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A COUNTER-EXAMPLE IN THE THEORY OF ALMOST PERIODIC DIFFERENTIAL EQUATIONS

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ABSTRACT. We show by means of an explicit example that a bounded solution of the real two-term differential equation of the second order, $x'' + a(t)x = 0$, $t \in \mathbf{R}$, where $a(t)$ is Bohr almost periodic, is not almost periodic. This disproves various claims which have appeared in the literature. We also provide such an example for the general linear equation of order k, $k \geq 2$, with almost periodic coefficients.

1. Introduction. It is known that if $a(t)$ is a piecewise continuous periodic function on $\mathbf{R}$, any bounded solution of the equation

$$x'' + a(t)x = 0, \qquad t \in \mathbf{R}, \tag{1.1}$$

is (Bohr) almost periodic [**2, 5**, p. 101] on account of Floquet theory. It is natural to expect such a result to hold in the event that $a(t)$ is merely almost-periodic and, indeed, some authors (e.g., [**1**]) have used this claim in proofs.

However, it is shown in [**4**, p. 333] (cf., also [**5**, p. 97]) that this weaker result is false at least in the case of first-order linear equations but their construction does not seem to adapt itself so easily to the second order case or higher order cases. In this note we present a counterexample to the statement that boundedness by itself implies almost periodicity in the second order case and subsequently for the general case of a linear equation of order k with almost periodic coefficients, $k > 2$. We note, in passing, that in the constant coefficient case, the equation

$$x^{(k)} + a_1(t)x^{(k-1)} + \cdots + a_k(t)x = 0, \qquad t \in \mathbf{R}, \tag{1.2}$$

has a bounded solution if and only if it is almost-periodic [**6**] and in this case all higher order derivatives up to order k are almost-periodic as well [**3**, p. 48].

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2. Construction of the counterexample. We begin with a lemma which displays the character of almost periodic (a.p. for brevity) solutions of a linear a.p. system.

Lemma 2.1. *Let $A(t)$ be an a.p. matrix (i.e., every entry is a.p.), and let $X(t)$ be an a.p. solution of the vector system $X'(t) = A(t)X(t)$. Then either* $\inf_{t\in\mathbf{R}}|X(t)| > 0$ *or else $X(t)$ is the trivial solution.*

Proof. See [**5**, p. 85]. □

Theorem 2.2. *There is a solution of an a.p. equation* (1.1) *which is bounded but not almost periodic.*

Proof. We define a sequence $\{f_n(t)\}$ of functions by

$$f_n(t) = -(1/n^2)\sin(t/n^3), \qquad t \in \mathbf{R}.$$

Then f_n is odd, periodic of period $2\pi n^3$ and

$$\int_0^t f_n(s)\,ds \le 0, \quad \text{all } t \in \mathbf{R}. \tag{2.1}$$

In addition,

$$\sum_{n=1}^{\infty}\sum_{p=0}^{2} ||f_n^{(p)}||_\infty < \infty$$

where $f_n^{(p)}$ is the p-th derivative of f_n as usual. Thus, each one of the functions $g(t), \dots, g''(t)$ defined by successively differentiating the series

$$g(t) = \sum_{n=1}^{\infty} f_n(t) \tag{2.2}$$

is almost periodic, as each such function is the uniform limit on $\mathbf{R}$ of a sequence of a.p. functions, and so is itself a.p., [**2**]. Furthermore,

$$\int_0^t g(s)\,ds \le 0, \qquad t \in \mathbf{R}. \tag{2.3}$$

So the function $x(t)$ defined by

$$x(t) = \exp\left\{\int_0^t g(s)\,ds\right\} \tag{2.4}$$

is a nontrivial bounded solution of the differential equation

$$x'' - (g^2(t) + g'(t))x = 0, \tag{2.5}$$

with an a.p. coefficient and with the additional property that

$$\inf_{t \in \mathbf{R}} |x^2(t) + x'^2(t)| = 0.$$

Now, $X(t) = \operatorname{col}(x(t), x'(t))$ solves $X'(t) = A(t)X(t)$ for an appropriate a.p. matrix $A(t)$.

Thus $\inf_{t\in\mathbf{R}}|X(t)| = 0$, and so by Lemma 2.1, $X(t)$ cannot be a.p., i.e., neither one of $x(t), x'(t)$ can be a.p. □

The other linearly independent solution of (2.5) must be unbounded by a result in [**7**, p. 104]. This now leads to the open question: Let $a(t)$ be a.p. on $\mathbf{R}$. If *every* solution of (1.1) is bounded on $\mathbf{R}$, are all solutions necessarily a.p.?

In this generality, the answer to this question is not known to us.

3. A counter-example for the general linear equation of degree k. We sketch the idea. Fix an integer $k > 2$. Define a sequence f_n by

$$f_n(t) = -(1/n^k)\sin(t/n^{k+1}), \qquad t \in \mathbf{R}.$$

Then, as before, f_n is odd, periodic of period $2\pi n^{k+1}$ and (2.1) holds. Furthermore,

$$\sum_{n=1}^{\infty}\sum_{p=0}^{k} ||f_n^{(p)}||_\infty < \infty.$$

Next the functions $g(t), g'(t), g''(t), \dots, g^{(k)}(t)$ defined by successively differentiating the series $g(t)$ in (2.2) are a.p. as each $g^{(j)}$ is the uniform limit of a sequence of a.p. functions as before. Once again (2.3) holds while the function $x(t)$ defined by (2.4) is a bounded solution of (2.5).

Since $g \in C^k(\mathbf{R})$, our solution $x(t)$ is $C^{k+1}(\mathbf{R})$. We can differentiate (2.5) another $(k-2)$ times and find a general expression of the form (1.2) with a.p. coefficients, having our solution $x(t)$ which is *a priori* bounded but cannot be a.p. because

$$\liminf_{t\to\infty} \int_0^t g(s)\,ds = -\infty,$$

as can be seen by choosing the subsequence $t_n = \pi n^{k+1}$. An application of Lemma 2.1 once again gives the required conclusion.

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