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SPATIALLY NONTEMPERATE PSEUDODIFFERENTIAL OPERATORS, SPHERE EXTENSIONS AND FREDHOLM THEORY

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0. Introduction. There are two main approaches to the global study of pseudodifferential operators, henceforth abbreviated ΨDO's, on a noncompact manifold which, in what follows, will always be the Euclidean n-space $\mathbf{R}^n$. In the first (and most commonly used) approach, the "calculus method," one starts with certain classes of "symbols" having suitable growth conditions with respect to appropriate weight functions, and one then assigns a ΨDO to each symbol via the Fourier inversion formula. The detailed analysis of this method is given, e.g., by L. Hörmander (cf. [**5**, vol. III, and its bibliography]). Going in the opposite direction, the second approach, the "Gelfand theory method," begins by constructing certain "comparison" C^*-algebras of ΨDO's based on suitable Schrödinger-type operators, and then uses Gelfand theory to attach a "symbol" to each ΨDO. This method is developed in detail by H.O. Cordes (cf. [3] and the references therein). Our goal in this paper is to look at the relation between the above methods, by following both of them in a rather general situation. In Section 1 we follow the "calculus method," introducing a class of symbols, not necessarily temperate in the space variables, with weight function

$$h(x,\xi) = (q(x) + |\xi|^2)^{1/2} \tag{0.1}$$

where $q(x)$ is a smooth (i.e., $C^\infty(\mathbf{R}^n)$) function satisfying

$$q(x) \geq 1, \quad \forall\, x \in \mathbf{R}^n, \qquad \lim_{|x|\to\infty} q(x) = \infty, \tag{0.2}$$

and

$$|\partial_x q(x)| = o(q(x)), \quad \text{as } |x| \to \infty, \tag{0.3}$$

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with $\partial_x = (\partial_{x_1}, \dots, \partial_{x_n}), \partial_{x_j} = \partial/\partial x_j$. As will be seen in Lemma 1.1 below, condition (0.3) may, in fact, be replaced by the stronger condition

$$(0.4) \qquad |\partial_x^\beta q(x)| = o(q(x)), \quad \text{as } |x| \to \infty, \quad \forall\, 0 \neq \beta \in \mathbf{Z}_+^n,$$

with $\partial_x^\beta = \partial_{x_1}^{\beta_1} \dots \partial_{x_n}^{\beta_n}$.

Conditions (0.2) and (0.4) are assumed to hold throughout the paper. Using a suitable set of generators, a C^*-algebra $\mathcal{A}_b$ of ΨDO's with compact commutators is then constructed and studied. In Section 2, we construct and study another C^*-algebra, also with compact commutators, this time using the "Gelfand theory method," where, with $q(x)$ as above, our basic Schrödinger operator will be $H = -\Delta + q$. In Section 3 we prove that the algebras $\mathcal{A}_b$ and $\mathcal{A}_c$ are in fact identical and provide us with a large class of "sphere extensions," one for each potential $q(x)$. The connection between the symbols attached to ΨDO's by means of the above methods is then considered in Section 4, where we also discuss necessary and sufficient Fredholm criteria.

1. The C^*-algebra $\mathcal{A}_b$ (The Beals Algebra). In this section we shall summarize the results of [**13**], where a C^*-algebra $\mathcal{A}_b$ of ΨDO's will be constructed using the calculus approach suggested to the author by Professor R. Beals. Let $q(x) \in C^\infty(\mathbf{R}^n)$ satisfy (0.2) and (0.3). Then we have

Lemma 1.1. (a) *$q(x) = o(e^{\varepsilon|x|})$, as $|x| \to \infty$, for all $\varepsilon > 0$.*

(b) *There exists $\tilde{q} \in C^\infty(\mathbf{R}^n)$, such that $\lim \tilde{q}(x)/q(x) = 1$, as $|x| \to \infty$, and that $\tilde{q}$ satisfies* (0.2) *and* (0.4).

(c) *Let $h(x,\xi) = (q(x) + |\xi|^2)^{1/2}$, and $\tilde{h}(x,\xi) = (\tilde{q}(x) + |\xi|^2)^{1/2}$, with $\tilde{q}$ as in* (b). *Then as $|x| + |\xi| \to \infty$, $\lim \tilde{h}(x,\xi)/h(x,\xi) = 1$, and for all α, $\beta \in \mathbf{Z}_+^n$, $|\alpha| = \Sigma\alpha_j$,*

$$(1.1) \qquad \tilde{h}(x,\xi)^{|\alpha|-1} |\tilde{h}_{(\beta)}^{(\alpha)}(x,\xi)| = O(1);$$

(and $o(1)$, if $\beta \neq 0$), where $\tilde{h}_{(\beta)}^{(\alpha)} = \partial_\xi^\alpha \partial_x^\beta \tilde{h}$.

Definition 1.2. Let $h(x,\xi)$ be as in (0.1), with (0.2) and (0.4) satisfied. Then, for each $m \in \mathbf{R}$, the class $S^m = S^m(\mathbf{R}^n \times \mathbf{R}^n)$, of *symbols of order m* is the set of all $a \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ such that for all multi-indices $\alpha, \beta \in \mathbf{Z}^n_+$, we have, as $|x| + |\xi| \to \infty$,

$$h(x,\xi)^{|\alpha|-m} |a^{(\alpha)}_{(\beta)}(x,\xi)| = O(1); \tag{1.2}$$

and $o(1)$, if $\beta \neq 0$. One then defines $S^{-\infty} = \cap S^m$, and $S^\infty = \cup S^m$.

Remark 1.3. (a) Equation (1.1) implies that $h \in S^1$.

(b) S^m is a Fréchet space, if the suprema of the left sides of the relations (1.2) are used as semi-norms.

(c) If $a \in S^m$, $b \in S^{m'}$, then $a^{(\alpha)}_{(\beta)} \in S^{m-|\alpha|}$, $ab \in S^{m+m'}$. Moreover, the map $(a,b) \to ab$ is continuous.

The corresponding classes of ΨDO's are defined in the following

Theorem 1.4. *Let $a \in S^m$. Then, with $u \in \mathcal{S}$ and $\hat{u}(\xi) = \int e^{-ix.\xi} u(x)\, dx$,*

$$a(x,D)u(x) = (2\pi)^{-n} \int e^{ix.\xi} a(x,\xi)\hat{u}(\xi)\, d\xi \in C^\infty(\mathbf{R}^n),$$

and the bilinear map $(a,u) \to a(x,D)u$ is continuous. Also, the commutators with $D_j = -i\partial/\partial x_j$, and x_j are given by

$$\begin{aligned} [a(x,D), D_j] &= i a_{(j)}(x,D), \\ [a(x,D], x_j] &= -i a^{(j)}(x,D), \end{aligned} \tag{1.3}$$

with $a_{(j)} = \partial a/\partial x_j$, and $a^{(j)} = \partial a/\partial \xi_j$. $a(x,D)$ is called a ΨDO of order m with "symbol" $a(x,\xi)$, and we define the corresponding class of operators

$$\Psi^m = \{a(x,D) : a \in S^m\}.$$

Remark 1.5. Notice that $q(x)$ is not necessarily temperate. Consider, for example, $q(x) = \exp(\langle x\rangle^{1/2})$, where $\langle x\rangle = (1+|x|^2)^{1/2}$, so that for an operator $a(x,D) \in \Psi^m$, $m > 0$, and a function $u \in \mathcal{S}$, $a(x,D)u \notin \mathcal{S}$ in general. However, if $m \leq 0$, then $a(x,D) : \mathcal{S} \to \mathcal{S}$. In fact, if we consider Hörmander's basic symbol class,

$$S^m_{1,0} = \{a \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n) : |a^{(\alpha)}_{(\beta)}(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}\},$$

then $S^m \subseteq S^m_{1,0}$, for all $m \leq 0$, and the global calculus in [**5**, vol. III] can be applied. The main facts regarding the operators in Ψ^m, $m \leq 0$, are summarized in the following theorems. The proofs may be found in [**13**], except for the positivity of $q^{1/2}h^{-1}(x, D) + \mathcal{K}$ in Theorem 1.7, which will follow from the proof of Corollary 3.4.

Theorem 1.6. (a) *Let $a(x, D) \in \Psi^m$, $b(x, D) \in \Psi^{m'}$, $m \leq 0$, $m' \leq 0$. Then the adjoint operator $a(x, D)^* = a^*(x, D) \in \Psi^m$, and we have the asymptotic expansion*

$$a^*(x, \xi) \sim \sum_\alpha \partial_\xi^\alpha D_x^\alpha \bar{a}(x, \xi)/\alpha!.$$

Also, $a(x, D)b(x, D) = c(x, D) \in \Psi^{m+m'}$, with the asymptotic expansion

$$c(x, \xi) \sim \sum_\alpha \partial_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi)/\alpha!.$$

(b) *Every $a(x, \xi) \in \Psi^0$ is a bounded operator of $\mathcal{H} = L^2(\mathbf{R}^n)$, and the norm $||a(x, D)||$ is bounded by a finite number of semi-norms of a in S^0. In particular, Ψ^0 is a $*$-subalgebra of the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators of the Hilbert space $\mathcal{H}$.*

(c) *If $a \in S^0$ and $\lim a(x, \xi) = 0$, as $|x| + |\xi| \to \infty$, then $a(x, D) \in \mathcal{K}$, where $\mathcal{K} = \mathcal{K}(\mathcal{H})$ is the ideal of compact operators of $\mathcal{H}$.*

(d) *$\Psi^m \subseteq \mathcal{K}$ for all $m < 0$.*

(e) *If $a \in S^0$ is real-valued, i.e., $a = \bar{a}$, then $a(x, D)^* - a(x, D) \in \Psi^{-1} \subseteq \mathcal{K}$.*

(f) *If $a(x, D), b(x, D) \in \Psi^0$, then the commutator $[a(x, D), b(x, D)] \in \Psi^{-1} \subseteq \mathcal{K}$.*

(g) *If $a \in S^0$, then $a(x, D)$ has essential spectrum*

$$M_a = \text{ess-sp}\,(a(x, D)) = \{\mu : \lim a(x^\nu, \xi^\nu) = \mu\},$$

for some sequence (x^ν, ξ^ν), with $|x^\nu| + |\xi^\nu| \to \infty$.

Next, with $\langle x \rangle = (1 + |x|^2)^{1/2}$ and $1 \leq j \leq n$, consider the symbols

$$\lambda_j(x) = x_j/\langle x \rangle, \qquad a_0(x, \xi) = q(x)^{1/2} h(x, \xi)^{-1}, \qquad a_j(x, \xi) = \xi_j h(x, \xi)^{-1}. \tag{1.4}$$

These are easily seen to be symbols in S^0, with corresponding operators

$$(1.5)\qquad \begin{aligned} x_j/\langle x\rangle, \qquad a_0(x,D) &= q^{1/2}(x)h^{-1}(x,D),\\ a_j(x,D) &= h^{-1}(x,D)D_j, \end{aligned}$$

where we use the notation $h^{-k} = 1/h^k$.

Theorem 1.7. (a) *$a_0(x,D) + \mathcal{K}$ is positive and $a_j(x,D) + \mathcal{K}$ is self-adjoint, $j = 1, \dots, n$.*

(b) *The joint essential spectrum of the operators* (1.5) *is given by*

$$\begin{aligned} \mathbf{M} &= j - \text{ess-sp}\,(\lambda_1(x), \dots, \lambda_n(x), A_0, A_1, \dots, A_n)\\ &\cong \partial(\mathbf{B}^n \times S^n_+) \cong S^{2n-1}. \end{aligned}$$

Here, the operators in (1.5) *are denoted $\lambda_j(x) = x_j\langle x\rangle^{-1}$, $A_0 = a_0(x,D)$, and $A_j = a_j(x,D)$. Also $\mathbf{B}^n$ is the ball-compactification of $\mathbf{R}^n$, induced by the homeomorphism $\lambda : x \to x\langle x\rangle^{-1}$ of $\mathbf{R}^n$ onto the open unit ball $\overset{o}{B}{}^n = \{x : |x| < 1\}$. Finally, ∂ denotes the boundary and S^n_+ is the upper hemisphere*

$$S^n_+ = \{(t_0, t) \in \mathbf{R}^{n+1} : 0 \le t_0 \le 1, t_0^2 + |t|^2 = 1\}.$$

(c) *With notations as in* (b), *consider the C^*-algebra*

$$(1.6)\qquad \mathcal{A}_b = C^*(\mathcal{K}, \lambda_1(x), \dots, \lambda_n(x), A_0, A_1, \dots, A_n),$$

where $C^*(\dots)$ denotes the unital C^*-algebra generated by $(\dots)$. Then $\mathcal{A}_b$ has compact commutators, and we have

$$\mathcal{A}_b/\mathcal{K} \cong C(\partial(\mathbf{B}^n \times S^n_+)) \cong C(S^{2n-1}).$$

2. The C^*-algebra $\mathcal{A}_c$ (The Cordes Algebra). In this section we summarize the main results of [**11**] and use the technique of "comparison algebras," due to H.O. Cordes, to construct another C^*-algebra of ΨDO's with compact commutators. As before, we assume that $q(x) \in C^\infty(\mathbf{R}^n)$ satisfies (0.2) and (0.4), and consider the Schrödinger operator

$$H_0 = -\Delta + q, \qquad \text{Dom}\,(H_0) = C_0^\infty(\mathbf{R}^n),$$

on the Hilbert space $\mathcal{H} = L^2$. Then all the operators H_0^m, $m = 1, 2, 3, \dots$, are essentially self-adjoint on $C_0^\infty(\mathbf{R}^n)$, and the corresponding closures, denoted by H^m, are self-adjoint with dense range. Now consider the operator $\Lambda = H^{-1/2}$, which is positive and compact, and introduce the C^*-algebra

$$\mathcal{A}_c = C^*(\mathcal{K}, x_1\langle x\rangle^{-1}, \dots, x_n\langle x\rangle^{-1}, q^{1/2}\Lambda, D_1\Lambda, \dots, D_n\Lambda). \tag{2.1}$$

For the following facts about $\mathcal{A}_c$, we refer to [**11**] (cf. also [**3**]).

Theorem 2.1. (a) *The C^*-algebra $\mathcal{A}_c$ of* (2.1) *has compact commutators*: $[\mathcal{A}_c, \mathcal{A}_c] \subseteq \mathcal{K}$. *In particular,* $[q^{1/2}, \Lambda^2] = \Lambda K_0 \Lambda$ *and* $[D_j, \Lambda^2] = \Lambda K_j \Lambda$, K_0, $K_j \in \mathcal{K}$, $1 \le j \le n$.

(b) $q^{1/2}\Lambda + \mathcal{K}$ *is positive and* $D_j\Lambda + \mathcal{K}$ *is self-adjoint for* $j = 1, \dots, n$.

(c) $\mathcal{A}_c/\mathcal{K} \cong C(\mathbf{M})$, *with* $\mathbf{M} \cong \partial(\mathbf{B}^n \times S_+^n) \cong S^{2n-1}$.

(d) *Let* $\gamma : \mathcal{A}_c/\mathcal{K} \xrightarrow{\sim} C(\mathbf{M})$ *be the Gelfand map and* $\pi : \mathcal{A}_c \to \mathcal{A}_c/\mathcal{K}$ *the natural projection. Define the* symbol homomorphism $\sigma = \gamma \circ \pi : \mathcal{A}_c \to C(\mathbf{M})$, *and for each* $A \in \mathcal{A}_c$, *its* symbol (*or* σ-symbol) *by* $\sigma_A = \sigma(A) \in C(\mathbf{M})$. *Then, for each maximal ideal* $m = (s, t_0, t) \in \partial(\mathbf{B}^n \times S_+^n) = \mathbf{M}$, *and each compact* $K \in \mathcal{K}$,

$$\begin{aligned} \sigma_{x_j/\langle x\rangle}(m) &= s_j, & \sigma_{q^{1/2}\Lambda}(m) &= t_0, \\ \sigma_{D_j\Lambda}(m) &= t_j, & \sigma_K(m) &= 0. \end{aligned} \tag{2.2}$$

(e) *If* $\tilde{q} \in C^\infty(\mathbf{R}^n)$ *also satisfies* (0.2) *and* (0.4), *and if* $\tilde{q}(x)/q(x) \to 1$, *as* $|x| \to \infty$, *then with* $\widetilde{H} = -\Delta + \tilde{q}$, *and* $\tilde{\Lambda} = \widetilde{H}^{-1/2}$, *we have*

$$H^\rho [H^{-r}, \widetilde{H}^{-t}] H^\tau \in \mathcal{K}, \tag{2.3}$$

for all $r, t, \rho, \tau \in [0, 1]$, *satisfying* $\rho + \tau < r + 1/2$. *As a consequence, we have* $q^{1/2}\Lambda - \tilde{q}^{1/2}\tilde{\Lambda} \in \mathcal{K}$, *and* $D_j\Lambda - D_j\tilde{\Lambda} \in \mathcal{K}$, *and if* $\tilde{\mathcal{A}}_c$ *is the corresponding* C^**-algebras as in* (2.1), *we have* $\mathcal{A}_c = \tilde{\mathcal{A}}_c$.

3. $\mathcal{A}_b = \mathcal{A}_c$ **(set theories equality of the Beals and Cordes algebras).** We will prove the following theorem which shows that the C^*-algebras $\mathcal{A}_b$ and $\mathcal{A}_c$ defined by (1.6) and (2.1) are, in fact, identical.

Theorem 3.1. *With notations as in Sections* 1 *and* 2, *we have*

$$q^{1/2}h^{-1}(x,D) \equiv q^{1/2}\Lambda \pmod{\mathcal{K}}, \tag{3.1}$$

$$h^{-1}(x,D)D_j \equiv D_j h^{-1}(x,D) \equiv D_j\Lambda \pmod{\mathcal{K}}, \quad 1 \le j \le n. \tag{3.2}$$

In particular, $\mathcal{A}_b = \mathcal{A}_c$.

Remark 3.2. The calculus in Section 1 was given for S^m, $m \le 0$, because the symbols in S^m, $m > 0$, are not in general temperate in x-variables. In particular, if $A \in \Psi^m$, $B \in \Psi^{m'}$, then AB is not in general defined for $m' > 0$. However, it is obviously defined for all $m' \le 0$, and we will use such compositions in what follows.

Theorem 3.1 will follow from Corollaries 3.4 and 3.6 below.

Lemma 3.3. *There is a compact* ΨDO $K \in \Psi^{-1} \subseteq \mathcal{K}$ *such that*

$$h^{-2}(x,D) = \Lambda^2 + \Lambda^2 K. \tag{3.3}$$

Proof. Differentiating formally under the integral sign in $h^{-2}(x,D)u$, for a function $u \in \mathcal{S}$, we get

$$\begin{aligned} Hh^{-2}(x,D)u(x) &= (-\Delta + q(x))h^{-2}(x,D)u(x) \\ &= (2\pi)^{-n}\int H((q+|\xi|^2)^{-1}e^{ix.\xi})\hat{u}(\xi)\,d\xi. \end{aligned}$$

Now

$$e^{-ix.\xi}(q-\Delta)(e^{ix.\xi}(q(x)+|\xi|^2)^{-1}) = 1 + k(x,\xi),$$

where

$$\begin{aligned} k(x,\xi) = \Sigma(q_{x_jx_j} + 2i\xi_j q_{x_j})&(q+|\xi|^2)^{-2} \\ &- 2\Sigma(q_{x_j})^2(q+|\xi|^2)^{-3} \in S^{-1}. \end{aligned}$$

Thus, with $K = k(x,D) \in \Psi^{-1}$, we have $Hh^{-2}(x,D) = I + K$, from which (3.3) follows at once. □

Corollary 3.4. *We have* $q^{1/2}h^{-1}(x,D) \equiv q^{1/2}\Lambda \pmod{\mathcal{K}}$, *i.e.,* (3.1) *holds.*

Proof. From (3.3) we get

$$qh^{-2}(x,D) = q\Lambda^2 + q\Lambda^2 K. \tag{3.4}$$

Now, by the calculus in Section 1, we have

$$qh^{-2}(x,D) \equiv (q^{1/2}h^{-1}(x,D))^2 \pmod{\mathcal{K}},$$
$$q^{1/2}h^{-1}(x,D) \equiv (q^{1/4}h^{-1/2}(x,D))^2 \pmod{\mathcal{K}},$$

and the operators $q^{1/2}h^{-1}(x,D)$ and $q^{1/4}h^{-1/2}(x,D)$ are both self-adjoint (mod$\mathcal{K}$), because they have real-valued symbols. In particular, the operator $q^{1/2}h^{-1}(x,D) + \mathcal{K}$ is positive. On the other hand, by the compactness of commutators in Section 2, we have

$$q\Lambda^2 \equiv \Lambda q \Lambda \equiv (q^{1/2}\Lambda)^*(q^{1/2}\Lambda) \pmod{\mathcal{K}},$$

and $q^{1/2}\Lambda + \mathcal{K}$ is positive. But then (3.4) implies

$$(q^{1/2}h^{-1}(x,D))^2 \equiv (q^{1/2}\Lambda)^2 \pmod{\mathcal{K}},$$

from which (3.1) follows by taking square roots. □

Lemma 3.5. *The operators* $(1-\Delta)^{1/2}h^{-1}(x,D)$ *and* $(1-\Delta)^{1/2}\Lambda$ *are both positive* (mod$\mathcal{K}$)*, and we have*

$$(1-\Delta)^{1/2}h^{-1}(x,D) \equiv (1-\Delta)^{1/2}\Lambda \pmod{\mathcal{K}}. \tag{3.5}$$

Proof. First note that $\langle\xi\rangle = (1+|\xi|^2)^{1/2} \in S^1_{1,0}$, and $h^{-1}(x,\xi) \in S^{-1}_{1,0}$, and we have $(1-\Delta)^{1/2}h^{-1}(x,D) = c(x,D)$, with

$$c(x,\xi) = (2\pi)^{-n} \iint \langle\eta\rangle h^{-1}(y,\xi) e^{-i(x-y).(\xi-\eta)}\, dy\, d\eta,$$

where, again, $\langle\eta\rangle = (1+|\eta|^2)^{1/2}$. Also, $h^{-1}(x,D)(1-\Delta)^{1/2} = d(x,D)$, with

$$d(x,\xi) = \langle\xi\rangle h^{-1}(x,\xi).$$

Now we have the asymptotic expansion

$$c(x,\xi) \sim d(x,\xi) + \sum_{\alpha \neq 0} \partial_\xi^\alpha \langle \xi \rangle D_x^\alpha h^{-1}(x,\xi)/\alpha!,$$

which implies that

$$[(1-\Delta)^{1/2}, h^{-1}(x,D)] \in \Psi^{-1} \subseteq \mathcal{K}.$$

However, $h^{-1}(x,D)^* - h^{-1}(x,D) \in \Psi^{-2}$, and $\Psi^{-2}(1-\Delta)^{1/2} \subseteq \Psi^{-1} \subseteq \mathcal{K}$, so that the operator $(1-\Delta)^{1/2}h^{-1}(x,D) + \mathcal{K}$ is self-adjoint, and we have

$$(3.6) \qquad (1-\Delta)h^{-2}(x,D) \equiv ((1-\Delta)^{1/2}h^{-1}(x,D))^2 \pmod{\mathcal{K}}.$$

Similarly, using the calculus, we can prove that $(1-\Delta)^{1/4}h^{-1/2}(x,D) + \mathcal{K}$ is self-adjoint, and we have

$$(3.7) \quad (1-\Delta)^{1/2}h^{-1}(x,D) \equiv ((1-\Delta)^{1/4}h^{-1/2}(x,D))^2 \pmod{\mathcal{K}},$$

so that the operator $(1-\Delta)^{1/2}h^{-1}(x,D) + \mathcal{K}$ is indeed positive. Next, note that by the compactness of commutators in Section 2, we have

$$(1-\Delta)\Lambda^2 \equiv \Lambda(1-\Delta)\Lambda \equiv ((1-\Delta)^{1/2}\Lambda)^*(1-\Delta)^{1/2}\Lambda \pmod{\mathcal{K}}.$$

On the other hand, the positivity of $(1-\Delta)^{1/2}\Lambda + \mathcal{K}$ is proved by observing that, as in the proof of Lemma 2.2 in [**11**], we can show that, for $r, t, \rho, \tau \in [0,1]$, with $\rho + \tau < r + 1/2$, one has, as in (2.3),

$$(1-\Delta)^\rho [(1-\Delta)^{-r}, (q-\Delta)^{-t}](1-\Delta)^\tau \in \mathcal{K}.$$

This implies that

$$(1-\Delta)^{1/2}\Lambda \equiv ((1-\Delta)^{1/4}\Lambda^{1/2})^2 \pmod{\mathcal{K}}$$

and

$$[(1-\Delta)^{1/4}, \Lambda^{1/2}] \in \mathcal{K},$$

from which the positivity of $(1-\Delta)^{1/2}\Lambda + \mathcal{K}$ follows at once. Now, from (3.3), we have

$$(1-\Delta)h^{-2}(x,D) = (1-\Delta)\Lambda^2 + (1-\Delta)\Lambda^2 K,$$

which, in view of (3.6) and (3.7), implies

$$((1-\Delta)^{1/2}h^{-1}(x,D))^2 \equiv ((1-\Delta)^{1/2}\Lambda)^2 \quad (\mathrm{mod}\ \mathcal{K}),$$

and (3.5) is obtained by taking square roots. $\square$

Corollary 3.6. *We have*

$$h^{-1}(x,D)D_j \equiv D_j h^{-1}(x,D) \equiv D_j\Lambda \quad (\mathrm{mod}\ \mathcal{K}), \qquad 1 \le j \le n,$$

i.e., (3.2) *holds.*

Proof. First, by (1.3), $[h^{-1}(x,D), D_j] = ih^{-1}_{(j)}(x,D) \in \Psi^{-1} \subseteq \mathcal{K}$. Next, using (3.5), we have

$$\begin{aligned} D_j h^{-1}(x,D) &= D_j(1-\Delta)^{-1/2}(1-\Delta)^{1/2}h^{-1}(x,D) \\ &\equiv D_j(1-\Delta)^{-1/2}(1-\Delta)^{1/2}\Lambda = D_j\Lambda \quad (\mathrm{mod}\ \mathcal{K}), \end{aligned}$$

which completes the proof of the corollary and of Theorem 3.1. $\square$

Remark 3.7. It follows from the results of this section that $\mathcal{A}_q = \mathcal{A}_b = \mathcal{A}_c$ is in fact a *sphere extension*, in the sense that the short sequence

$$0 \to \mathcal{K} \to \mathcal{A}_q \to C(S^{2n-1}) \to 0$$

is exact (cf., e.g., [**4**]). In other words, we get a sphere extension $\mathcal{A}_q$ for each smooth potential $q(x)$ satisfying the conditions (0.2) and (0.4).

4. Fredholm theory in $\mathcal{A}_q = \mathcal{A}_b = \mathcal{A}_c$. To obtain Fredholm criteria in the C^*-algebra $\mathcal{A}_q$, we will first explore the relation between the *Gelfand theory symbol*, σ_A, of a ΨDO $A = a(x,D) \in \mathcal{A}_q$ and its *calculus symbol*, $a(x,\xi)$.

Theorem 4.1. *With the open upper hemisphere and open unit ball defined, respectively, by*

$$\overset{\circ}{S}{}^n_+ = \{(t_0,t) \in (0,1] \times [-1,1]^n : t_0^2 + |t|^2 = 1\},$$

and

$$\overset{\circ}{B}{}^n = \{x : |x| < 1\},$$

consider the map $\kappa : \mathbf{R}^n \times \mathbf{R}^n \to \overset{\circ}{B}{}^n \times \overset{\circ}{S}{}^n_+$, *defined by*

$$\kappa(x,\xi) = (x\langle x\rangle^{-1}, a_0(x,\xi), a_1(x,\xi), \dots, a_n(x,\xi)),$$

with the symbols $a_0, a_1, \dots, a_n$ *defined by* (1.4). *Then* κ *is a homeomorphism, and for each operator* $a(x,D) \in \mathcal{A}_q$, *the* σ*-symbol of this operator is given by the restriction*

$$\sigma_{a(x,D)} = a \circ \kappa^{-1} | \partial(B^n \times S^n_+), \tag{4.1}$$

where the continuous extension of $a \circ \kappa^{-1}$ *to* $B^n \times S^n_+$ *is still denoted* $a \circ \kappa^{-1}$. *In particular,* $a(x,D) \in \mathcal{A}_q$ *is Fredholm, if and only if* $a \circ \kappa^{-1}$ *never vanishes on* $\partial(B^n \times S^n_+) \cong S^{2n-1}$.

Proof. That κ is a homeomorphism is obvious. In fact, we have

$$\kappa^{-1}(s,t_0,t) = (s(1-|s|^2)^{-1/2}, \qquad \frac{t}{t_0} q^{1/2}(s(1-|s|^2)^{-1/2})),$$
$$(s,t_0,t) \in \overset{\circ}{B}{}^n \times \overset{\circ}{S}{}^n_+.$$

Also, with notations as in (1.4) and $m = (s,t_0,t)$,

$$\lambda_j \circ \kappa^{-1}(m) = s_j, \qquad a_0 \circ \kappa^{-1}(m) = t_0, \qquad a_j \circ \kappa^{-1}(m) = t_j. \tag{4.2}$$

Next, consider the function algebra

$$\mathcal{B} = C^*(\lambda_1(x), \dots, \lambda_n(x), a_0(x,\xi), a_1(x,\xi), \dots, a_n(x,\xi)).$$

Then, using κ, we have $\mathcal{B} \cong C(\mathbf{B}^n \times S^n_+)$, so that (4.1) follows from Theorem 1.7, (2.2), Theorem 3.1, and (4.2) for the generators of the algebra, and hence for other operators as well. $\square$

Now, for each $N \geq 0$, define the Nth Sobolev space $\mathcal{H}_N$, by

$$\mathcal{H}_N = \mathrm{Dom}\,(\Lambda^{-N}) = \mathrm{Dom}\,(H^{N/2}),$$

and consider the linear partial differential operator

$$L = \sum_{|\alpha|+k \le N} a_{\alpha,k}(x) q^{k/2} D^{\alpha}, \qquad k \in \mathbf{Z}_+, \tag{4.3}$$

where each $a_{\alpha,k} \in C^{\infty}(\mathbf{R}^n)$ is bounded, and has a continuous extension to $\mathbf{B}^n$.

Lemma 4.2. *The operators $L\Lambda^N$ and $Lh^{-N}(x,D)$ are in $\mathcal{A}_q$, and we have*

$$L\Lambda^N \equiv Lh^{-N}(x,D) \quad (\mathrm{mod}\ \mathcal{K}). \tag{4.4}$$

More precisely, we have

$$L\Lambda^N \equiv \sum_{|\alpha|+k=N} a_{\alpha,k}(q^{1/2}\Lambda)^k (D_1\Lambda)^{\alpha_1} \cdots (D_n\Lambda)^{\alpha_n} \quad (\mathrm{mod}\ \mathcal{K}), \tag{4.5}$$

and with notations as in Theorem 1.7 (b)*,*

$$Lh^{-N}(x,D) \equiv L(h^{-1}(x,D))^N \equiv \sum_{|\alpha|+k=N} a_{\alpha,k} A_0^k A_1^{\alpha_1} \cdots A_n^{\alpha_n} \quad (\mathrm{mod}\ \mathcal{K}). \tag{4.6}$$

Proof. First of all, (4.4) follows from (4.5), (4.6) and Theorem 3.1. Now (4.5) is a consequence of the compactness of commutators in Section 2 and induction. Finally, (4.6) follows from the calculus in Section 1 and induction. For instance, one sees that for all α, with $|\alpha| + k < N$, the operators $q^{k/2} D^{\alpha} h^{-N}(x,D)$ are all compact, while for $|\alpha| + k = N$,

$$\begin{aligned} q^{k/2} D^{\alpha} h^{-N}(x,D) &\equiv q^{k/2} h^{-N}(x,D) D^{\alpha} \\ &\equiv A_0^k (h^{-1}(x,D))^{|\alpha|} D^{\alpha} \\ &\equiv A_0^k A_1^{\alpha_1} \cdots A_n^{\alpha_n} \quad (\mathrm{mod}\ \mathcal{K}), \end{aligned}$$

which completes the proof of the lemma. □

Corollary 4.3. *Let L be as* (4.3). *Then, for each maximal ideal $m = (s, t_0, t) \in \partial(\mathbf{B}^n \times S^n_+) \cong S^{2n-1}$, we have*

$$\sigma_{Lh^{-N}(x,D)}(s, t_0, t) = \sigma_{L\Lambda^N}(s, t_0, t) = \sum_{|\alpha|+k=N} a_{\alpha,k}(s) t_0^k t^\alpha,$$

where the continuous extension of $a_{\alpha,k}$ to $\mathbf{B}^n$ is still denoted $a_{\alpha,k}$. In particular, L with $\mathrm{Dom}\,(L) = \mathcal{H}_N$, *the Nth Sobolev space, is Fredholm if and only if*

$$\sum_{|\alpha|+k=N} a_{\alpha,k}(s) t_0^k t^\alpha \neq 0, \quad \forall\, (s, t_0, t) \in \partial(\mathbf{B}^n \times S^n_+) \cong S^{2n-1}.$$

Proof. This follows at once from Lemma 4.2 and Theorem 4.1. □

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