s) \neq 0,$$

and so (15) gives that

$$g_l^1(t,s) = g_l^2(t,s)$$

for $t, s \in [a+1, b+1]$ and $1 \leq l \leq n$. Hence, $G^{D_1}(t,s) = G^{D_2}(t,s)$ on $[a, b+2] \times [a+1, b+1]$. $\square$

Corollary. *Suppose that for $1 \le i,\ j \le n$, we have*

$$\begin{aligned} |r_{ij}^1(t)| &\le |r_{ij}^2(t)|, \qquad t \in [a+1, b+1] \\ |c_{ij}^1| &\le |c_{ij}^2| \end{aligned}$$

and

$$\sum_{\tau=a}^{b+1} \frac{1}{d_i^1(\tau)} \sum_{\tau=a}^{b+1} \frac{1}{d_i^2(\tau)} \le \frac{4}{d_i^2(a) d_i^2(b+1)}. \tag{16}$$

Then the least positive eigenvalues of λ_1 and λ_2 of (10) and (11), respectively, satisfy $\lambda_2 \le \lambda_1$. Further, if $\lambda_1 = \lambda_2$, then $R_1(t) = R_2(t)$ on $[a+1, b+1]$; in addition, if $R_2(t)$ is symmetric, we also have that $B_1 = B_2$ and $G^{D_1}(t,s) = G^{D_2}(t,s)$ on $[a, b+2] \times [a+1, b+1]$.

Proof. We need only show that inequality (9) holds, and the result will follow from Theorem 7.

Let $1 \le i \le n$ and $a+1 \le s_1 \le s_2 \le b+1$. Taking $a = \sum_{\tau=a}^{s_1-1} 1/d_i^1(\tau)$ and $b = \sum_{\tau=s_1}^{b+1} 1/d_i^1(\tau)$ in inequality (8), we get that

$$\begin{aligned} &\frac{\sum_{\tau=a}^{s_1-1} \frac{1}{d_i^1(\tau)} \sum_{\tau=s_2}^{b+1} \frac{1}{d_i^1(\tau)}}{\sum_{\tau=a}^{b+1} \frac{1}{d_i^1(\tau)}} \sum_{\tau=a}^{b+1} \frac{1}{d_i^2(\tau)} \\ &\qquad\le \frac{\sum_{\tau=a}^{s_1-1} \frac{1}{d_i^1(\tau)} \sum_{\tau=s_1}^{b+1} \frac{1}{d_i^1(\tau)}}{\sum_{\tau=a}^{b+1} \frac{1}{d_i^1(\tau)}} \sum_{\tau=a}^{b+1} \frac{1}{d_i^2(\tau)} \\ &\qquad= \frac{ab}{a+b} \sum_{\tau=a}^{b+1} \frac{1}{d_i^2(\tau)} \\ &\qquad\le \frac{1}{4}(a+b) \sum_{\tau=a}^{b+1} \frac{1}{d_i^2(\tau)} \\ &\qquad= \frac{1}{4} \sum_{\tau=a}^{b+1} \frac{1}{d_i^1(\tau)} \sum_{\tau=a}^{b+1} \frac{1}{d_i^2(\tau)} \\ &\qquad\le \frac{1}{d_i^2(a)} \frac{1}{d_i^2(b+1)}, \end{aligned}$$

where the last inequality follows from (16). Hence,

$$\sum_{\tau=a}^{s_1-1} \frac{1}{d_i^1(\tau)} \sum_{\tau=s_2}^{b+1} \frac{1}{d_i^1(\tau)} \sum_{\tau=a}^{b+1} \frac{1}{d_i^2(\tau)} \le \frac{1}{d_i^2(a)} \frac{1}{d_i^2(b+1)} \sum_{\tau=a}^{b+1} \frac{1}{d_i^1(\tau)}$$
$$\le \sum_{\tau=a}^{s_1-1} \frac{1}{d_i^2(\tau)} \sum_{\tau=s_2}^{b+1} \frac{1}{d_i^2(\tau)} \sum_{\tau=a}^{b+1} \frac{1}{d_i^1(\tau)},$$

and so (9) holds. $\square$

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