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ESSENTIAL CRITICAL POINTS IN PRODUCT MANIFOLDS

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ABSTRACT. First, an analog of the deformation lemma is proved for functions $f : M \times P \to \mathbf{R}$, where M is an infinite-dimensional Finsler manifold, P is a space of parameters, and f is continuous in both variables and locally Lipschitz in the first one. Taking P to be a compact manifold one derives a result generalizing both the Chang's extension of the deformation lemma for locally Lipschitz functions and the Wille's displacement lemma concerning essential critical points of continuous (possibly nondifferentiable) functions. Examples of theorems on multiplicity of critical points obtained that way are presented. At the end, an example of application to periodic boundary value problems is discussed.

1. Introduction. In a recent paper [**14**] F. Wille gave an elegant definition of an essential critical point of a continuous real-valued function defined on a Banach manifold. That definition (recalled in Section 3) does not refer to any concept of differentiability and it has a nice geometrical flavor. Moreover, if M is a C^1 manifold and a function $f : M \to \mathbf{R}$ is of class C^1, then any essential critical point of f in the sense of Wille is a critical point. The main result of Wille is the following extension of the classical Lusternik-Schnirelman theorem:

Theorem 1.1. *Let M be a metrizable Banach manifold modelled on X, without a boundary, and let $f : M \to \mathbf{R}$ be continuous. Assume that $f^{-1}[a,b]$ is compact for all $a, b \in \mathbf{R}$, $a \le b$, and that f is bounded below* (*or above*). *Then there are at least* $\operatorname{cat}(M)$ *essential critical points of f.*

We recall that $\operatorname{cat}(M)$ is the least integer (or ∞ if such integers do not exist) such that M can be covered by n closed subsets which are contractible in M.

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It is instantly noticed that the compactness of $f^{-1}[a,b]$ (replacing the classical Palais-Smale condition) forces M to be locally compact, therefore finite dimensional. Thus, Wille's result is not really applicable to problems in analysis and differential equations where infinite dimensional spaces must be considered. This paper is motivated by the following two observations:

i) First, even if f is a C^1 function satisfying the Palais-Smale condition, results concerning lower estimates of a number of essential critical points are stronger than those on critical points since we conclude on the cardinality of a smaller set. Due to the geometrical flavor, a conclusion on the number of essential critical points is more informative.

ii) Next, in certain applications, the considered Banach manifolds are products $E \times P$ of an infinite dimensional Banach space and a compact manifold (see, e.g., [**7**]). The result of Wille suggests that we should not need differentiability assumptions and the Palais-Smale condition for the total derivative but only for the partial derivative with respect to the variable in E.

Proofs of critical points theorems, such as Theorem 1.1, are based on the so-called Deformation Lemma (cf. [**2, 11, 12**]). We based our results on Chang's generalization of the Deformation Lemma [**1**] for locally Lipschitz functions and Clarke's generalized derivative. In Section 2 we prove an analog of that lemma for a function $f : M \times P \to \mathbf{R}$, where M is a Banach manifold, P a metric space of parameters, and f is locally Lipschitz with respect to the variable in M. We assume that P is also a manifold and we prove a deformation lemma concerning essential critical points of f in the product manifold $M \times P$. Some corollaries are derived: in the particular case when M is a singleton, we get Wille's result and, when P is a singleton, we get Chang's results with conclusions generalized to essential critical points. We give two examples of critical points theorems obtained with the use of our deformation lemma. The reader is referred to [**5**], where some results of critical point theory are derived from an abstract "deformation lemma" assumed as an axiom. In Section 4, an example of application to a periodic boundary value problem is provided. By showing that the trivial solution is not an essential critical point of a corresponding functional, we prove the existence of nontrivial solutions.

2. Deformation Lemma for functions with parameters. Let E be a Banach space, P a metric space, and $f : E \times P \to \mathbf{R}$ a continuous function which is locally Lipschitz in $x \in E$ uniformly in $p \in P$. More explicitly, we assume that the following condition (L) is verified:

(L) For any $(x_0, p_0) \in E \times P$, there exist an open neighborhood U of (x_0, p_0) in $E \times P$ and a constant $k > 0$ such that, for all $(x,p), (y,p) \in U$, $|f(x,p) - f(y,p) \leq k||x-y||$.

If f satisfies (L), then the *Clarke's generalized gradient* $\partial_x f$ (cf. [**3, 4**]) of the function $x \mapsto f(x,p)$ is well-defined for each p. We recall that $\partial_x f(x,p)$ is the set of those $w \in E^*$ that $\langle w, v\rangle \leq f_x^0(x;p;v)$ for all $v \in E$, where

$$f_x^0(x;p;v) = \overline{\lim_{\substack{h\to 0\\ t\searrow 0}}} \frac{1}{t}[f(x+h+tv) - f(x+h)].$$

We also recall that, for any $(x,p) \in E \times P$, $\partial_x f(x,p)$ is a nonempty convex w^* compact subset of E^*. By similar arguments as those in [**1**], the set-valued map $(x,p) \mapsto \partial_x f(x,p)$ is w^*-upper semicontinuous and the function

$$\lambda(x,p) = \min\{||w||_{E^*} \mid w \in \partial_x f(x,p)\}$$

is lower semicontinuous, i.e., $\underline{\lim}_{(x,p)\to(x_0,p_0)} \lambda(x,p) = \lambda(x_0,p_0)$ for any $(x_0, y_0) \in E \times P$.

Definition. We say that f satisfies the *Palais-Smale condition* (PS) if any sequence $\{(x_n, p_n)\} \subset E \times P$ such that $\{f(x_n, p_n)\}$ is bounded and $\lambda(x_n, p_n) \to 0$ has a convergent subsequence.

We introduce the following notations

$$\begin{aligned} K_x &:= \{(x,p) \in E \times P \mid 0 \in \partial_x f(x,p)\};\\ K_x^c &:= K_x \cap f^{-1}(c), \qquad c \in \mathbf{R};\\ f^c &:= f^{-1}((-\infty, c]);\\ B(c,\varepsilon,\delta) &:= \{(x,p) \in E \times P \mid c - \varepsilon < f(x,p) < c + \varepsilon\\ &\qquad\qquad \text{and } \operatorname{dist}((x,p), K_x^c) > \delta\}. \end{aligned}$$

As a consequence of the preceding discussion, if f satisfies (PS), then K_x^c is compact.

The following lemma can be proved by similar arguments as those used in Section 3 of [**1**]:

Lemma 2.1. *Let $f : E \times P \to \mathbf{R}$ be a continuous function satisfying* (L) *and* (PS)*. Then, for any $\delta > 0$, there exist $b, \bar{\varepsilon} > 0$ and a continuous vector field $V : E \times P \to E$ satisfying* (L) *such that the following conditions are verified*:

(i) $\lambda(x, p) \geq b$ *for all* $(x, p) \in B(c, 2\bar{\varepsilon}, \delta)$;

(ii) $\langle x_p^*, V(x, p)\rangle \geq b/2$ *for all* $(x, p) \in B(c, 2\bar{\varepsilon}, \delta)$, $x_p^* \in \partial_x f(x, p)$;

(iii) $||V(x, p)|| \leq 1$ *for all* $(x, p) \in E \times P$;

(iv) $V(x, p) = 0$ *if* $(x, p) \notin B(c, \bar{\varepsilon}, 2\delta)$.

We are now able to prove the following theorem.

Theorem 2.2 (Deformation Theorem). *Let $f : E \times P \to \mathbf{R}$ be a continuous function satisfying* (L) *and* (PS)*, and let U be an open neighborhood of K_x^c, $U = \varnothing$ if $K_x^c = \varnothing$. Then, for any $\varepsilon_0 > 0$ there exist $0 < \varepsilon < \varepsilon_0$ and a continuous map $\eta : E \times P \times [0, 1] \to E \times P$ with the following properties*:

(i) η_0 *is the identity on* $E \times P$ (*where* $\eta_t(x, p) := \eta(x, p, t)$);

(ii) η_t *is a homeomorphism for all* $t \in [0, 1]$;

(iii) $\eta_t(x, p) = (x, p)$ *for all* $(x, p) \in E \times P - f^{-1}([c - \varepsilon_0, c + \varepsilon_0])$ *and all* $t \in [0, 1]$;

(iv) $f(\eta_1(x, p)) \leq f(x, p)$ *for all* $(x, p) \in E \times P$;

(v) $\eta_1(f^{c+\varepsilon} - U) \subset f^{c-\varepsilon}$.

Proof. By the fundamental existence and uniqueness theorem for differential equations (cf. [**13**]), the initial value problem

$$y'(t) = -V(y(t), p), \qquad y(0) = p$$

has a unique solution $y : \mathbf{R} \to E$, for any $(x, p) \in E \times P$. By the Kurzweil-Vorel theorem [**8**] (see also [**13**]), the solution $y(t) = \eta^1(x, p, t)$

is continuous in $(x,p,t) \in E \times P \times \mathbf{R}$. We may define $\eta : E \times P \times \mathbf{R} \to E \times P$, $\eta(x,p,t) := (\eta^1(x,p,t), p)$ and complete the proof by the same arguments as in [**1**]. $\square$

Let now M be a C^1 Finsler manifold modelled on E, i.e., a paracompact manifold with a Finsler structure $||\cdot|| : TM \to [0,\infty)$, let $\rho : M \times M \to [0,\infty)$ be the metric on M defined by the Finsler structure (cf. [**11**]), and $f : M \times P \to \mathbf{R}$ a continuous function satisfying the Lipschitz condition

(L$'$) For any $(x_0, p_0) \in E \times P$, there exists an open neighborhood U of (x_0, p_0) in $M \times P$ and a constant $k > 0$ such that, for all (x,p), $(y,p) \in U, |f(x,p) - f(y,p)| \le k\rho(x-y)$.

Then $f \circ (\phi \times \mathrm{id}) : E \times P \to \mathbf{R}$ satisfies (L) for any chart $\phi : E \to M$ and we may define $\partial_x f(x,p) \subset (T_x M)^*$ by the same amalgamation formula as in [**10**]. The correctness of the definition is a consequence of the "chain rule" for compositions of locally Lipschitz functions with diffeomorphisms, the proof of which is left to the reader.

Proposition 2.3. *Let U be open in E, $\phi : U \to E$ a C^1 diffeomorphism and $f : \phi(U) \to \mathbf{R}$ a Lipschitz function. Then*

(a) $(f \circ \phi)^0(x,v) = f^0(\phi(x), d\phi(x)v)$,

(b) $\partial(f \circ \phi)(x) = \partial f(\phi(x)) \circ d\phi(x)$.

Given a continuous $f : M \times P \to \mathbf{R}$ satisfying (L$'$) we define $\lambda(x,p) := \min\{||w||_x^* \mid w \in \partial_x f(x,p)\}$, and the Palais-Smale condition can be formulated as before with M in the place of E. Similarly as for C^1 functions, our deformation lemma (Theorem 2.2) can be extended to functions on $M \times P$:

Theorem 2.4. *Let M be a C^2 Finsler manifold, P a metric space, $f : M \times P \to \mathbf{R}$ a continuous function satisfying* (L$'$) *and* (PS), *and let U be an open neighborhood of K_x^c in $M \times P$. The conclusion of* Theorem 2.2 *holds with E replaced by M.*

3. Essential critical points in product manifolds. Let E be a Banach space, D an open subset of E, and $f : D \to \mathbf{R}$ a continuous function. We recall from [**14**] that $x_0 \in D$ is called an *essential regular point* of f if there exists a homeomorphism $h : U_{x_0} \to V_{x_0}$ of open neighborhoods U_{x_0}, V_{x_0} of x_0 with the property $h(x_0) = x_0$, such that $f \circ h$ is a nonconstant affine functional restricted to U_{x_0}. A point x_0 is an *essential critical point* of f if it is not essentially regular. It is known (cf. canonical form theorem for a regular point [**10**]) that if f is of class C^1 and x^0 is a regular point then it is essentially regular. Consequently, any essential critical point of a C^1 function is one of its critical points.

Next, let M be a Banach manifold for E and $f : M \to \mathbf{R}$ continuous. A point $x_0 \in M$ is an *essentially regular point* of f if $\phi(x_0)$ is an essentially regular point of $f \circ \phi^{-1}$ for at least one chart ϕ. It is easy to see that if that happens for one chart ϕ, then it is true for all charts. An *essential critical point* of $f : M \to \mathbf{R}$ is again defined as one which is not essentially regular. The following result is due to Wille:

Lemma 3.1 (Local Displacement [**14**]). *Let M be a metrizable Banach manifold, $f : M \to \mathbf{R}$ a continuous function, and let $x_0 \in M$ be an essentially regular point of f. Then there exists a homotopy $\eta : M \times [0,1] \to M$ and a neighborhood U_{x_0} of x_0 with the following properties*:

(i) *η_0 is the identity on M (where $\eta_t(x) := \eta(x,t)$);*

(ii) *η_t is a homeomorphism for all $t \in [0,1]$;*

(iii) *$\eta_t(x) = x$ for all $x \in M - U_{x_0}$ and all $t \in [0,1]$;*

(iv) *$f(\eta_t(x)) < f(x)$ for all $x \in U_{x_0}$ and all $t \in (0,1]$.*

We now let M be a C^2 Finsler manifold, P a finite dimensional topological manifold and $f : M \times P \to \mathbf{R}$ a continuous function satisfying the conditions (L$'$) and (PS) of the previous section. Since $M \times P$ is now a Banach manifold, we may consider the set

$$K_{\mathrm{ess}} = \{(x,p) \in M \times P \mid (x,p) \text{ is an essential critical point of } f\}.$$

It is easily proved that K_{ess} is closed in $M \times P$. If P is a C^1 manifold, f is a C^1 function, and $K = \{(x,p) \in M \times P \mid Df(x,p) = 0\}$ is the set of its critical points, then $K_{\text{ess}} \subset K \subset K_x$. We may now establish our key result:

Theorem 3.2. *Let M be a C^2 Finsler manifold, P a metrizable Banach manifold, $f : M \times P \to \mathbf{R}$ a continuous function satisfying* (L′) *and* (PS), *and let U be an open neighborhood of $K_{\text{ess}} \cap K_x^c$ in $M \times P$, $U = \varnothing$ if $K^c = \varnothing$. Then the conclusion of Theorem* 2.2 *holds with $E \times P$ replaced by $M \times P$.*

Proof. For each $(x_0, p_0) \in K_x^c - U$ let $U_{(x_0,p_0)}$ and $\eta^{(x_0,p_0)} : M \times P \times [0,1] \to M \times P$ be the local displacement as given by Lemma 3.1 for the manifold $M \times P$. We may assume, without a loss of generality, that $U_{(x_0,y_0)} \subset f^{-1}([c - \varepsilon_0, c + \varepsilon_0])$. Since $f(\eta_1^{(x_0,p_0)}(x,p)) < f(x,p)$ for all $(x,p) \in U_{(x_0,p_0)}$, we may take

$$0 < \delta_{(x_0,p_0)} := \frac{1}{4}[f(x_0,p_0) - f(\eta_1^{(x_0,p_0)}(x_0,p_0))].$$

Then there exists an open $U'_{(x_0,p_0)} \subset U_{(x_0,p_0)}$ such that

$$f(\eta_1^{(x_0,p_0)}(x,p)) < f(x,p) - 2\delta_{(x_0,p_0)}$$

for all $(x,p) \in U'_{(x_0,p_0)}$. Since $K_x^c - U$ is compact, its covering $\{U'_{(x,p)}\}_{(x,p) \in K_x^c - U}$ has a finite subcovering $\{U'_{(x_1,p_1)}, U'_{(x_2,p_2)}, \dots, U'_{(x_n,p_n)}\}$. Then $\widetilde{U} = U'_{(x_1,p_1)} \cup U'_{(x_2,p_2)} \cup \dots \cup U'_{(x_n,p_n)}$ is an open neighborhood of K_x^c. Let $\bar{\eta} : M \times P \times [0,1] \to M \times P$ and $0 < \tilde{\varepsilon} < \varepsilon_0$ as in the conclusion of Theorem 2.2 for $\widetilde{U}$ replacing U. We now put

$$\begin{aligned} \varepsilon &:= \min\{\tilde{\varepsilon}, \delta_{(x_1,p_1)}, \delta_{(x_2,p_2)}, \dots, \delta_{(x_n,p_n)}\} \\ \eta_t &:= \tilde{\eta}_t \circ \eta_t^{(x_n,p_n)} \circ \eta_t^{(x_{n-1},p_{n-1})} \circ \dots \circ \eta_t^{(x_1,p_1)}. \end{aligned}$$

It is easily verified that η has the desired properties (i) through (v). □

We shall now discuss some consequences of the Deformation Lemma. We first obtain two special cases of Theorem 3.2.

Corollary 3.3 (Wille's Global Displacement Lemma). *Let P be a finite dimensional topological manifold and $f : P \to \mathbf{R}$ a continuous* *function such that* $f^{-1}([a,b])$ *is compact for any compact interval* $[a,b] \in \mathbf{R}$. *Then the conclusion of Theorem* 2.2 *holds with* $E \times P$ *replaced by* P.

Proof. We apply Theorem 3.2 with $M = \{0\}$. The (PS) condition is trivially satisfied. $\square$

Corollary 3.4. *Let* M *be a* C^2 *Finsler manifold,* $f : M \to \mathbf{R}$ *a locally Lipschitz function satisfying* (PS), *and let* U *be an open neighborhood of the set* $K \cap K_{\text{ess}}$ *where*

$$K = \{x \in M \mid 0 \in \partial f(x)\}.$$

Then the conclusion of Theorem 2.2 *holds with* $E \times P$ *replace by* M.

Proof. We now let P be a singleton, in the statement of Theorem 3.2. $\square$

Our extension of the Deformation Lemma automatically includes extensions of many results on multiplicity of critical points which are based on that lemma. We reach analogous lower estimates of multiplicity of critical points which are essential critical points. Here are some examples:

Theorem 3.5 (cf. [**11**, Theorem 7.2]). *Let* M *be a* C^2 *Finsler manifold,* P *a metrizable Banach manifold, and* $f : M \times P \to \mathbf{R}$ *a continuous function satisfying* (L′) *and* (PS). *Then there exists at least* $\operatorname{cat}(M \times P)$ *of points in the set* $K_{\text{ess}} \cap K_x$.

Theorem 3.6 (cf. [**7**, Theorem 4.2]). *Let* M, P *be as above. Suppose that for two reals* $a < b$, $K_{\text{ess}} \cap K_x^a = K_{\text{ess}} \cap K_x^b = \varnothing$. *Then there is at least* $\operatorname{cat}_{M \times P, f^a}(f^b)$ *points in the set* $K_{\text{ess}} \cap K_x \cap f^{-1}([a,b])$.

For the definition of the relative category $\operatorname{cat}_{X,Y}(A)$, we refer the reader to [**7**].

In a similar way, Theorems 3.2, 3.3 and 3.4 in [**1**] can be extended to essential critical points.

4. An application to differential equations. We consider the periodic boundary value problem

$$\text{(P)} \qquad x'' = f(x,t), \qquad x(0) = x(T), \qquad t \in [0,T],$$

where $f : \mathbf{R}^2 \to \mathbf{R}$ is continuous with the continuous partial derivative f_t and $f(0,t) = 0$ for all t. Obviously, (P) has a trivial solution $x \equiv 0$. We shall impose further restrictions on f under which the existence of one or more nontrivial solutions can be deduced. Assume that

(i) There exist constants α, β, c such that $F(x,t) \geq \alpha x^2 - \beta$ for all $|x| \geq c$, where $F(x,t) = \int_0^x f(s,t)\, ds$.

(ii) There exist $r > 0$ such that $f(x,t) > 0$ for all $0 < |x| < r$ and all $t \in [0,T]$.

Let $H_T^1 = W_T^{1,2}$ be the Sobolev space of order one of T-periodic functions (cf. [**9**]), and let $\phi : H_T^1 \to \mathbf{R}$ be given by

$$\phi(x) = \int_0^T \frac{x'^2}{2} + F(x,t)\, dt, \qquad x \in H_T^1.$$

It is verified that $\phi \subset C^1(H_T^1, \mathbf{R})$ and that any $x \in H_T^1$ is a weak solution of (P) if and only if x is a critical point of ϕ (cf. [**9**, Theorem 1.4]). Moreover, by standard arguments similar to those in the proof of [**9**, Proposition 4.1], ϕ is bounded from below and it satisfies (PS). Therefore ϕ has at least one essential critical point x where it assumes a minimum value. However, we have the following

Lemma 4.1. *The trivial solution $x \equiv 0$ is an essential regular point of ϕ.*

Proof. Let $\bar{x} = (1/T)\int_0^T x(t)\, dt$ and $\tilde{x} = x - \bar{x}$. We define $\psi(x) = \tilde{x} + \phi(x)$. Then ψ is continuous in x, $\psi(0) = 0$ and $\phi(x) = \alpha \circ \psi(x)$ where $\alpha(x) = \bar{x}$ is a nontrivial linear functional. The conclusion will follow from $h = \psi^{-1}$ if we show that ψ is a homeomorphism from some neighborhood of the origin in H_T^1 onto another one. Note that $|\tilde{x}(t)| \leq \sqrt{T}||x'||_{L^2} \leq \sqrt{T}||x||_{H_T^1}$. Let $0 < \varepsilon < r/\sqrt{T}$ and $B_\varepsilon = \{x \in H_T^1 : ||x||_{H_T^1} < \varepsilon\}$. We define $G : B_\varepsilon \times [-r + \sqrt{T}\varepsilon, r - \sqrt{T}\varepsilon]$ by

$$G(x,s) = \int_0^T F(\tilde{x} + s, t)\, dt.$$

Note that $\psi(x) = \tilde{x} + (1/2)||x'||_2^2 + G(\tilde{x}, \bar{x})$. It follows from (ii) that G is strictly increasing in S. Moreover, by suitably reducing ε one can show that

$$G(\tilde{y}, -r + \sqrt{T}\varepsilon) < \bar{y} - \frac{1}{2}||y'||_2^2 < G(\tilde{y}, r - \sqrt{T}\varepsilon)$$

for all $y \in B_\varepsilon$. By the intermediate value theorem, for any $y \in B_\varepsilon$, there exists a unique $\rho(y) \in \mathbf{R}$ continuous in y, such that $G(\tilde{y}, \rho(y)) = \bar{y} - (1/2)||y'||_2^2$. It is verified that $h(y) := \tilde{y} + \rho(y)$ is the inverse of $\psi : B_\varepsilon \cap \psi^{-1}(B_\varepsilon) \to B_\varepsilon$. □

Lemma 4.1 implies that (P) has at least one nontrivial solution; however, the use of essential critical points can be easily avoided since one can show by more elementary arguments that $\phi(0)$ is not a minimum. We shall give an example of ϕ which has at least three essential critical points (two are local minima and one "saddle point"). Then Lemma 4.1 is a simple argument for the existence of at least three nontrivial solutions of (P). Indeed, let

$$f(x,t) = 6Rx^2(x^2-1)(2x-1) + 3a\sin^2 x \cos x \cos t, \qquad x, t \in \mathbf{R},$$

where R and a are positive constants such that $8\pi^2(31R + 5a) \leq 5$. We take $T = 2\pi$. Then

$$F(x,t) = \frac{R}{5}(10x^6 - 6x^5 - 15x^4 + 10x^3) + a\sin^3 x \cos t,$$

and f verifies the previous assumptions. Moreover, one can show that $\phi(\pm 1) \leq p < 0 \leq \phi(\tilde{x})$ for all $x \in H_T^1$, where $p = -2\pi R/5$. Let $q = \sup\{\phi(s) : -1 \leq s \leq 1\}$, $A = \phi^q$, $B = \phi^p$. Then $q > 0$, $B \subset A$, 1 and -1 are in different connected components of B and in the same connected component of A. It follows from [**6**, Theorem 2.2] and [**7**, Theorem 4.2] that $\operatorname{cat} B \geq 2$ and $\operatorname{cat}_{A,B}(A) \geq 1$. Therefore, ϕ has at least two essential critical points in B and at least one in $A - B$. Consequently, (P) has at least three nontrivial solutions.

Note that since f is smooth and 2π-periodic in t, the solutions of (P) extend to classical periodic solutions of $x'' = f(x,t)$ in $\mathbf{R}$.

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