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THE CARTESIAN CLOSED TOPOLOGICAL HULL OF THE CATEGORY OF APPROACH UNIFORM SPACES

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ABSTRACT. The category **AUnif** of approach uniform spaces and uniform contractions properly combines uniform spaces and extended pseudo-metric spaces but (like **Unif**) lacks convenience, such as cartesian closedness. This paper therefore considers its cartesian closed topological hull, which is first described as a subcategory of **SAULim**, the category of semi-approach uniform limit spaces and uniform contractions. This hull is then also given a description inside the topological universe hull of **AUnif** and is shown to be a reasonable generalization of the corresponding hull of **Unif**. Furthermore, some referencing notes are provided with respect to similar results that can be obtained when starting from q**AUnif** (where symmetry assumptions are omitted).

1. Introduction. It is often desirable and useful for a (concrete) category to have extra properties in addition to just being nicely topological, such as being cartesian closed topological (CCT). However, many categories are not cartesian closed, which has inspired a theory of CCT extensions of such (failing) categories, where the least such CCT extension of a given concrete category, the CCT hull of a category, is especially interesting.

For instance, in [**2**], Adámek and Reiterman constructed the CCT hull of **Unif**, the category of uniform spaces (and uniformly continuous maps), and in [**3**], they described the CCT hull of the category $(p)\mathbf{MET}^{(\infty)}$ of (extended pseudo-)metric spaces (and nonexpansive maps). Later the author added to these results by describing the CCT hull of the category q**Unif** of quasi-uniform spaces (and uniformly continuous maps) [**16**] (and thereby also adding to an alternative characterization of the CCT hull of **Unif** by Alderton and Schwarz [**4**]) and by describing the CCT hull of the category $pq\mathbf{MET}^{\infty}$ of extended

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pseudo-quasi-metric spaces (and nonexpansive maps) [**14**], where *quasi* refers in both cases to the omitting of symmetry assumptions.

The categories of extended pseudo-(quasi-)metric spaces and (quasi-) uniform spaces were combined by Lowen and Windels into the category (q)**AUnif** of (quasi-)approach uniform spaces (and uniform contractions) [**11**], [**24**], which allows a quantified view on (quasi-)uniform spaces, such as on uniform properties [**12**] or on completion [**13**].

It is the intention of this paper to describe the CCT hull of **AUnif** as a subcategory of **SAULim**, the category of semi-approach uniform limit spaces (and uniform contractions) and (necessarily) even as a subcategory of the topological universe (= topological quasitopos) hull of **AUnif** which (as shown in [**18**]) is its final hull in **PsAULim**, the category of pseudo-approach uniform limit spaces (and uniform contractions), whose objects are essentially described by ultrafilters [**17**], [**15**]. In particular, the argumentation is constructed such that most of it can be immediately used to obtain analogous results regarding the CCT hull of q**AUnif**, where notes are provided to point out differences as they occur (and where the full details in such cases can be found in [**19**]). Besides that it will be shown that this hull is a reasonable generalization of the hull of the (nonquantified) **Unif**; it may also be interesting to note that the descriptions here have also been inspired by those obtained in the case of $p(q)\mathbf{MET}^\infty$ [**3**], [**14**] and are reminiscent of what is described in [**15**].

2. Preliminaries. A *topological* construct will stand for a concrete category over **Set** which is a *well-fibered topological c-construct* in the sense of [**1**], i.e., each structured source has an initial lift, every set carries only a set of structures and each constant map (or empty map) between two objects is a morphism. Also recall that a construct **A** is *CCT* (*cartesian closed topological*) if **A** is a topological construct which has *canonical function spaces*, i.e., for every pair (A, B) of **A**-objects the set $\hom(A, B)$ can be supplied with the structure of an **A**-object, denoted by $[A, B]$, such that

(a) the evaluation map $\mathrm{ev} : A \times [A, B] \to B$ is an **A**-morphism.

(b) for each **A**-object C and **A**-morphism $f : A \times C \to B$, the map $f^* : C \to [A, B]$ defined by $f^*(c)(a) = f(a, c)$ is an **A**-morphism (f^* is called the *transpose* of f). Note that, given $f : A \times C \to B$, the transpose $f^* : C \to [A,B]$ is the map which makes the following diagram commute:

$$\begin{tikzcd} A \times [A,B] \arrow[r, "\mathrm{ev}"] & B \\ A \times C \arrow[u, "1\times f^*"] \arrow[ur, "f"'] & \end{tikzcd}$$

In general, categorical concepts and terminology used in this paper (and possibly not recalled here), in particular regarding categorical topology, can be found in [**1**] and [**20**]. Furthermore, a functor shall always be assumed to be concrete (unless this is clearly not the case from its definition) and subcategories to be full and isomorphism-closed. The *CCT hull* of a construct $\mathbf{A}$ (shortly denoted by $\mathrm{CCTH}(\mathbf{A})$) (if it exists) is defined as the smallest CCT construct $\mathbf{B}$ in which $\mathbf{A}$ is finally dense (see [**8**]), where $\mathbf{A}$ is finally dense in $\mathbf{B}$ if each $\mathbf{B}$-object is a final lift of some structured sink in $\mathbf{A}$. Also from [**8**], recall that given a CCT construct $\mathbf{C}$ in which $\mathbf{A}$ is finally dense, the CCT hull of $\mathbf{A}$ is the full subconstruct of $\mathbf{C}$ determined by

$$\mathrm{CCTH}(\mathbf{A}) := \{C \in \mathbf{C} \mid \text{there exists an initial source } (f_i : C \to [A_i, B_i])_{i\in I} \text{ where } \forall\, i \in I : A_i, B_i \in \mathbf{A}\}.$$

In short, the CCT hull of $\mathbf{A}$ is the initial hull in $\mathbf{C}$ of the power-objects of $\mathbf{A}$-objects. A more recent survey of such properties and hull concepts can be found in [**7**] and [**22**]. First, some necessities regarding (approach) uniform spaces and generalizations thereof need to be recalled.

Given a set X, $\mathbf{F}(X)$ stands for the set of all filters on X; if $\mathcal{F} \in \mathbf{F}(X)$, then $\mathbf{U}(\mathcal{F})$ stands for the set of all ultrafilters on X finer than $\mathcal{F}$. In particular, $\mathbf{U}(X) := \mathbf{U}(\{X\})$ stands for the set of all ultrafilters on X. Given $A \subset X$, we recall that $\operatorname{stack} A := \{B \subset X \mid A \subset B\}$ and if A consists of a single point a, we also denote $\dot{a} := \operatorname{stack} a := \operatorname{stack} A$. If $\mathcal{F} \in \mathbf{F}(X^2)$, then $\mathcal{F}^{-1}$ denotes the filter generated by $\{F^{-1} \mid F \in \mathcal{F}\}$ where, given $F \subset X^2$, it holds that $F^{-1} := \{(y,x) \mid (x,y) \in F\}$. If $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X^2)$, then $\mathcal{F} \circ \mathcal{G}$ (the *composite* of $\mathcal{F}$ and $\mathcal{G}$) is defined to be the filter on X^2 generated by the filterbasis $\{F \circ G \mid F \in \mathcal{F}, G \in \mathcal{G}\}$, where $F \circ G := \{(x,z) \in X^2 \mid \exists y \in X : (x,y) \in G \text{ and } (y,z) \in F\}$. Besides the "normal" (cartesian) product of sets, maps, filters, etc., we also define the following special product of filters. If $\mathcal{F} \in \mathbf{F}(X^2)$ and $\mathcal{G} \in \mathbf{F}(Y^2)$, then $\mathcal{F} \otimes \mathcal{G}$ denotes the filter generated by $\{F \otimes G \mid F \in \mathcal{F}, G \in \mathcal{G}\}$ where, given $F \subset X^2$ and $G \subset Y^2$, the set $F \otimes G$ is given by $F \otimes G := \{((x,y),(x',y')) \mid (x,x') \in F, (y,y') \in G\}$. Also, given a set X, Δ_X denotes the diagonal of X^2, that is, the set $\{(x,x) \mid x \in X\}$. Given $F \subset X$, we let

$$\mathbf{S}_q(X,F) := \{\mathcal{F} \in \mathbf{F}(X^2) \mid \mathcal{F} \subset \operatorname{stack} \Delta_F \text{ and } F \times F \in \mathcal{F}\}$$

and

$$\mathbf{S}(X,F) := \{\mathcal{F} \in \mathbf{S}_q(X,F) \mid \mathcal{F}^{-1} = \mathcal{F}\},$$

elements of which are called *quasi-semi-uniformities* (*on* F) and *semi-uniformities* (*on* F), respectively. Also let $\mathbf{S}_q(X) := \cup_{F \subset X} \mathbf{S}_q(X,F)$ and $\mathbf{S}(X) := \cup_{F \subset X} \mathbf{S}(X,F)$ denote the collection of quasi-semi-uniformities (in X) and semi-uniformities (in X), respectively, and observe that the set $F \subset X$ such that $\mathcal{F} \in \mathbf{S}_q(X,F)$ is uniquely determined by $\mathcal{F} \in \mathbf{S}_q(X)$, i.e., $\mathbf{S}_q(X,F) \cap \mathbf{S}_q(X,G) = \varnothing$ whenever $F \neq G$. Indeed, if $\mathcal{F} \in \mathbf{S}_q(X,F)$, $\mathcal{G} \in \mathbf{S}_q(X,G)$ and $\mathcal{F} \subset \mathcal{G}$, then it follows that $\Delta_G \subset F \times F$, hence $G \subset F$. Consequently, $\mathbf{S}_q(X,F) \cap \mathbf{S}_q(X,G) \neq \varnothing$ implies that $F = G$.

An *approach uniform space* is a pair $(X, (\mathcal{U}_\varepsilon)_{\varepsilon \in \mathbf{R}^+})$ where $(\mathcal{U}_\varepsilon)_{\varepsilon \in \mathbf{R}^+}$ is a *uniform tower* on X, meaning a family of filters $(\mathcal{U}_\varepsilon)_{\varepsilon \in \mathbf{R}^+}$ on $X \times X$ such that

(UT1) for all $\varepsilon \in \mathbf{R}^+$ and for all $U \in \mathcal{U}_\varepsilon$: $\Delta_X \subset U$.

(UT2) For all $\varepsilon \in \mathbf{R}^+$ and for all $U \in \mathcal{U}_\varepsilon$: $U^{-1} \in \mathcal{U}_\varepsilon$.

(UT3) For all ε and for all $\varepsilon' \in \mathbf{R}^+$: $\mathcal{U}_\varepsilon \circ \mathcal{U}_{\varepsilon'} \supset \mathcal{U}_{\varepsilon+\varepsilon'}$.

(UT4) For all $\varepsilon \in \mathbf{R}^+$: $\mathcal{U}_\varepsilon = \cup_{\alpha > \varepsilon} \mathcal{U}_\alpha$.

Thus, a uniform tower is a stack of semi-uniformities satisfying (UT3) and (UT4).

Using the prefix *quasi* in the sequel will indicate that (UT2) need not necessarily by satisfied, while *semi* indicates that (UT3) need not necessarily be satisfied.

A map $f : (X, (\mathcal{U}_\varepsilon)_{\varepsilon \in \mathbf{R}^+}) \to (Y, (\mathcal{U}'_\varepsilon)_{\varepsilon \in \mathbf{R}^+})$ between quasi-semi-approach uniform spaces is called a *uniform contraction* if it fulfills the property that for all $\varepsilon \in \mathbf{R}^+$:

$$f : (X, \mathcal{U}_\varepsilon) \longrightarrow (Y, \mathcal{U}'_\varepsilon) \text{ is uniformly continuous (i.e., } \mathcal{U}'_\varepsilon \subset (f \times f)(\mathcal{U}_\varepsilon)).$$

Quasi-semi-approach uniform spaces together with uniform contractions form a category, denoted by qs**AUnif**, having q**AUnif**, s**AUnif** and **AUnif** as subcategories, which are described and studied in [**24**] (see also [**11**]).

In [**24**], [**17**], it is shown that (q)**AUnif** is a subconstruct of (q)**SAULim**, the category of (quasi)-semi-approach uniform limit spaces, the objects of which are described by means of a concept of a (*quasi*-)*semi-approach uniform limit structure* (*on* X), which is a map $\eta : \mathbf{F}(X^2) \to [0, \infty]$ satisfying

(SAUCS$_1$) For all $x \in X$: $\eta(\dot{x} \times \dot{x}) = 0$.

(SAUCS$_2$) For all $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X^2)$: $\mathcal{F} \subset \mathcal{G} \Rightarrow \eta(\mathcal{G}) \leq \eta(\mathcal{F})$.

(SAUCS$_3$) For all $\mathcal{F} \in \mathbf{F}(X^2)$: $\eta(\mathcal{F}) = \eta(\mathcal{F}^{-1})$.

(SAULS) For all $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X^2)$: $\eta(\mathcal{F} \cap \mathcal{G}) \leq \eta(\mathcal{F}) \vee \eta(\mathcal{G})$.

The pair (X, η) is a *semi-approach uniform limit space* and again (but now in this context), the prefix *quasi* indicates that (SAUCS$_3$) need not necessarily be satisfied, which yields, in particular, a *quasi-semi-approach uniform limit space*.

A map $f : (X, \eta_X) \to (Y, \eta_Y)$ between quasi-semi-approach uniform limit spaces (X, η_X) and (Y, η_Y) is said to be a *uniform contraction* provided that

$$\forall \mathcal{F} \in \mathbf{F}(X^2) : \eta_Y((f \times f)(\mathcal{F})) \leq \eta_X(\mathcal{F}).$$

Quasi-semi-approach uniform limit spaces and uniform contractions form the objects and morphisms of a topological construct, denoted q**SAULim**. More information on this category (and various of its subcategories) can be found in [**17**] and [**24**], while only properties that are required in the sequel are recalled here.

A (*quasi*)-*approach uniform limit* is a (quasi-)semi-approach uniform limit that additionally satisfies the property

(AULS) For all $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X^2) : \eta(\mathcal{F} \circ \mathcal{G}) \leq \eta(\mathcal{F}) + \eta(\mathcal{G})$

and a (quasi-)(semi-)approach uniform limit $\eta : \mathbf{F}(X^2) \to [0, \infty]$ is called *principal* (and (X, η) a *principal* (quasi-)(semi-)approach uniform limit space) if it additionally satisfies the condition

(PrSAULS) For any family $(\mathcal{F}_j)_{j \in J} \in \prod_{j \in J} \mathbf{F}(X^2) : \eta\left(\cap_{j \in J} \mathcal{F}_j\right) \leq \sup_{j \in J} \eta(\mathcal{F}_j)$, which ensures it to be equivalent to a (quasi-)(semi-)uniform tower in the sense of the following result, that allows us to conclude that $(q)(s)\mathbf{AUnif} \hookrightarrow (q)\mathbf{SAULim}$.

Proposition 2.1 [**24**]. (1) *Given a set X and a principal (quasi-)(semi-)approach uniform limit structure η on X,*

$$\mathcal{U}'_\varepsilon := \mathcal{U}(\eta)_\varepsilon := \bigcap_{\substack{\mathcal{F}\in\mathbf{F}(X^2)\\ \eta(\mathcal{F})\le\varepsilon}} \mathcal{F}, \quad \forall\, \varepsilon \in \mathbf{R}^+$$

defines a (quasi-)(semi-)uniform tower $(\mathcal{U}(\eta)_\varepsilon)_{\varepsilon\in\mathbf{R}^+}$ (on X), and vice versa, if $(\mathcal{U}_\varepsilon)_{\varepsilon\in\mathbf{R}^+}$ is a (quasi-)(semi-)uniform tower on X, then

$$\eta' := \eta((\mathcal{U}_\varepsilon)_{\varepsilon\in\mathbf{R}^+})(\mathcal{F}) := \min\{\alpha \in \mathbf{R}^+ \mid \mathcal{U}_\alpha \subset \mathcal{F}\} \quad \forall\, \mathcal{F} \in \mathbf{F}(X^2)$$

is a principal (quasi-)(semi-) approach uniform limit structure on X, such that

$$\eta((\mathcal{U}'_\varepsilon)_{\varepsilon\in\mathbf{R}^+}) = \eta \quad \text{and} \quad (\mathcal{U}(\eta')_\varepsilon)_{\varepsilon\in\mathbf{R}^+} = (\mathcal{U}_\varepsilon)_{\varepsilon\in\mathbf{R}^+}.$$

(2) *If (X, η_X) and (Y, η_Y) are principal quasi-semi-approach uniform limit spaces, then the following are equivalent*:

(a) *$f : (X, \eta_X) \to (Y, \eta_Y)$ is a uniform contraction.*

(b) *$f : (X, (\mathcal{U}(\eta_X)_\varepsilon)_{\varepsilon\in\mathbf{R}^+}) \to (Y, (\mathcal{U}(\eta_Y)_\varepsilon)_{\varepsilon\in\mathbf{R}^+})$ is a uniform contraction.*

To obtain some results (elegantly), there are some other concepts regarding approach uniform spaces (and variations thereof) to be introduced from [**24**] such as a characterization by means of uniform gauges (see also [**23**]).

To this end, first recall that an *extended pseudo-(quasi-)metric d (on X)* is a map $d : X \times X \to [0, \infty]$ satisfying

(1) For all $x \in X : d(x, x) = 0$.

(2) Symmetry: for all $x, y \in X : d(x, y) = d(y, x)$.

(3) Triangle inequality: for all $x, y, z \in X : d(x, z) \le d(x, y) + d(y, z)$.

(and the prefix *quasi* again means that symmetry need not necessarily be satisfied). A (*quasi-*)*uniform gauge* on a set X is then a collection $\mathcal{D}$ of extended pseudo-(quasi-)metrics (on X) which is closed under finite suprema and which is saturated, in the sense that $\mathcal{D}$ contains every extended pseudo-(quasi-)metric d such that

$$\forall\, \varepsilon > 0,\ \forall\, N < \infty,\ \exists e \in \mathcal{D} : d \wedge N \leq e + \varepsilon.$$

Before proceeding, a matter of notation; in the sequel, for any $\gamma : Z \to [0, \infty]$ and $\alpha \in [0, \infty]$, let $\{\gamma < \alpha\} := \{z \in Z \mid \gamma(z) < \alpha\}$ and $\{\gamma \leq \alpha\} := \{z \in Z \mid \gamma(z) \leq \alpha\}$.

Proposition 2.2 [**24**]. (1) *Given a set X and a* (*quasi-*)*uniform tower $(\mathcal{U}_\varepsilon)_{\varepsilon \in \mathbf{R}^+}$ on X, $\mathcal{D}' := \mathcal{D}((\mathcal{U}_\varepsilon)_{\varepsilon \in \mathbf{R}^+})$ defined by*

$$\mathcal{D}((\mathcal{U}_\varepsilon)_{\varepsilon \in \mathbf{R}^+}) := \{d \mid d \textit{ is extended pseudo-(quasi-)metric and } \forall\, \varepsilon \in \mathbf{R}^+, \forall\, \alpha > \varepsilon : \{d < \alpha\} \in \mathcal{U}_\varepsilon\}$$

is a (*quasi-*)*uniform gauge on X, and vice versa, if $\mathcal{D}$ is a* (*quasi-*) *uniform gauge on X, then $(\mathcal{U}'_\varepsilon)_{\varepsilon \in \mathbf{R}^+} := T_{ut}(\mathcal{D})$ defined by*

$$\mathcal{U}'_\varepsilon := \langle \big\{ \{d < \alpha\} \mid d \in \mathcal{D},\ \alpha > \varepsilon \big\} \rangle$$

is a (*quasi-*)*uniform tower on X, such that*

$$T_{ut}(\mathcal{D}') := (\mathcal{U}_\varepsilon)_{\varepsilon \in \mathbf{R}^+} \quad \textit{and} \quad \mathcal{D}((\mathcal{U}'_\varepsilon)_{\varepsilon \in \mathbf{R}^+}) = \mathcal{D}.$$

(2) *If $(X, (\mathcal{U}^X_\varepsilon)_{\varepsilon \in \mathbf{R}^+})$ and $(Y, (\mathcal{U}^Y_\varepsilon)_{\varepsilon \in \mathbf{R}^+})$ are* (*quasi-*)*approach uniform spaces, then the following are equivalent*:

(a) *$f : (X, (\mathcal{U}^X_\varepsilon)_{\varepsilon \in \mathbf{R}^+}) \to (Y, (\mathcal{U}^Y_\varepsilon)_{\varepsilon \in \mathbf{R}^+})$ is a uniform contraction.*

(b) *For all $d \in \mathcal{D}((\mathcal{U}^Y_\varepsilon)_{\varepsilon \in \mathbf{R}^+}) : d \circ (f \times f) \in \mathcal{D}((\mathcal{U}^X_\varepsilon)_{\varepsilon \in \mathbf{R}^+})$.*

Next, some facts need to be recalled regarding convenient extensions of (q)**AUnif**.

Proposition 2.3 [**17**]. q**SAULim** *is a cartesian closed topological construct. Moreover, given a source $(f_i : X \to (X_i, \eta_i))_{i \in I}$, the initial lift η_X on X is given by*

$$\eta_X(\mathcal{F}) := \sup_{i \in I} \eta_i((f_i \times f_i)(\mathcal{F})), \quad \mathcal{F} \in \mathbf{F}(X^2).$$

Also, given quasi-semi-approach uniform limit spaces (X,η_X) *and* (Y,η_Y)*, the function space* $(Z,\eta) := [(X,\eta_X),(Y,\eta_Y)]$ *(in* q**SAULim***) is given by*

$$\begin{aligned}\eta(\Psi) &:= \inf\{\alpha\in[0,\infty] \mid \forall\,\mathcal{F}\in\mathbf{F}(X^2) : \eta_Y(\Psi(\mathcal{F}))\le\eta_X(\mathcal{F})\vee\alpha\},\\ &\qquad\qquad \Psi\in\mathbf{F}(Z^2))\\ &= \sup\{\eta_Y(\Psi(\mathcal{F})) \mid \mathcal{F}\in\mathbf{F}(X^2) \text{ and } \eta_Y(\Psi(\mathcal{F}))>\eta_X(\mathcal{F})\},\end{aligned}$$

(*where* $\Psi(\mathcal{F}) := (\mathrm{ev}\times\mathrm{ev})(\mathcal{F}\otimes\Psi)$ *and* $\mathrm{ev} : X\times\hom((X,\eta_X),(Y,\eta_Y)) \to Y$). *The following relations hold* (*where* $r(c) : \mathbf{A}\to\mathbf{B}$ *means that* $\mathbf{A}$ *is a bi*(*co*)*reflective subconstruct of* $\mathbf{B}$)*:*

$$\begin{tikzcd} q\mathbf{AUnif} \arrow[r, "r"] & qs\mathbf{AUnif} \arrow[r, "r"] & q\mathbf{SAULim} \\ \mathbf{AUnif} \arrow[r, "r"] \arrow[u, "r", "c"'] & s\mathbf{AUnif} \arrow[r, "r"] \arrow[u, "r", "c"'] & \mathbf{SAULim} \arrow[u, "r", "c"'] \end{tikzcd} .$$

In particular, all indicated subconstructs are topological constructs and the bireflectors of the bottom row are obtained as restrictions of the respective bireflector of the top row, such as the (q)**AUnif***-bireflector*

$$R : (q)\mathbf{SAULim} \longrightarrow (q)\mathbf{AUnif} : (X,\eta) \longmapsto (X,(q)\mathbf{AUnif}(\eta)).$$

Furthermore, the embedding of **SAULim** *in* q**SAULim** *not only preserves initial sources, but also function spaces* (*hence, function spaces in* **SAULim** *are obtained by just forming them in* q**SAULim**).

It should now be noted that, henceforth, we will concentrate on the symmetric (non-quasi) situation, but in such a way that the interested reader is invited to consider the applicability of the sequel to obtain results in a quasi situation. Whenever necessary, he will be guided with appropriate notes to point out some occurring differences, whereas full details can be found in [**19**].

As mentioned earlier, in order to describe the CCT hull of a construct, one should start by having some finally dense CCT extension available. The following results, shown in [**17**], indicate such an appropriate candidate in the present setting.

Definition 2.4. Let **saug** be the full subconstruct of **SAULim** consisting of *semi-approach uniformly generated spaces*, i.e., semi-approach uniform limit spaces (X,η) satisfying

$$(\mathbf{saug})\, \forall\, \mathcal{F} \in \mathbf{F}(X^2),\ \exists \mathcal{H} \in \mathbf{S}(X) : (\mathcal{H} \subset \mathcal{F} \text{ and } \eta(\mathcal{H}) = \eta(\mathcal{F})),$$

i.e.,

$$\forall\, \mathcal{F} \in \mathbf{F}(X^2) : \eta(\mathcal{F}) = \min_{\substack{\mathcal{H}\in\mathbf{S}(X)\\ \mathcal{H}\subset\mathcal{F}}} \eta(\mathcal{H})$$

$$(\mathbf{saug}_\Delta)\ \forall\, H \subset X : \eta(\operatorname{stack} \Delta_H) < \infty \Rightarrow \eta(\operatorname{stack} \Delta_H) = 0.$$

Proposition 2.5. (1) **saug** *is the final hull of* **AUnif** *in* **SAULim**.

(2) **saug** *is a cartesian closed topological construct. Moreover, given a source* $(f_i : X \to (X_i, \eta_i))_{i\in I}$ *(in* **saug**)*, the initial lift* η_X *on* X *is given by*

$$\eta_X(\mathcal{H}) := \sup_{i\in I} \eta_i((f_i \times f_i)(\mathcal{H})), \quad \mathcal{H} \in \mathbf{S}_q(X).$$

Given $(X, \eta_X), (Y, \eta_Y) \in$ **saug**, *the function space* $(Z, \eta) := [(X, \eta_X), (Y, \eta_Y)]$ *(in* **saug**) *is the* **saug**-*bicoreflection of the* **SAULim**-*function space and is given by*

$$\eta(\Psi) := \infty \ \textit{if}\ \exists \mathcal{H} \in \mathbf{S}_q(X) : \eta_Y((\operatorname{stack} \Delta_\psi)(\mathcal{H})) > \eta_X(\mathcal{H}),$$
$$\Psi \in \mathbf{S}_q(Z, \psi),$$

otherwise

$$\begin{aligned}\eta(\Psi) &:= \inf\{\alpha \in [0,\infty] \mid \forall\, \mathcal{H} \in \mathbf{S}_q(X) : \eta_Y(\Psi(\mathcal{H})) \le \eta_X(\mathcal{H}) \vee \alpha\} \\ &= \sup\{\eta_Y(\Psi(\mathcal{H})) \mid \mathcal{H} \in \mathbf{S}_q(X) \textit{ and } \eta_Y(\Psi(\mathcal{H})) > \eta_X(\mathcal{H})\}.\end{aligned}$$

It is also shown in [**17**] that **SAULim** has various relations to several "nonquantified," classical constructs in the following sense.

First recall (from [**5**], [**21**]), that a *semi-uniform limit space* $(X, \mathbf{L})$ consists of a set X and $\mathbf{L} \subset \mathbf{F}(X^2)$ satisfying

(SUC$_1$) for all $x \in X : \dot{x} \times \dot{x} \in \mathbf{L}$.

(SUC$_2$) For all $\mathcal{F} \in \mathbf{L}$, for all $\mathcal{G} \in \mathbf{F}(X^2) : \mathcal{F} \subset \mathcal{G} \Rightarrow \mathcal{G} \in \mathbf{L}$.

(SUC$_3$) For all $\mathcal{F} \in \mathbf{F}(X^2) : \mathcal{F} \in \mathbf{L} \Rightarrow \mathcal{F}^{-1} \in \mathbf{L}$.

(SUL) For all $\mathcal{F}, \mathcal{G} \in \mathbf{L} : \mathcal{F} \cap \mathcal{G} \in \mathbf{L}$,

and that $f : (X, \mathbf{L}_X) \to (Y, \mathbf{L}_Y)$ is called *uniformly continuous* if and only if $\mathcal{F} \in \mathbf{L}_X$ implies that $(f \times f)(\mathcal{F}) \in \mathbf{L}_Y$.

Clearly, the category **SULim** of semi-uniform limit spaces and uniformly continuous maps is concretely isomorphic to the full subconstruct of **SAULim** consisting of objects (X, η) such that $\eta(\mathbf{F}(X^2)) \subset \{0, \infty\}$ (by, for instance, associating $(X, \eta^{-1}(0))$ to (X, η)).

Proposition 2.6. *The following diagram holds*:

$$\begin{tikzcd}
\mathbf{AUnif} \arrow[r, "r"] & s\mathbf{AUnif} \arrow[r, "r"] & \mathbf{saug} \arrow[r, "c"] & \mathbf{SAULim} \\
\mathbf{Unif} \arrow[r, "r"] \arrow[u, "r"', "c"] & s\mathbf{Unif} \arrow[r, "r"] \arrow[u, "r"', "c"] & \mathbf{sug} \arrow[r, "c"] \arrow[u, "r"', "c"] & \mathbf{SULim} \arrow[u, "r"', "c"]
\end{tikzcd}$$

where all constructs on the bottom level are obtained as the restriction of the corresponding top level construct to **SULim***, yielding several "non-quantified," classical constructs.* (*For further background see also* [**6**]*,* [**9**]*,* [**10**]*,* [**5**] *and* [**21**])*, and where all bicoreflectors from the top level to the respective bottom level are restrictions of the bicoreflector*

$$C : \mathbf{SAULim} \longrightarrow \mathbf{SULim} : (X, \eta) \longmapsto (X, \eta_0)$$

where

$$\eta_0 : \mathbf{F}(X^2) \longrightarrow [0, \infty] : \mathcal{F} \longmapsto \begin{cases} 0 & \textit{if } \eta(\mathcal{F}) = 0, \\ \infty & \textit{otherwise.} \end{cases}$$

Definition 2.7. The *right-order topology* $\mathcal{T}_r$ on $[0, \infty]$ is the topology whose open sets are $\{\,]a, \infty] \mid a \in [0, \infty]\} \cup \{[0, \infty]\}$, hence for any $A \subset [0, \infty]$, $\mathrm{cl}_r(A) = [0, \sup A]$.

3. The CCT hull of AUnif.

Definition 3.1. Let X be a set and $E \subset X^2$ such that $\Delta_X \subset E$. Define a Čech closure operator $E(-)$, called *E-enlargement* in X^2, by

$$E(-) : \mathcal{P}(X^2) \longrightarrow \mathcal{P}(X^2) : A \longmapsto E(A) := E \circ A \circ E.$$

Since it holds for any $A \subset B \subset X^2$ that $E(A) \subset E(B)$, it follows that $\{E(F) \mid F \in \mathcal{F}\}$ is a filterbasis whenever $\mathcal{F} \in \mathbf{F}(X^2)$. The filter generated by it will be denoted $E(\mathcal{F})$ and is called the *E-closure* of $\mathcal{F}$.

Also recall the following concept (for instance, used in [**3**]) which has been inspiring in defining the foregoing E-enlargement in X^2.

Definition 3.2. Let X be a set and $E \subset X^2$ such that $\Delta_X \subset E$. Define the Čech closure operator $E(-)$, called *E-enlargement* in X, by

$$E(-) : \mathcal{P}(X) \longrightarrow \mathcal{P}(X) : A \longmapsto E(A) := \{x \in X \mid \exists a \in A : (x, a) \in E\}.$$

There is the following connection between these E-enlargements.

Proposition 3.3. *Let $\Delta_X \subset E \subset X \times X$ be symmetric (i.e., $E^{-1} = E$). Then $\mathcal{H} \in \mathbf{S}_q(X, H)$ implies that $E(\mathcal{H}) \in \mathbf{S}_q(X, E(H))$.*

Proof. First observe that as E is symmetric, it follows that $\Delta_{E(H)} \subset E(\Delta_H)$ (since, for any $x \in E(H)$, there exists $h \in H$ such that $(x, h) \in E$, $(h, h) \in \Delta_H$ and $(h, x) \in E$) and $E(H \times H) = E(H) \times E(H)$. Indeed, if $(x, y) \in E(H \times H)$, then there exists $(h, h') \in H \times H$ such that $(x, h) \in E$ and $(h', y) \in E$; hence $(x, y) \in E(H) \times E(H)$. Conversely, if $(x, y) \in E(H) \times E(H)$, then there exists $(h, h') \in H \times H$ such that $(x, h) \in E$ and $(y, h') \in E = E^{-1}$; hence $(x, y) \in E(H \times H)$.

Since $H \times H \in \mathcal{H}$, it already follows that $E(H) \times E(H) \in E(\mathcal{H})$. To show that $E(\mathcal{H}) \subset \operatorname{stack} \Delta_{E(H)}$, let $G \in \mathcal{H}$, then $\Delta_H \subset G$; hence $\Delta_{E(H)} \subset E(\Delta_H) \subset E(G)$.

Definition 3.4. Let $U : \mathbf{SAULim} \to \mathbf{Unif}$ be the composition of the **AUnif**-bireflection followed by the **Unif**-bicoreflection (which, by Proposition 2.6, is just the restriction of the **SULim**-bicoreflection in **SAULim**). Also, given $(X, \eta) \in \mathbf{SAULim}$, let $U(\eta)$ (by abuse of notation) also denote the resulting uniformity (on X) of $U(X, \eta)$.

Definition 3.5. Let $(X,\eta) \in \mathbf{SAULim}$, $E \in U(\eta)$, $\mathcal{H} \in \mathbf{S}_q(X)$ and $\mathcal{W} \in \mathbf{U}(\mathcal{H})$ and define

$$V_{E,\mathcal{W}}(\mathcal{H}) := \{\mathcal{G} \in \mathbf{S}_q(X) \mid E(\mathcal{G}) \subset \mathcal{W}\}$$

and

$$V_E(\mathcal{H}) := \{\mathcal{G} \in \mathbf{S}_q(X) \mid E(\mathcal{G}) \subset \mathcal{H}\}$$

and

$$\mathrm{cl}^u_X : \mathcal{P}(\mathbf{S}_q(X)) \longrightarrow \mathcal{P}(\mathbf{S}_q(X)) : \Theta \longmapsto \mathrm{cl}^u_X(\Theta),$$

where

$$\mathrm{cl}^u_X(\Theta) := \{\mathcal{H} \in \mathbf{S}_q(X) \mid \forall\, E \in U(\eta), \forall\, \mathcal{W} \in \mathbf{U}(\mathcal{H}) : V_{E,\mathcal{W}}(\mathcal{H}) \cap \Theta \neq \varnothing\}$$

and

$$\mathrm{cl}_X : \mathcal{P}(\mathbf{S}_q(X)) \longrightarrow \mathcal{P}(\mathbf{S}_q(X)) : \Theta \longmapsto \mathrm{cl}_X(\Theta),$$

where

$$\mathrm{cl}_X(\Theta) := \{\mathcal{H} \in \mathbf{S}_q(X) \mid \forall\, E \in U(\eta) : V_E(\mathcal{H}) \cap \Theta \neq \varnothing\}.$$

Next, define

$$\mathrm{cl}^{u,a}_X : \mathcal{P}(\mathbf{S}_q(X)) \longrightarrow \mathcal{P}(\mathbf{S}_q(X)) : \Theta \longmapsto \mathrm{cl}^{u,a}_X(\Theta)$$

where

$$\mathrm{cl}^{u,a}_X := \big\{\mathcal{H} \in \mathbf{S}_q(X,H) \mid \mathcal{H} \in \mathrm{cl}^u_X(\Theta) \text{ and } \\ \operatorname{stack}\Delta_H \in \mathrm{cl}_X(\{\operatorname{stack}\Delta_G \mid \eta(\operatorname{stack}\Delta_G) = 0\})\big\}.$$

The following result is most useful in handling (in the present setting) the *immer* elusive (and/or illusive) ultrafilters.

Lemma 3.6. *If $\mathcal{F} \in \mathbf{F}(X)$ and $\Psi \subset \mathbf{F}(X)$, then the following are equivalent*:

(1) *For all $\mathcal{W} \in \mathbf{U}(\mathcal{F})$, there exists $\mathcal{G} \in \Psi : \mathcal{G} \subset \mathcal{W}$.*

(2) *For any family $(\sigma(\mathcal{G}))_{\mathcal{G}\in\Psi}$ such that $\sigma(\mathcal{G}) \in \mathcal{G}$, $\mathcal{G} \in \Psi$, there exists a finite set $\Psi' \subset \Psi$ such that $\cup_{\mathcal{G}\in\Psi'}\sigma(\mathcal{G}) \in \mathcal{F}$.*

Proof. $1 \Rightarrow 2$. Let $(\sigma(\mathcal{G}))_{\mathcal{G}\in\Psi}$ be a family such that $\sigma(\mathcal{G}) \in \mathcal{G}$, $\mathcal{G} \in \Psi$. Suppose the conclusion does not hold, then it follows that the family $\mathcal{F} \cup \{X \setminus \sigma(\mathcal{G}) \mid \mathcal{G} \in \Psi\}$ has the finite intersection property and is therefore contained in some ultrafilter $\mathcal{W} \in \mathbf{U}(\mathcal{F})$. By (1), there exists $\mathcal{G} \in \Psi$ such that $\mathcal{G} \subset \mathcal{W}$. This implies that both $\sigma(\mathcal{G}) \in \mathcal{G} \subset \mathcal{W}$ and $X \setminus \sigma(\mathcal{G}) \in \mathcal{W}$, which is a contradiction.

$2 \Rightarrow 1$. Suppose (1) does not hold. Then there exists some $\mathcal{W} \in \mathbf{U}(\mathcal{F})$ such that for all $\mathcal{G} \in \Psi : \mathcal{G} \not\subset \mathcal{W}$, which implies that for all $\mathcal{G} \in \Psi$ there exists $\sigma(\mathcal{G}) \in \mathcal{G} : \sigma(\mathcal{G}) \notin \mathcal{W}(*)$. Applying (2) on the family $(\sigma(\mathcal{G}))_{\mathcal{G}\in\Psi}$ yields a finite set $\Psi' \subset \Psi$ such that $\cup_{\mathcal{G}\in\Psi'}\sigma(\mathcal{G}) \in \mathcal{F}$. As $\mathcal{F} \subset \mathcal{W}$ and $\mathcal{W}$ is an ultrafilter, there is some $\mathcal{G} \in \Psi' : \sigma(\mathcal{G}) \in \mathcal{W}$ which contradicts $(*)$. $\square$

Although they will actually not be needed in the sequel, let us nevertheless consider some properties of the "closures" cl and cl^u.

Proposition 3.7. *Let* $(X, \eta) \in \mathbf{saug}$. *Then*

(1) *for all* $\Theta \subset \mathbf{S}_q(X) : \Theta \subset \mathrm{cl}_X^{(u)}(\Theta)$.

(2) *For all* Θ, $\Psi \subset \mathbf{S}_q(X) : \Theta \subset \Psi \Rightarrow \mathrm{cl}_X^{(u)}(\Theta) \subset \mathrm{cl}_X^{(u)}(\Psi)$.

(3) *For all* $\Theta \subset \mathbf{S}_q(X) : \mathrm{cl}_X^{(u)}(\mathrm{cl}_X^{(u)}(\Theta)) = \mathrm{cl}_X^{(u)}(\Theta)$.

(4) *For all* Θ, $\Psi \subset \mathbf{S}_q(X) : \mathrm{cl}_X(\Theta \cup \Psi) = \mathrm{cl}_X(\Theta) \cup \mathrm{cl}_X(\Psi)$.

Proof. (1) and (2) are easily verified.

As for (3), the inclusion, $\supset$, clearly holds. Conversely, let $\mathcal{H} \in \mathrm{cl}_X(\mathrm{cl}_X(\Theta))$ and let $E \in U(\eta)$. Then there exists $\mathcal{G} \in \mathrm{cl}_X(\Theta)$ such that $E(\mathcal{G}) \subset \mathcal{H}$, which in turn provides us with $\mathcal{G}' \in \Theta$ such that $E(\mathcal{G}') \subset \mathcal{G}$. Consequently, $E^2(\mathcal{G}') = E(E(\mathcal{G}')) \subset \mathcal{H}$, which shows the required (as $\{E^2 \mid E \in U(\eta)\}$ is a basis for the uniformity $U(\eta)$).

Now let $\mathcal{H} \in \mathrm{cl}_X^u(\mathrm{cl}_X^u(\Theta))$ and let $E \in U(\eta)$ and $\mathcal{W} \in \mathbf{U}(\mathcal{H})$. Then there exists $\mathcal{G} \in \mathrm{cl}_X^u(\Theta)$ such that $E(\mathcal{G}) \subset \mathcal{W}$. It follows that there exists $\mathcal{V} \in \mathbf{U}(\mathcal{G})$ such that $E(\mathcal{V}) \subset \mathcal{W}$. Indeed, assume otherwise; then for any $\mathcal{V} \in \mathbf{U}(\mathcal{G})$, there exists $G_{\mathcal{V}} \in \mathcal{V}$ such that $E(G_{\mathcal{V}}) \notin \mathcal{W}$. Letting $\Psi := \mathbf{U}(\mathcal{G})$ and $\mathcal{F} := \mathcal{G}$, then (1) of the foregoing lemma is satisfied (by construction), consequently (2) of that lemma implies that there exist $\mathcal{V}_1, \dots, \mathcal{V}_n$ such that $G_{\mathcal{V}_1} \cup \dots \cup G_{\mathcal{V}_n} \in \mathcal{G}$. Hence, $E(G_{\mathcal{V}_1} \cup \dots \cup G_{\mathcal{V}_n}) \in$

$\mathcal{W}$. As $\mathcal{W}$ is an ultrafilter, it must be that $E(G_{\mathcal{V}_i}) \in \mathcal{W}$ for some $1 \leq i \leq n$, which is a contradiction. Let $\mathcal{V} \in \mathbf{U}(\mathcal{G})$ now be such that $E(\mathcal{V}) \subset \mathcal{W}$. Then it follows from $\mathcal{G} \in \mathrm{cl}_X^u(\Theta)$ that there exists $\mathcal{G}' \in \Theta$ such that $E(\mathcal{G}') \subset \mathcal{V}$. Consequently, $E^2(\mathcal{G}') = E(E(\mathcal{G}')) \subset \mathcal{W}$, which again shows the required (as $\{E^2 \mid E \in U(\eta)\}$ is a basis for the uniformity $U(\eta)$).

(4) Again, the inclusion, $\supset$, is clear. Conversely, assume that $\mathcal{H} \notin \mathrm{cl}_X(\Theta) \cup \mathrm{cl}_X(\Psi)$. Then there exist $E_\Theta, E_\Psi \in U(\eta)$ such that for all $\mathcal{G} \in \Theta : E_\Theta(\mathcal{G}) \not\subset \mathcal{H}$ and for all $\mathcal{G} \in \Psi : E_\Psi(\mathcal{G}) \not\subset \mathcal{H}$. Let $E := E_\Theta \cap E_\Psi \in U(\eta)$. Then for all $\mathcal{G} \in \Theta : E_\Theta(\mathcal{G}) \subset E(\mathcal{G})$ and for all $\mathcal{G} \in \Psi$: $E_\Psi(\mathcal{G}) \subset E(\mathcal{G})$. Consequently, for all $\mathcal{G} \in \Theta \cup \Psi : E(\mathcal{G}) \not\subset \mathcal{H}$, which shows that $\mathcal{H} \notin \mathrm{cl}_X(\Theta \cup \Psi)$. ◻

Proposition 3.8. *Let* $(X, \eta) \in \mathbf{saug}$. *For any* $\Theta \subset \mathbf{S}_q(X)$, *let*

$$\Theta_\Delta := \{\mathrm{stack}\, \Delta_{H_1 \cup \cdots \cup H_n} \mid \exists \mathcal{H}_1, \ldots, \mathcal{H}_n \in \Theta : \mathcal{H}_i \in \mathbf{S}(X, H_i),\ H_i \subset X\};$$

then the following hold:

(1) $\mathcal{H} \in \mathrm{cl}_X^u(\Theta) \cap \mathbf{S}_q(X, H)$, $H \subset X$, *implies that* $\mathrm{stack}\, \Delta_H \in \mathrm{cl}_X(\Theta_\Delta)$.

(2) $\mathcal{H} \in \mathrm{cl}_X(\Theta) \cap \mathbf{S}_q(X, H)$, $H \subset X$, *implies that* $\mathrm{stack}\, \Delta_H \in \mathrm{cl}_X(\Theta_\Delta)$.

(3) *For any* $0 \leq K < \infty$:

$$\mathcal{H} \in \mathrm{cl}_X^{u,a}(\{\eta \leq K\} \cap \mathbf{S}_{(q)}(X)\}) \Longleftrightarrow \mathcal{H} \in \mathrm{cl}_X^u(\{\eta \leq K\} \cap \mathbf{S}_{(q)}(X)\}).$$

Proof. (1) To show that $\mathrm{stack}\, \Delta_H \in \mathrm{cl}_X(\Theta_\Delta)$, let $E \in U(\eta)$, and it can be assumed that E is symmetric (without loss of generality in this situation). Since $\mathcal{H} \in \mathrm{cl}_X^u(\Theta)$, it holds for any $\mathcal{W} \in \mathbf{U}(\mathcal{H})$ that there exists $\mathcal{G}_\mathcal{W} \in \Theta \cap \mathbf{S}_q(X, G_\mathcal{W})$ such that $E(\mathcal{G}_\mathcal{W}) \subset \mathcal{W}$. In particular, it follows from Proposition 3.3 that $E(G_\mathcal{W}) \times E(G_\mathcal{W}) \in \mathcal{W}$. Letting $\mathcal{F} := \mathcal{H}$ and $\Psi := \mathbf{U}(\mathcal{H})$ satisfies (by definition) (1) of Lemma 3.6; hence application of (2) of this lemma to the family $(E(G_\mathcal{W}) \times E(G_\mathcal{W}))_{\mathcal{W} \in \Psi}$ leads to

$$n \in \mathbf{N}_0$$

and

$$\mathcal{G}_{\mathcal{W}_1}, \ldots, \mathcal{G}_{\mathcal{W}_n} \in \Theta : (E(G_{\mathcal{W}_1}) \times E(G_{\mathcal{W}_1})) \cup \cdots \cup (E(G_{\mathcal{W}_n}) \times E(G_{\mathcal{W}_n})) \in \mathcal{H}.$$

Clearly, this implies that

$$H \subset E(G_{\mathcal{W}_1}) \cup \cdots \cup E(G_{\mathcal{W}_n}) = E(G_{\mathcal{W}_1} \cup \cdots G_{\mathcal{W}_n}),$$

hence, by Proposition 3.3,

$$E(\operatorname{stack}\Delta_{G_{\mathcal{W}_1}\cup\cdots\cup G_{\mathcal{W}_n}}) \subset \operatorname{stack}\Delta_{E(G_{\mathcal{W}_1}\cup\cdots\cup G_{\mathcal{W}_n})} \subset \operatorname{stack}\Delta_H.$$

Since $\mathcal{G}_{\mathcal{W}_1}, \dots, \mathcal{G}_{\mathcal{W}_n} \in \Theta$, this shows that $\operatorname{stack}\Delta_H \in \mathrm{cl}_X(\Theta_\Delta)$.

(2) This follows immediately from (1) and the fact that $\mathrm{cl}_X(\Theta) \subset \mathrm{cl}^u_X(\Theta)$. The implication $3, \Rightarrow$ is clear.

$3, \Leftarrow$. Let $\Theta := \{\eta \le K\} \cap \mathbf{S}_{(q)}(X)$. Then it follows from $K < \infty$ and $(\mathbf{saug}_\Delta)$ that $\Theta_\Delta \subset \{\operatorname{stack}\Delta_G \mid \eta(\operatorname{stack}\Delta_G) = 0\}$. Hence, by (1), $\mathcal{H} \in \mathrm{cl}^u_X(\Theta)$ implies that $\operatorname{stack}\Delta_H \in \mathrm{cl}_X(\Theta_\Delta) \subset \mathrm{cl}_X(\{\operatorname{stack}\Delta_G \mid \eta(\operatorname{stack}\Delta_G) = 0\})$ (where $\mathcal{H} \in \mathbf{S}_q(X, H)$, $H \subset X$), which combined with $\mathcal{H} \in \mathrm{cl}^u_X(\{\eta \le K\} \cap \mathbf{S}_{(q)}(X))$ leads to $\mathcal{H} \in \mathrm{cl}^{u;a}_X(\{\eta \le K\} \cap \mathbf{S}_{(q)}(X))$. $\square$

Note. The foregoing result does not hold in a quasi setting (as might be expected); however, the following one does, albeit with a different argumentation.

Proposition 3.9. *Let* $(X, \eta) \in \mathbf{saug}$ *and* $H \subset X$. *Then the following are equivalent*:

(1) $\operatorname{stack}\Delta_H \in \mathrm{cl}_X(\{\operatorname{stack}\Delta_G \mid \eta(\operatorname{stack}\Delta_G) = 0\})$.

(2) $\operatorname{stack}\Delta_H \in \mathrm{cl}_X(\{\mathcal{H} \in \mathbf{S}_q(X) \mid \eta(\mathcal{H}) = 0\})$.

(3) $\operatorname{stack}\Delta_H \in \cap_{K>0}\mathrm{cl}_X(\{\mathcal{H} \in \mathbf{S}_q(X) \mid \eta(\mathcal{H}) \le K\})$.

(4) $\operatorname{stack}\Delta_H \in \mathrm{cl}^u_X(\{\operatorname{stack}\Delta_G \mid \eta(\operatorname{stack}\Delta_G) = 0\})$.

(5) $\operatorname{stack}\Delta_H \in \mathrm{cl}^u_X(\{\mathcal{H} \in \mathbf{S}_q(X) \mid \eta(\mathcal{H}) = 0\})$.

(6) $\operatorname{stack}\Delta_H \in \cap_{K>0}\mathrm{cl}^u_X(\{\mathcal{H} \in \mathbf{S}_q(X) \mid \eta(\mathcal{H}) \le K\})$.

(7) *For all* $E \in U(\eta)$, *there exist* $G \subset X : \eta(\operatorname{stack}\Delta_G) = 0$ *and* $H \subset (E \cap E^{-1})(G)$.

Proof. Clearly, $1 \Rightarrow 2$, $2 \Rightarrow 3$, $4 \Rightarrow 5$ and $5 \Rightarrow 6$. Since also $1 \Rightarrow 4$ and $3 \Rightarrow 6$, it suffices to show that $6 \Rightarrow 1$, $7 \Rightarrow 1$ and $1 \Rightarrow 7$.

$6 \Rightarrow 1$. Letting $\Theta := \{\mathcal{H} \in \mathbf{S}_q(X) \mid \eta(\mathcal{H}) \leq K\}$ (where $\infty > K > 0$), it follows from $(\mathbf{saug}_\Delta)$ that $\Theta_\Delta \subset \{\operatorname{stack}\Delta_G \mid \eta(\operatorname{stack}\Delta_G) = 0\}$. Hence, the foregoing proposition implies that $\operatorname{stack}\Delta_H \in \operatorname{cl}_X(\{\operatorname{stack}\Delta_G \mid \eta(\operatorname{stack}\Delta_G) = 0\})$.

$7 \Rightarrow 1$. Let $E \in U(\eta)$. Then it follows from (7) that there exists $G \subset X$ such that $\eta(\operatorname{stack}\Delta_G) = 0$ and $H \subset (E \cap E^{-1}(G)$, hence

$$\begin{aligned} E(\operatorname{stack}\Delta_G) &\subset (E \cap E^{-1})(\operatorname{stack}\Delta_G) \quad (\text{as } E \cap E^{-1} \subset E), \\ &\subset \operatorname{stack}\Delta_{(E\cap E^{-1})(G)} \quad (\text{by Proposition 3.3}), \\ &\subset \operatorname{stack}\Delta_H \quad (\text{as } H \subset (E \cap E^{-1})(G)), \end{aligned}$$

which shows that $\operatorname{stack}\Delta_H \in \operatorname{cl}_X(\{\operatorname{stack}\Delta_G \mid \eta(\operatorname{stack}\Delta_G) = 0\})$.

$1 \Rightarrow 7$. Let $E \in U(\eta)$. Then there exists a symmetric $E' \in U(\eta)$ such that $E' \subset E$ and for which it follows from (1) that there exists $G \subset X$ such that $\eta(\operatorname{stack}\Delta_G) = 0$ and $E'(\operatorname{stack}\Delta_G) \subset \operatorname{stack}\Delta_H$. Hence, by Proposition 3.3, $H \subset E'(G) \subset (E \cap E^{-1})(G)$. $\square$

Proposition 3.10. *Let* $(X, \eta) \in \mathbf{SAULim}$ *and* $\mathcal{H} \in \mathbf{S}_q(X)$. *Then the following hold for any* $0 \leq K \leq \infty$.

(1) $\mathcal{H} \in \operatorname{cl}^u_X(\{\eta \leq K\} \cap \mathbf{S}_q(X)) \Leftrightarrow \mathcal{H} \cap \mathcal{H}^{-1} \in \operatorname{cl}^u_X(\{\eta \leq K\} \cap \mathbf{S}(X))$.

(2) $\mathcal{H} \in \operatorname{cl}_X(\{\eta \leq K\} \cap \mathbf{S}_q(X)) \Leftrightarrow \mathcal{H} \cap \mathcal{H}^{-1} \in \operatorname{cl}_X(\{\eta \leq K\} \cap \mathbf{S}(X))$.

Proof. The implications $1, \Leftarrow$ and $2, \Leftarrow$ are easily verified.

$1, \Rightarrow$. Let $E \in U(\eta)$. Hence, as before, it can be assumed that E is symmetric. Also let $\mathcal{W} \in \mathbf{U}(\mathcal{H} \cap \mathcal{H}^{-1})$. Hence, either $\mathcal{W} \in \mathbf{U}(\mathcal{H})$ or $\mathcal{W} \in \mathbf{U}(\mathcal{H}^{-1})$, and in the latter case, $\mathcal{W}^{-1} \in \mathbf{U}(\mathcal{H})$. As $\mathcal{H} \in \operatorname{cl}^u_X(\{\eta \leq K\} \cap \mathbf{S}_q(X))$, there exists $\mathcal{G} \in \mathbf{S}_q(X)$ such that $\eta(\mathcal{G}) \leq K$ and $E(\mathcal{G} \cap \mathcal{G}^{-1}) \subset E(\mathcal{G}) \subset \mathcal{W}$ (in the first case) or $E(\mathcal{G} \cap \mathcal{G}^{-1}) \subset E(\mathcal{G}) \subset \mathcal{W}^{-1}$ (in the latter case). In either case, it also holds that $\eta(\mathcal{G} \cap \mathcal{G}^{-1}) \leq K$ (as $(X, \eta) \in \mathbf{SAULim}$), $\mathcal{G} \cap \mathcal{G}^{-1} \in \mathbf{S}(X)$ and $E(\mathcal{G} \cap \mathcal{G}^{-1})^{-1} = E(\mathcal{G} \cap \mathcal{G}^{-1})$ (since E and $\mathcal{G} \cap \mathcal{G}^{-1}$ are symmetric). Consequently, in either case, $E(\mathcal{G} \cap \mathcal{G}^{-1}) \subset \mathcal{W}$, which shows that $\mathcal{H} \cap \mathcal{H}^{-1} \in \operatorname{cl}^u_X(\{\eta \leq K\} \cap \mathbf{S}(X))$.

$2, \Rightarrow$. Let $E \in U(\eta)$ (and again assume that E is symmetric). As $\mathcal{H} \in \operatorname{cl}_X(\{\eta \leq K\} \cap \mathbf{S}_q(X))$, there exists $\mathcal{G} \in \mathbf{S}_q(X)$ such that

$\eta(\mathcal{G}) \le K$ and $E(\mathcal{G} \cap \mathcal{G}^{-1}) \subset E(\mathcal{G}) \subset \mathcal{H}$. Since it again holds that $\eta(\mathcal{G} \cap \mathcal{G}^{-1}) \le K$ (as $(X, \eta) \in$ **SAULim**), $\mathcal{G} \cap \mathcal{G}^{-1} \in \mathbf{S}(X)$ and $E(\mathcal{G} \cap \mathcal{G}^{-1})^{-1} = E(\mathcal{G} \cap \mathcal{G}^{-1})$, it follows that $E(\mathcal{G} \cap \mathcal{G}^{-1}) \subset \mathcal{H} \cap \mathcal{H}^{-1}$, which shows that $\mathcal{H} \cap \mathcal{H}^{-1} \in \mathrm{cl}_X(\{\eta \le K\} \cap \mathbf{S}(X))$. $\square$

Proposition 3.11. *Let* $(X, \eta) \in$ **saug**. *Then the following are equivalent*:

(1) $\eta : (\mathbf{S}_q(X), \mathrm{cl}_X^{u,a}) \to ([0, \infty], \mathcal{T}_r)$ *is continuous.*

(2) $\eta : (\mathbf{S}(X), \mathrm{cl}_X^{u,a}) \to ([0, \infty], \mathcal{T}_r)$ *is continuous.*

(3) $\eta : (\mathbf{S}_q(X), \mathrm{cl}_X^{u}) \to ([0, \infty], \mathcal{T}_r)$ *is continuous.*

(4) $\eta : (\mathbf{S}(X), \mathrm{cl}_X^{u}) \to ([0, \infty], \mathcal{T}_r)$ *is continuous.*

Proof. $1 \Leftrightarrow 3$ and $2 \Leftrightarrow 4$ follow immediately from Proposition 3.8 (3). Clearly $3 \Rightarrow 4$ holds.

As for $4 \Rightarrow 3$, let $\mathcal{H} \in \mathrm{cl}_X^u(\{\eta \le K\} \cap \mathbf{S}_q(X))$. Hence, by the previous proposition, $\mathcal{H} \cap \mathcal{H}^{-1} \in \mathrm{cl}_X^u(\{\eta \le K\} \cap \mathbf{S}(X))$. It then follows from (3) that $\eta(\mathcal{H} \cap \mathcal{H}^{-1}) \le K$; consequently, $\eta(\mathcal{H}) \le \eta(\mathcal{H} \cap \mathcal{H}^{-1}) \le K$. $\square$

Note. As might again be expected, the foregoing equivalences do not hold in a quasi setting, in which only the first property is of importance, which is in fact the one that will be primarily used in the sequel.

Definition 3.12. Let **EpiAUnif** be the full subconstruct of **saug** whose objects satisfy one of the foregoing equivalent conditions.

Now we are in a position to state the following claim.

Theorem 3.13. **EpiAUnif** *is the cartesian closed topological hull of* **AUnif**.

Before proceeding to show this in several steps, let it be noted that the (perhaps not so apparent) relation and position with respect to the topological universe (TU) hull of **AUnif** shall be considered in the next section.

Step 1. It needs to be shown that $\mathbf{AUnif} \subset \mathbf{EpiAUnif}$.

Proposition 3.14. $\mathbf{AUnif}$ *is a subconstruct of* $\mathbf{EpiAUnif}$.

Proof. Let $(X,\eta) = (X,(\mathcal{U}_\varepsilon)_{\varepsilon\in\mathbf{R}^+}) \in \mathbf{AUnif}$, and let $\mathcal{H} \in \mathrm{cl}_X^{u,a}(\Theta) \subset \mathrm{cl}_X^u(\Theta)$, where $\Theta := \{\eta \le K\} \cap \mathbf{S}_q(X)$, $0 \le K < \infty$. To show that $\eta(\mathcal{H}) \le K$, i.e., $\mathcal{U}_K \subset \mathcal{H}$ (by Proposition 2.1), let $E \in \mathcal{U}_K$ and $\mathcal{W} \in \mathbf{U}(\mathcal{H})$. By property (UT3) of a uniform tower, there exist $E_0 \in \mathcal{U}_0$ and $E_K \in \mathcal{U}_K$ such that $E_0 \circ E_K \circ E_0 \subset E$. It then follows from $\mathcal{H} \in \mathrm{cl}_X^u(\Theta)$ that there exists $\mathcal{G} \in \Theta$ such that $E_0(\mathcal{G}) \subset \mathcal{W}$. As $\eta(\mathcal{G}) \le K$, it holds that $\mathcal{U}_K \subset \mathcal{G}$, hence $E_K \in \mathcal{G}$; consequently, $E_0(E_K) \subset E \in \mathcal{W}$. Since this is the case for any $\mathcal{W} \in \mathbf{U}(\mathcal{H})$, it follows that $E \in \mathcal{H}$ which shows that $\mathcal{U}_K \subset \mathcal{H}$. □

Step 2. Next we show that $\mathbf{EpiAUnif}$ is a cartesian closed topological construct.

The following general observation about ultrafilters will be useful.

Lemma 3.15. *Let X, Y and Z be sets.*

(1) *Let $f : X \to Y$ be a map. If $\mathcal{F} \in \mathbf{F}(X)$ and $\mathcal{W} \in \mathbf{U}(f(\mathcal{F}))$, then there exists a $\mathcal{V} \in \mathbf{U}(\mathcal{F})$ such that $f(\mathcal{V}) = \mathcal{W}$.*

(2) *Let $g : X \times Y \to Z$ be a map. If $\mathcal{F} \in \mathbf{F}(X \times X)$, $\mathcal{G} \in \mathbf{F}(Y \times Y)$ and $\mathcal{W} \in \mathbf{U}((g \times g), \mathcal{F} \otimes \mathcal{G}))$, then there exists a $\mathcal{Z} \in \mathbf{U}(\mathcal{G})$ such that $(g \times g)(\mathcal{F} \otimes \mathcal{Z}) \subset \mathcal{W}$.*

Proof. (1) Let $F \in \mathcal{F}$ and $W \in \mathcal{W}$. Since $\varnothing \neq (f(F) \cap W) \subset f(F \cap f^{-1}(W))$, it follows that $\varnothing \neq (F \cap f^{-1}(W))$. Hence, there exists $\mathcal{V} \in \mathbf{U}(\mathcal{F}) \cap \mathbf{U}(f^{-1}(\mathcal{W}))$ which also implies that $\mathcal{W} \subset f(f^{-1}(\mathcal{W})) \subset f(\mathcal{V})$ and even (since $\mathcal{W}$ is ultra) $f(\mathcal{V}) = \mathcal{W}$.

(2) By (1) we find an ultrafilter $\mathcal{V}$ on $(X \times Y)^2$ such that $\mathcal{F} \otimes \mathcal{G} \subset \mathcal{V}$ and $(g \times g)(\mathcal{V}) = \mathcal{W}$. By $\mathcal{F} \otimes \mathcal{G} \subset \mathcal{V}$, we have $\mathcal{G} \subset \mathcal{Z} := (\mathrm{pr}_Y \times \mathrm{pr}_Y)(\mathcal{V})$. Furthermore, $(g \times g)(\mathcal{F} \otimes \mathcal{Z}) \subset (g \times g)((\mathrm{pr}_X \times \mathrm{pr}_X)(\mathcal{V}) \otimes (\mathrm{pr}_Y \times \mathrm{pr}_Y)(\mathcal{V})) \subset (g \times g)(\mathcal{V}) = \mathcal{W}$. □

Proposition 3.16. *Let $f : (X, \eta_X) \to (Y, \eta_Y)$ be a uniform* *contraction between semi-approach uniform limit spaces. Then it holds that*

(1) $\bar{f} : (\mathbf{S}_q(X), \mathrm{cl}^u_X) \to (\mathbf{S}_q(Y), \mathrm{cl}^u_Y) : \mathcal{H} \mapsto (f \times f)(\mathcal{H})$ *is continuous.*

(2) $\bar{f} : (\mathbf{S}_q(X), \mathrm{cl}_X) \to (\mathbf{S}_q(Y), \mathrm{cl}_Y) : \mathcal{H} \mapsto (f \times f)(\mathcal{H})$ *is continuous.*

(3) $\bar{f} : (\mathbf{S}_q(X), \mathrm{cl}^{u,a}_X) \to (\mathbf{S}_q(Y), \mathrm{cl}^{u,a}_Y) : \mathcal{H} \mapsto (f \times f)(\mathcal{H})$ *is continuous.*

Proof. (1) Let $\Theta \subset \mathbf{S}_q(X)$ and $\mathcal{H} \in \mathrm{cl}^u_X(\Theta)$. To show that $(f \times f)(\mathcal{H}) \in \mathrm{cl}^u_X(\bar{f}(\Theta))$, let $E \in U(\eta_Y)$ and $\mathcal{W} \in \mathbf{U}((f \times f)(\mathcal{H}))$. It follows from the foregoing lemma that there exists $\mathcal{V} \in \mathbf{U}(\mathcal{H})$ such that $(f \times f)(\mathcal{V}) = \mathcal{W}$, and since $(f \times f)^{-1}(E) \in U(\eta_X)$ (by the uniform continuity of $f : (X, U(\eta_X)) \to (Y, U(\eta_Y))$), there exists $\mathcal{G} \in \Theta$ such that $(f \times f)^{-1}(E)(\mathcal{G}) \subset \mathcal{V}$. Now observe that $E((f \times f)(\mathcal{G})) \subset (f \times f)((f \times f)^{-1}(E)(\mathcal{G}))$ (indeed, for any $G \in \mathcal{G}$ we have: $(f \times f)((f \times f)^{-1}(E)(G)) \subset E((f \times f)(G)))$, consequently, $E((f \times f)(\mathcal{G})) \subset (f \times f)((f \times f)^{-1}(E)(\mathcal{G})) \subset (f \times f)(\mathcal{V}) = \mathcal{W}$, which shows that $(f \times f)(\mathcal{H}) \in \mathrm{cl}^u_Y(\bar{f}(\Theta))$.

(2) Analogous to the previous (but now just without ultrafilters).

(3) Let $\Theta \subset \mathbf{S}_q(X)$ and $\mathcal{H} \in \mathrm{cl}^{u,a}_X(\Theta)$ (where $\mathcal{H} \in \mathbf{S}_q(X, H)(H \subset X)$), hence $\mathcal{H} \in \mathrm{cl}^u_X(\Theta)$ and $\mathrm{stack}\, \Delta_H \in \mathrm{cl}_X(\{\mathrm{stack}\, \Delta_G \mid \eta_X(\mathrm{stack}\, \Delta_G) = 0\})$. It then follows from the previous items and the uniform contractivity of f that

$$(f \times f)(\mathcal{H}) \in \mathrm{cl}^u_Y(\bar{f}(\Theta))$$

and

$$\begin{aligned} \mathrm{stack}\, \Delta_{f(H)} &= (f \times f)(\mathrm{stack}\, \Delta_H) \\ &\in \mathrm{cl}_Y(\{(f \times f)(\mathrm{stack}\, \Delta_G) \mid \eta_X(\mathrm{stack}\, \Delta_G) = 0\}) \\ &\subset \mathrm{cl}_Y(\{\mathrm{stack}\, \Delta_G \mid \eta_Y(\mathrm{stack}\, \Delta_G) = 0\}), \end{aligned}$$

consequently $(f \times f)(\mathcal{H}) \in \mathrm{cl}^{u,a}_Y(\bar{f}(\Theta))$. $\square$

Proposition 3.17. **EpiAUnif** *is bireflective in* **saug***, in particular,* **EpiAUnif** *is a topological construct.*

Proof. It suffices to show that **EpiAUnif** is initially closed in **saug**. To this end, let $(f_i : (X, \eta_X) \to (X_i, \eta_i))_{i \in I}$ be initial in **saug** and all $(X_i, \eta_i) \in \mathbf{EpiAUnif}$. To show that $(X, \eta_X) \in \mathbf{EpiAUnif}$, let $\mathcal{H} \in \mathrm{cl}_X^{u,a}(\{\eta_X \le K\} \cap \mathbf{S}_q(X))$. Then it follows from the previous proposition and the fact that all f_i, $i \in I$, are uniform contractions that

$$\begin{aligned}(f_i \times f_i)(\mathcal{H}) &\in \mathrm{cl}_{X_i}^{u,a}(\bar{f}_i(\{\eta_X \le K\} \cap \mathbf{S}_q(X))) \\ &\subset \mathrm{cl}_{X_i}^{u,a}(\{\eta_i \le K\} \cap \mathbf{S}_q(X_i)).\end{aligned}$$

As $(X_i, \eta_i) \in \mathbf{EpiAUnif}$, it follows from $\eta_i((f_i \times f_i)(\mathcal{H})) \le K$, $i \in I$, and the description of initial lifts (in **saug**) given in Proposition 2.5 that $\eta_X(\mathcal{H}) \le K$. $\square$

Proposition 3.18. *Let (X, η_X) and (Y, η_Y) be semi-approach uniform limit spaces and let $(Z, \eta) := [(X, \eta_X), (Y, \eta_Y)]_{\mathbf{SAULim}}$. Let $H \subset X$ and $\Delta_Y \subset E \subset Y^2$ and denote $F(H, E) := \{(f, g) \in Z^2 \mid \forall x \in H : (f(x), g(x)) \in E\}$. Then, letting $\mathcal{U}_\varepsilon^H := \langle\{F(H, E) \mid E \in \mathcal{U}_\varepsilon^Y\}\rangle$ (where $(Y, (\mathcal{U}_\varepsilon^Y)_{\varepsilon \in \mathbf{R}^+})$ is the* **AUnif***-bireflection of (Y, η_Y)), we define a uniform tower $(\mathcal{U}_\varepsilon^H)_{\varepsilon \in \mathbf{R}^+}$ such that $\eta_X(\mathit{stack}\,\Delta_H) = 0$ implies that $\mathcal{U}_\varepsilon^H \subset \mathcal{U}_\varepsilon^Z$, $\varepsilon \in \mathbf{R}^+$ (where $(Z, (\mathcal{U}_\varepsilon^Z)_{\varepsilon \in \mathbf{R}^+})$ is the* **AUnif***-bireflection of (Z, η)).*

Proof. It is easily verified that $(\mathcal{U}_\varepsilon^H)_{\varepsilon \in \mathbf{R}^+}$ is a uniform tower (on Z) (observe for instance that $F(H, E) \circ F(H, E') \subset F(H, E \circ E')$).

To prove the latter claim, let $\eta_X(\mathrm{stack}\,\Delta_H) = 0$. Then it needs to be shown that $1_Z : (Z, \mathbf{AUnif}(\eta)) \to (Z, (\mathcal{U}_\varepsilon^H)_{\varepsilon \in \mathbf{R}^+})$ is a uniform contraction. By bireflection-properties, it then suffices to show that $1_Z : (Z, \eta) \to (Z, (\mathcal{U}_\varepsilon^H)_{\varepsilon \in \mathbf{R}^+})$ is a uniform contraction (since $(Z, (\mathcal{U}_\varepsilon^H)_{\varepsilon \in \mathbf{R}^+}) \in \mathbf{AUnif}$).

To this end, let $\Psi \in \mathbf{F}(Z^2)$ and $\alpha := \eta(\Psi)$, and let $E \in \mathcal{U}_\alpha^Y$. Since $\eta_X(\mathrm{stack}\,\Delta_H) = 0$, it follows from the description of η in Proposition 2.3 that $\eta_Y(\Psi(\mathrm{stack}\,\Delta_H)) \le \alpha$; hence $E \in \mathcal{U}_\alpha^Y \subset \Psi(\mathrm{stack}\,\Delta_H)$ (by uniform contractivity of $1_Y : (Y, \eta_Y) \to (Y, \mathbf{AUnif}(\eta_Y))$ and by Proposition 2.1). Consequently, there exists $\psi \in \Psi$ such that $(\mathrm{ev} \times \mathrm{ev})(\Delta_H \otimes \psi) \subset E$, implying that $\psi \subset F(H, E)$ and therefore $F(H, E) \in \Psi$. Thus, it has been shown that $F(H, E) \in \Psi$ for all $E \in \mathcal{U}_\alpha^Y$; hence $\mathcal{U}_\alpha^H \subset \Psi$ and therefore, by Proposition 2.1, $\eta^H(\Psi) \le \alpha = \eta(\Psi)$ (where η^H corresponds to the tower $(\mathcal{U}_\varepsilon^H)_{\varepsilon \in \mathbf{R}^+}$). $\square$

Proposition 3.19. *Let* $(X,\eta_X),(Y,\eta_Y)\in$ **saug** *and let* $(Z,\eta):=[(X,\eta_X),(Y,\eta_Y)]$ *(in* **saug***). If* $\mathcal{H}\in\mathbf{S}_q(X,H)$ $(H\subset X)$ *such that* $\eta_X(\mathcal{H})<\infty$*, then it holds that*:

(1) $\overline{\mathcal{H}}:(\mathbf{S}_q(Z),\mathrm{cl}^u_Z)\to(\mathbf{S}_q(Y),\mathrm{cl}^u_Y):\Psi\mapsto\Psi(\mathcal{H})$ *is continuous.*

(2) $\overline{\mathcal{H}}:(\mathbf{S}_q(Z),\mathrm{cl}_Z)\to(\mathbf{S}_q(Y),\mathrm{cl}_Y):\Psi\mapsto\Psi(\mathcal{H})$ *is continuous.*

(3) $\overline{\mathcal{H}}:(\mathbf{S}_q(Z),\mathrm{cl}^{u,a}_Z)\to(\mathbf{S}_q(Y),\mathrm{cl}^{u,a}_Y):\Psi\mapsto\Psi(\mathcal{H})$ *is continuous.*

Proof. (1) Let $\Theta\subset\mathbf{S}_q(Z)$ and $\Psi\in\mathrm{cl}^u_Z(\Theta)$. To show that $\Psi(\mathcal{H})\in\mathrm{cl}^u_Y(\overline{\mathcal{H}}(\Theta))$, let $E\in U(\eta_Y)$ and $\mathcal{W}\in\mathbf{U}(\Psi(\mathcal{H}))$. It follows from Lemma 3.15 that there exists $\mathcal{Z}\in\mathbf{U}(\Psi)$ such that $\mathcal{Z}(\mathcal{H})=(\mathrm{ev}\times\mathrm{ev})(\mathcal{H}\otimes\mathcal{Z})\subset\mathcal{W}$. Since $\eta_X(\mathcal{H})<\infty$ (and $\mathcal{H}\in\mathbf{S}_q(X,H)$), it follows from ($\mathbf{saug}_\Delta$) that $\eta_X(\mathrm{stack}\,\Delta_H)=0$; hence, using notations as in the previous proposition, $F(H,E)\in\mathcal{U}^H_0\subset\mathcal{U}^Z_0\subset U(\eta)$ (where the latter inclusion follows from Proposition 2.5; the latter function space involved is a bicoreflection of the first one involved). As $\Psi\in\mathrm{cl}^u_X(\Theta)$, there exists $\Phi\in\Theta$ such that $F(H,E)(\Phi)\subset\mathcal{Z}$. Now observe that $E(\Phi(\mathcal{H}))\subset(F(H,E)(\Phi))(\mathcal{H})$ (indeed, for any $\phi\in\Phi$ and $H\times H\supset G\in\mathcal{H}$ we have: $(F(H,E)(\phi))(G)\subset E(\phi(G))$). Consequently, $E(\Phi(\mathcal{H}))\subset(F(H,E)(\Phi))(\mathcal{H})\subset\mathcal{Z}(\mathcal{H})\subset\mathcal{W}$, which shows that $\Psi(\mathcal{H})\in\mathrm{cl}^u_Y(\overline{\mathcal{H}}(\Theta))$.

(2) Analogous to the previous (but just without ultrafilters).

(3) Let $\Theta\subset\mathbf{S}_q(Z)$ and $\Psi\in\mathrm{cl}^{u,a}_Z(\Theta)$ (where $\Psi\in\mathbf{S}_q(X,\psi)(\psi\subset Z)$); hence $\Psi\in\mathrm{cl}^u_Z(\Theta)$ and $\mathrm{stack}\,\Delta_\psi\in\mathrm{cl}_Z(\{\mathrm{stack}\,\Delta_\phi\mid\eta(\mathrm{stack}\,\Delta_\phi)=0\})$. It then follows from the previous items, the description of η in Proposition 2.5 and $\eta_X(\mathrm{stack}\,\Delta_H)=0$ that

$$\Psi(\mathcal{H})\in\mathrm{cl}^u_Y(\overline{\mathcal{H}}(\Theta))$$

and

$$\begin{aligned}\mathrm{stack}\,\Delta_{\psi(H)}&=(\mathrm{stack}\,\Delta_\psi)(\mathrm{stack}\,\Delta_H)\\&\in\mathrm{cl}_Y(\{(\mathrm{stack}\,\Delta_\phi)(\mathrm{stack}\,\Delta_H)\mid\eta_Z(\mathrm{stack}\,\Delta_\phi)=0\})\\&\subset\mathrm{cl}_Y(\{\mathrm{stack}\,\Delta_G\mid\eta_Y(\mathrm{stack}\,\Delta_G)=0\});\end{aligned}$$

consequently, $\Psi(\mathcal{H})\in\mathrm{cl}^{u,a}_Y(\overline{\mathcal{H}}(\Theta))$. ◻

Proposition 3.20. **EpiAUnif** *is closed under formation of function spaces in* **saug***. Moreover, if* $(X,\eta_X)\in$ **saug** *and* $(Y,\eta_Y)\in$

EpiAUnif, *then* $[(X,\eta_X),(Y,\eta_Y)] \in$ **EpiAUnif**. *In particular,* **EpiAUnif** *is a cartesian closed category.*

Proof. Let $(Z,\eta) := [(X,\eta_X),(Y,\eta_Y)]$ (in **saug**). To show that $(Z,\eta) \in$ **EpiAUnif**, let $\Psi \in \mathrm{cl}_Z^{u,a}(\{\eta \le K\} \cap \mathbf{S}_q(X))$ (where $\Psi \in \mathbf{S}_q(X,\psi)$, $\psi \subset Z$). To prove that $\eta(\Psi) \le K$, let $\mathcal{H} \in \mathbf{S}_q(X)$ be such that $\eta_X(\mathcal{H}) < \infty$ (which is an acceptable restriction in view of the description of η in Proposition 2.5). Since it also holds that $\mathrm{stack}\,\Delta_\psi \in \mathrm{cl}_Z(\{\mathrm{stack}\,\Delta_\phi \mid \eta(\mathrm{stack}\,\Delta_\phi) = 0\})$, it follows from the previous proposition (and $\eta_X(\mathcal{H}) < \infty$) and the description of η in Proposition 2.5 that

$$\begin{aligned}(\mathrm{stack}\,\Delta_\psi)(\mathcal{H}) &\in \mathrm{cl}_Y(\overline{\mathcal{H}}(\{\mathrm{stack}\,\Delta_\phi \mid \eta(\mathrm{stack}\,\Delta_\phi) = 0\}))\\ &\subset \mathrm{cl}_Y(\{\mathcal{G} \in \mathbf{S}_q(X) \mid \eta_Y(\mathcal{G})) \le \eta_X(\mathcal{H})\})\end{aligned}$$

and

$$\begin{aligned}\Psi(\mathcal{H}) &\in \mathrm{cl}_Y^{u,a}(\overline{\mathcal{H}}(\{\eta \le K\} \cap \mathbf{S}_q(X)))\\ &\subset \mathrm{cl}_Y^{u,a}(\{\eta_Y \le K \vee \eta_X(\mathcal{H})\} \cap \mathbf{S}_q(X)).\end{aligned}$$

As $(Y,\eta_Y) \in$ **EpiAUnif**, it follows that $\eta_Y((\mathrm{stack}\,\Delta_\psi)(\mathcal{H})) \le \eta_X(\mathcal{H})$ and $\eta_Y(\Psi(\mathcal{H})) \le K \vee \eta_X(\mathcal{H})$; hence, by description of function spaces (in **saug**) given in Proposition 2.5, we can conclude that $\eta(\Psi) \le K$. □

Step 3. The next goal is to show that proper "density" conditions are satisfied.

Definition 3.21. Recall that $d_{\mathbf{E}} : \mathbf{R} \times \mathbf{R} \to \mathbf{R}^+ : (x,y) \mapsto |x-y|$ is an extended pseudo-metric on $\mathbf{R}^+$. In particular, $\{\{(x,y) \in \mathbf{R}^2 \mid d_{\mathbf{E}}(x,y) < \alpha\} \mid \alpha > \varepsilon\}$ is a filterbasis that generates a semi-uniformity $\mathcal{U}_\varepsilon^{\mathbf{E}}$ such that $(\mathcal{U}_\varepsilon^{\mathbf{E}})_{\varepsilon \in \mathbf{R}^+}$ is a uniform tower, hence $(\mathbf{R}, (\mathcal{U}_\varepsilon^{\mathbf{E}})_{\varepsilon \in \mathbf{R}^+}) = (\mathbf{R}, \eta_{\mathbf{E}})$ is an approach uniform space.

Assume without restriction in the following that $X \neq \varnothing$.

Proposition 3.22. *Let* $(X,\eta) \in$ **EpiAUnif**. *Then the map*

$$j : (X,\eta) \longrightarrow [[(X,\eta),(\mathbf{R},\eta_{\mathbf{E}})],(\mathbf{R},\eta_{\mathbf{E}})] : x \longmapsto (f \mapsto f(x))$$

is an initial, uniform contraction (*considered in* **saug**).

Proof. First observe that $j := \mathrm{ev}^*_{(X,\eta),(\mathbf{R},\eta_{\mathbf{E}})}$ is the map which makes the following diagram commute:

$$\begin{tikzcd}
{[[(X,\eta),(\mathbf{R},\eta_{\mathbf{E}})],(\mathbf{R},\eta_{\mathbf{E}})] \times [(X,\eta),(\mathbf{R},\eta_{\mathbf{E}})]} \arrow[r, "\mathrm{ev}"] & (\mathbf{R},\eta_{\mathbf{E}}) \\
(X,\eta) \times [(X,\eta),(\mathbf{R},\eta_{\mathbf{E}})] \arrow[u, "j\times 1"] \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}$$

Hence, by properties of function spaces, j is a uniform contraction. In the following, also let

$$(\hom((X,\eta),(\mathbf{R},\eta_{\mathbf{E}})),\eta_H) := [(X,\eta),(\mathbf{R},\eta_{\mathbf{E}})],$$

and

$$(\hom([X,\eta),(\mathbf{R},\eta_{\mathbf{E}})],(\mathbf{R},\eta_{\mathbf{E}})),\eta_{HH}) := [[(X,\eta),(\mathbf{R},\eta_{\mathbf{E}})],(\mathbf{R},\eta_{\mathbf{E}})].$$

By Proposition 2.5, proving that j is initial can be done by showing that

$$\forall\, \mathcal{H} \in \mathbf{S}_q(X), \forall\, K > 0 : (\eta(\mathcal{H}) > K \Longrightarrow \eta_{HH}(j(\mathcal{H})) \geq K).$$

To this end, let $\mathcal{H} \in \mathbf{S}_q(X)$ be such that $\eta(\mathcal{H}) > K$, $0 < K < \infty$. It will be shown that $\eta_{HH}(j(\mathcal{H})) \geq K$ by defining an appropriate $\Psi \in \mathbf{S}_q(\hom((X,\eta),(\mathbf{R},\eta_{\mathbf{E}})))$ such that $\eta_{\mathbf{E}}(\Psi(\mathcal{H})) = \eta_{\mathbf{E}}((j(\mathcal{H}))(\Psi)) \geq K > \eta_{\mathbf{E}}(\Psi)$, hence, by Proposition 2.5, $\eta_{HH}(j(\mathcal{H})) \geq K$.

Definition of $\Psi \in \mathbf{S}_q(\hom((X,\eta),(\mathbf{R},\eta_{\mathbf{E}})))$. As $(X,\eta) \in$ **EpiAUnif** and $\eta(\mathcal{H}) > K$, it follows from Proposition 3.11 that $\mathcal{H} \notin \mathrm{cl}^u_X(\{\eta \leq K\} \cap \mathbf{S}_q(X))$; hence there exist $E \in U(\eta)$ and $\mathcal{W} \in \mathbf{U}(\mathcal{H})$ such that

$$(*) \qquad \forall\, \mathcal{G} \in \mathbf{S}_q(X) : (E(\mathcal{G}) \subset \mathcal{W} \Longrightarrow \eta(\mathcal{G}) > K).$$

Since $E \in U(\eta)$, it follows from Proposition 2.2 that there exist a symmetric $d' \in \mathcal{D}(\mathbf{AUnif}(\eta))$ and $0 < \delta < (K/2)$ such that $\{d' < \delta\} \subset E$. If we let $d := d' \wedge K \leq d'$, then it also holds that $d \in \mathcal{D}(\mathbf{AUnif}(\eta))$ (by saturatedness of a uniform gauge) and $\{d < \delta\} \subset E$. Now let

$$\begin{aligned}\Phi_1 &:= \{f : (X,\eta) \longrightarrow (\mathbf{R},\eta_{\mathbf{E}}) \mid \exists K_1, K_2 \in \mathbf{R},\\ &\qquad \exists x \in X : f = K_1 + (d(-,x) \wedge K_2)\},\\ \Phi_2 &:= \{f : (X,\eta) \longrightarrow (\mathbf{R},\eta_{\mathbf{E}}) \mid \exists K_1, K_2 \in \mathbf{R},\\ &\qquad \exists x \in X : f = K_1 - (d(x,-) \wedge K_2)\},\end{aligned}$$

and

$$\Phi := \{f \in \Phi_1 \cup \Phi_2 \mid 0 \leq f \leq K\}.$$

Observe that for all $f \in \Phi : f : (X,\eta) \to (\mathbf{R},\eta_{\mathbf{E}})$ is a uniform contraction. Indeed, let $f = K_1 + (d(-,x) \wedge K_2) \in \Phi_1$, then it follows from the symmetry of d that

$$\begin{aligned}|f(u) - f(v)| &= |K_1 + (d(u,x) \wedge K_2) - K_1 - (d(v,x) \wedge K_2)|\\ &\leq |d(u,x) - d(v,x)| \leq d(u,v).\end{aligned}$$

In case $f = K_1 - (d(x,-) \wedge K_2) \in \Phi_2$, the symmetry of d again implies

$$\begin{aligned}|f(u) - f(v)| &= |K_1 - (d(x,u) \wedge K_2) - K_1 + (d(x,v) \wedge K_2)|\\ &\leq |d(x,v) - d(x,u)| \leq d(u,v).\end{aligned}$$

In any case, given $f \in \Phi$, it holds that

$$(**) \qquad \forall \varepsilon > 0 : \{d < \varepsilon\} \subset (f \times f)^{-1}(\{d_{\mathbf{E}} < \varepsilon\})$$

It then follows from the characterization of uniform contractivity in terms of uniform towers and Proposition 2.2 that $f : (X, \mathbf{AUnif}(\eta)) \to (\mathbf{R},\eta_{\mathbf{E}})$ is a uniform contraction and therefore, by uniform contractivity of $1_X : (X,\eta) \to (X, \mathbf{AUnif}(\eta))$, $f : (X,\eta) \to (\mathbf{R},\eta_{\mathbf{E}})$ is a uniform contraction.

For any $G \subset X \times X$, let

$$F_\delta(G) := \{(f,g) \in \Phi \times \Phi \mid \forall\,(x,y) \in G : d_{\mathbf{E}}(f(x),g(y)) \leq K - \delta\},$$

and let

$$\hat{\Psi} := \{F_\delta(G) \mid G \subset X \times X, E(G) \notin \mathcal{W}\}.$$

Note that $\hat{\Psi}$ is a filterbasis on $\hom((X,\eta),(\mathbf{R},\eta_{\mathbf{E}}))$. Indeed, $\hat{\Psi}\neq\varnothing$, since for any $x\in X$, $E(\dot{x}\times\dot{x})\not\subset\mathcal{W}$, otherwise, by $(*)$, $\eta(\dot{x}\times\dot{x})>0$, a contradiction. Furthermore, such an $F_\delta(G)$ is never a void set, as it always contains pairs of constant (positive) functions. Also, it holds that $F_\delta(G_1)\cap F_\delta(G_2)=F_\delta(G_1\cup G_2)$, $E(G_1\cup G_2)=E(G_1)\cup E(G_2)$ and $\mathcal{W}$ is an ultrafilter, hence $\hat{\Psi}$ is a filterbasis. Now let Ψ' denote the filter generated by $\hat{\Psi}$ and define $\Psi:=\Psi'\cap\operatorname{stack}\Delta_\Phi\in\mathbf{S}_q(\hom((X,\eta),(\mathbf{R},\eta_{\mathbf{E}})),\Phi)$.

Proof of $\eta_H(\Psi)\leq K-\delta<K$. By Proposition 2.5, it needs to be shown that

$$\forall\,\mathcal{G}\in\mathbf{S}_q(X):\eta_{\mathbf{E}}((\operatorname{stack}\Delta_\Phi)(\mathcal{G}))\leq\eta(\mathcal{G})$$

and

$$\eta_{\mathbf{E}}(\Psi(\mathcal{G}))\leq(K-\delta)\vee\eta(\mathcal{G}),$$

or equivalently,

$$\forall\,\mathcal{G}\in\mathbf{S}_q(X):\eta_{\mathbf{E}}((\operatorname{stack}\Delta_\Phi(\mathcal{G}))\leq\eta(\mathcal{G})$$

and

$$\eta_{\mathbf{E}}(\Psi'(\mathcal{G}))\leq(K-\delta)\vee\eta(\mathcal{G}).$$

To this end, let $\mathcal{G}\in\mathbf{S}_q(X)$. Since $1_X:(X,\eta)\to(X,\mathbf{AUnif}(\eta))$ is a uniform contraction, it follows from Proposition 2.1 that $\mathbf{AUnif}(\eta)_{\eta(\mathcal{G})}\subset\mathcal{G}$. In particular, by Proposition 2.2, for all $\alpha>\eta(\mathcal{G}):\{d<\alpha\}\in\mathcal{G}$. As $(**)$ actually states that for all $\alpha>0$: $(\mathrm{ev}\times\mathrm{ev})(\{d<\alpha\}\otimes\Delta_\Phi)\subset\{d_{\mathbf{E}}<\alpha\}$, it follows that $\mathcal{U}^{\mathbf{E}}_{\eta(\mathcal{G})}\subset(\operatorname{stack}\Delta_\Phi)(\mathcal{G})$, hence $\eta_{\mathbf{E}}((\operatorname{stack}\Delta_\Phi)(\mathcal{G}))\leq\eta(\mathcal{G})$.

Regarding the latter, let $\mathcal{G}\in\mathbf{S}_q(X)$ and assume that $\eta_{\mathbf{E}}(\Psi'(\mathcal{G}))>K-\delta$; hence $\mathcal{U}^{\mathbf{E}}_{K-\delta}\not\subset\Psi'(\mathcal{G})(I)$. This implies that $E(\mathcal{G})\subset\mathcal{W}$. Indeed, if this were not the case, then there would exist $G\in\mathcal{G}$ such that $E(G)\notin\mathcal{W}$. Hence, $F_\delta(G)\in\Psi'$; consequently, $(\mathrm{ev}\times\mathrm{ev})(G\otimes F_\delta(G))\subset\{d_{\mathbf{E}}\leq K-\delta\}\in\Psi'(\mathcal{G})$. In particular, $\mathcal{U}^{\mathbf{E}}_{K-\delta}\subset\Psi'(\mathcal{G})$, which contradicts (I). Thus, it must indeed be that $E(\mathcal{G})\subset\mathcal{W}$, hence by $(*)$, $\eta(\mathcal{G})>K$. On the other hand, it also holds that for all $f\in\Phi:0\leq f\leq K$, which implies that $\{d_{\mathbf{E}}\leq K\}\in\Psi'(\mathcal{G})$, hence $\eta_{\mathbf{E}}(\Psi'(\mathcal{G}))\leq K<\eta(\mathcal{G})$. Thus, it must be that $\eta_{\mathbf{E}}(\Psi'(\mathcal{G}))\leq(K-\delta)\vee\eta(\mathcal{G})$.

Proof of $\eta_{\mathbf{E}}(\Psi(\mathcal{H})) \geq K$. As $\Psi(\mathcal{H}) \subset \Psi'(\mathcal{W})$, it will suffice to show that $\eta_{\mathbf{E}}(\Psi'(\mathcal{W})) \geq K$. Assume the contrary of the latter, i.e., $\mathcal{U}^{\mathbf{E}}_{K'} \subset \Psi'(\mathcal{W})$ (where $K' < K$). In particular, this implies the existence of $G \subset X \times X$ and $U \in \mathcal{W}$ such that

$$\text{(II)} \qquad E(G) \notin \mathcal{W} \quad \text{and} \quad (\text{ev} \times \text{ev})(U \otimes F_\delta(G)) \subset \{d_{\mathbf{E}} < K\}.$$

It then follows that $U \subset E(G)$. Indeed, let $(x, y) \notin E(G)$ and define

$$0 \leq f_2 := d(-, y) \wedge \delta \leq \frac{K}{2} \leq f_1 := K - (d(x, -) \wedge \delta) \leq K$$

(since $0 < \delta < (K/2)$), then $f_1, f_2 \in \Phi$ (by definition). Moreover, $(f_1, f_2) \in F_\delta(G)$. For, if this were not the case, then there would exist $(x', y') \in G$ such that

$$|f_1(x') - f_2(y')| = K - (d(x, x') \wedge \delta) - (d(y', y) \wedge \delta) > K - \delta,$$

hence $d(x, x') < \delta$ and $d(y', y) < \delta$. As $\{d < \delta\} \subset E$, it follows that $(x, x'), (y', y) \in E$; consequently, $(x, y) \in E(G)$, which contradicts the fact that $(x, y) \notin E(G)$. Hence, $(f_1, f_2) \in F_\delta(G)$ and $(f_1(x), f_2(y)) = (K, 0) \notin \{d_{\mathbf{E}} < K\}$; consequently, by (II), $(x, y) \notin U$. Therefore, $U \subset E(G)$, hence $E(G) \in \mathcal{W}$, which contradicts (II). Thus, it must be that $\eta_{\mathbf{E}}(\Psi'(\mathcal{W})) \geq K$. □

Note. In a quasi setting, one proceeds by first considering the extended pseudo-quasi-metric $d_{\mathbf{P}} : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+ : (x, y) \mapsto (x - y) \vee 0$, which leads to $\{\{(x, y) \in (\mathbf{R}^+)^2 \mid d_{\mathbf{P}}(x, y) < \alpha\} \mid \alpha > \varepsilon\}$ being a filterbasis that generates a quasi-semi-uniformity $\mathcal{U}^{\mathbf{P}}_\varepsilon$ such that $(\mathcal{U}^{\mathbf{P}}_\varepsilon)_{\varepsilon \in \mathbf{R}^+}$ is a quasi-uniform tower, hence $\mathbf{S} := (\mathbf{R}^+, (\mathcal{U}^{\mathbf{P}}_\varepsilon)_{\varepsilon \in \mathbf{R}^+}) = (\mathbf{R}^+, \eta_{\mathbf{P}})$ is a quasi-approach uniform space.

It is then shown (by also making use of Proposition 3.9) that $j : (X, \eta) \to [[(X, \eta), \mathbf{S}], \mathbf{S}] : x \mapsto (f \mapsto f(x))$ is an initial, uniform contraction (all considered in **qsaug**).

Step 4. Now we combine all previous steps and prove the final result.

Theorem 3.23. **EpiAUnif** *is the cartesian closed topological hull of* **AUnif***.*

Proof. Previous steps have already shown that **EpiAUnif** is a finally dense cartesian closed topological extension of **AUnif**, hence (by the preliminaries), it only remains to show that the class

$$H := \{[(X,\eta_X),(Y,\eta_Y)] \mid (X,\eta_X),(Y,\eta_Y) \in \mathbf{AUnif}\}$$

is initially dense in **EpiAUnif**. To this end, the previous proposition implies that, for any $(X,\eta) \in \mathbf{EpiAUnif}$, there is an initial map $j : (X,\eta) \to [[(X,\eta),(\mathbf{R},\eta_{\mathbf{E}})],(\mathbf{R},\eta_{\mathbf{E}})]$ and since the functor $[-,(\mathbf{R},\eta_{\mathbf{E}})] : \mathbf{EpiAUnif} \to \mathbf{EpiAUnif}$ transforms final epi-sinks into initial sources (see [**8**, Lemma 6]) (and by Proposition 2.5, $[(X,\eta),(\mathbf{R},\eta_{\mathbf{E}})]$ can be obtained as a final lift of an epi-sink involving **AUnif**-objects by adding constant maps if necessary), it follows that H is indeed initially dense in **EpiAUnif**. $\square$

4. Relation to other hulls. First of all, it must be that the CCT hull of **AUnif** is contained in the TU hull of **AUnif**, which is the full subconstruct **saug-PsAULim** of **saug** whose objects (X,η) satisfy one of the following equivalent conditions:

(1) For all $\mathcal{H} \in \mathbf{F}(X) : \eta(\mathcal{H}) = \sup_{\mathcal{W}\in\mathbf{U}(\mathcal{H})} \eta(\mathcal{W})$.

(2) For all $\mathcal{H} \in \mathbf{S}_q(X) : \eta(\mathcal{H}) = \sup_{\mathcal{W}\in\mathbf{U}(\mathcal{H})} \eta(\mathcal{W})$.

(3) For all $\mathcal{H} \in \mathbf{S}_q(X,H) : (\eta(\operatorname{stack}\Delta_H) = 0 \Rightarrow \eta(\mathcal{H}) = \sup_{\mathcal{W}\in\mathbf{U}(\mathcal{H})} \eta(\mathcal{W}))$.

On the one hand, this may not be very apparent from its definition, and on the other hand, would this allow us to simplify the description of the CCT hull?

To obtain satisfactory answers, it is necessary to perform a kind of "splitting up" of the closure $\mathrm{cl}_X^{u,a}$, to which end it needs to be adapted first.

Definition 4.1. Let

$$\mathrm{cl}_X^a : \mathcal{P}(\mathbf{S}_q(X)) \longrightarrow \mathcal{P}(\mathbf{S}_q(X)) : \Theta \longmapsto \mathrm{cl}_X^a(\Theta)$$

where

$$\begin{aligned}\mathrm{cl}_X^a := \{\mathcal{H} \in \mathbf{S}_q(X,H) \mid \mathcal{H} \in \mathrm{cl}_X(\Theta) \quad &\text{and} \\ \operatorname{stack}\Delta_H \in \mathrm{cl}_X(\{\operatorname{stack}\Delta_G \mid \eta(\operatorname{stack}\Delta_G) = 0\})\}.\end{aligned}$$

Proposition 4.2. *Let* $(X,\eta) \in \mathbf{saug}$; *then the following are equivalent*:

(1) $(X,\eta) \in \mathbf{EpiAUnif}$.

(2) $(X,\eta) \in$ **saug-PsAULim** *and* (X,η) *satisfies one of the equivalent conditions*:

- $\eta : (\mathbf{S}_q(X), cl^a_X) \to ([0,\infty], \mathcal{T}_r)$ *is continuous.*
- $\eta : (\mathbf{S}(X), cl^a_X) \to ([0,\infty], \mathcal{T}_r)$ *is continuous.*
- $\eta : (\mathbf{S}_q(X), cl_X) \to ([0,\infty], \mathcal{T}_r)$ *is continuous.*
- $\eta : (\mathbf{S}(X), cl_X) \to ([0,\infty], \mathcal{T}_r)$ *is continuous.*

Proof. The equivalence of the properties mentioned in (2) can be shown as in Proposition 3.11, but the first one shall again be primarily of importance.

$1 \Rightarrow 2$. Clearly it only needs to be shown that $(X,\eta) \in \mathbf{saug} - \mathbf{PsAULim}$. To this end, let $\mathcal{H} \in \mathbf{S}_q(X,H)$ such that $\eta(\operatorname{stack}\Delta_H) = 0$ and for all $\mathcal{W} \in \mathbf{U}(\mathcal{H}) : \eta(\mathcal{W}) \le K$. In particular, $\operatorname{stack}\Delta_H \in \mathrm{cl}_X(\{\operatorname{stack}\Delta_G \mid \eta(\operatorname{stack}\Delta_G) = 0\})$. Also, $\mathcal{H} \in \mathrm{cl}^u_X(\{\eta \le K\} \cap \mathbf{S}_q(X))$. Indeed, let $E \in U(\eta)$ and $\mathcal{W} \in \mathbf{U}(\mathcal{H})$. As $\eta(\mathcal{W}) \le K$, there exists $\mathcal{G} \in \mathbf{S}_q(X)$ such that $\eta(\mathcal{G}) \le K$ and $\mathcal{G} \subset \mathcal{W}$ (by **saug**), hence $E(\mathcal{G}) \subset \mathcal{G} \subset \mathcal{W}$. Thus, $\mathcal{H} \in \mathrm{cl}^{u,a}_X(\{\eta \le K\} \cap \mathbf{S}_q(X))$, and therefore $\eta(\mathcal{H}) \le K$, which shows that $\eta(\mathcal{H}) \le \sup_{\mathcal{W}\in\mathbf{U}(\mathcal{H})} \eta(\mathcal{W}) \le \eta(\mathcal{H})$ (where the latter inequality follows from (SAUCS_2)).

$2 \Rightarrow 1$. Let $\mathcal{H} \in \mathbf{S}_q(X,H)$ be such that $\mathcal{H} \in \mathrm{cl}^{u,a}_X(\{\eta \le K\} \cap \mathbf{S}_q(X))$ (where $0 \le K < \infty$). In particular, $\operatorname{stack}\Delta_H \in \mathrm{cl}_X(\{\operatorname{stack}\Delta_G \mid \eta(\operatorname{stack}\Delta_G) = 0\})$, hence $\operatorname{stack}\Delta_H \in \mathrm{cl}^a_X(\{\operatorname{stack}\Delta_G \mid \eta(\operatorname{stack}\Delta_G) = 0\})$; consequently, by continuity of $\eta : (\mathbf{S}_q(X), \mathrm{cl}^a_X) \to ([0,\infty], \mathcal{T}_r), \eta(\operatorname{stack}\Delta_H) = 0$.

To show that $\eta(\mathcal{H}) \le K$, it suffices to show that $\eta(\mathcal{W}) \le K$ for all $\mathcal{W} \in \mathbf{U}(\mathcal{H})$. To this end, let $\mathcal{W} \in \mathbf{U}(\mathcal{H})$ and let $E \in U(\eta)$, then it follows from $\mathcal{H} \in \mathrm{cl}^{u,a}_X(\{\eta \le K\} \cap \mathbf{S}_q(X))$ that there exists $\mathcal{G}_E \in \mathbf{S}_q(X)$ such that $E(\mathcal{G}_E) \subset \mathcal{W}$ and $\eta(\mathcal{G}_E) \le K$. Now let $\mathcal{H}_E := \mathcal{G}_E \cap \operatorname{stack}\Delta_H$, then also $\eta(\mathcal{H}_E) \le K$ and $\mathcal{W} \supset E(\mathcal{H}_E)_{|H^2} \in \mathbf{S}_q(X,H)$ (as $H \times H \in \mathcal{W}$). In particular, letting $\mathcal{G} := \vee_{E\in U(\eta)} E(\mathcal{H}_E)_{|H^2}$ is well-defined and is such that $\mathcal{G} \in \mathbf{S}_q(X,H)$ and $\mathcal{G} \subset \mathcal{W}$. Also, by construction, $\mathcal{G} \in \mathrm{cl}_X(\{\eta \le K\} \cap \mathbf{S}_q(X))$ and even $\mathcal{G} \in \mathrm{cl}^a_X(\{\eta \le K\} \cap \mathbf{S}_q(X))$; hence the continuity of $\eta : (\mathbf{S}_q(X), \mathrm{cl}_X^a) \to ([0,\infty], \mathcal{T}_r)$ implies that $\eta(\mathcal{G}) \le K$ and consequently $\eta(\mathcal{W}) \le \eta(\mathcal{G}) \le K$. □

Next, recall from [**16**] that CCTH(**Unif**) is the full subconstruct of **sug** $=$ **saug** $\cap$ **SAULim** consisting of objects $(X, \mathbf{L})$ satisfying (primarily) the property $\mathrm{cl}_X^u(\mathbf{L} \cap \mathbf{S}_q(X)) = \mathbf{L} \cap \mathbf{S}_q(X)$ (or the property $\mathrm{cl}_X^u(\mathbf{L} \cap \mathbf{S}(X)) = \mathbf{L} \cap \mathbf{S}(X)$), where the uniformity involved in cl_X^u is, in this case, the **Unif**-bireflection $\mathbf{Unif}(\mathbf{L})$ of $(X, \mathbf{L})$.

Proposition 4.3. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *and* $\mathbf{D}$ *be constructs such that*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "r", "R_{\mathbf{A}}"'] & \mathbf{B} \\ \mathbf{C} \arrow[u, "c", "C_{\mathbf{C}}"'] \arrow[r, "R_{\mathbf{C}}", "r"'] & \mathbf{D} \arrow[u, "C_{\mathbf{D}}", "c"'] \end{tikzcd}$$

(*where* $R_{\mathbf{A}}$, $R_{\mathbf{C}}$, $C_{\mathbf{C}}$ *and* $C_{\mathbf{D}}$ *denote the appropriate concrete* (*co*)*reflectors*), *then the following are equivalent*:

(1) $C_{\mathbf{C}}$ *is the restriction of* $C_{\mathbf{D}}$ *to* $\mathbf{A}$.

(2) $R_{\mathbf{C}}$ *is the restriction of* $R_{\mathbf{A}}$ *to* $\mathbf{D}$.

Proof. $1 \Rightarrow 2$. Let $D \in \mathbf{D}$. Then it suffices to show that $D' := R_{\mathbf{C}}(D)$ is also an $\mathbf{A}$-reflection of D. To this end, let $f : D \to A$ be a morphism (where $A \in \mathbf{A}$), hence $f : D \to C_{\mathbf{D}}(A)$ is a morphism. It follows from (1) that $C_{\mathbf{D}}(A) \in \mathbf{C}$, hence $f : D' \to C_{\mathbf{D}}(A)$ is a morphism (and also (1): $C_{\mathbf{D}}(A) \to A$ is a morphism); consequently, $f : D' \to A$ is a morphism.

$1 \Rightarrow 2$. This is "dual" to the previous implication. □

Proposition 4.4. CCTH(**AUnif**) $\cap$ **SULim**$=$ CCTH(**Unif**).

Proof. Let $(X, \eta) = (X, \mathbf{L}) \in$ **sug** (where $\mathbf{L} = \{\eta = 0\}$), then $(X, \eta) \in$ CCTH(**AUnif**) if and only if $\mathrm{cl}_X^{u,a}(\{\eta \le K\} \cap \mathbf{S}_q(X)) = \{\eta \le K\} \cap \mathbf{S}_q(X)$ for all $K \in \mathbf{R}^+$. As $(X, \eta) \in$ **SULim**, this is clearly equivalent to $\mathrm{cl}_X^{u,a}(\{\eta = 0\} \cap \mathbf{S}_q(X)) = \{\eta = 0\} \cap \mathbf{S}_q(X)$. Since it follows from Proposition 2.6 and the previous one that $\mathbf{AUnif}(\eta) = \mathbf{Unif}(\mathbf{L})$,

$$
\begin{tikzcd}
 & q\mathbf{AUnif} \arrow[r, "r"] & \mathrm{CCTH}(q\mathbf{AUnif}) \arrow[r, "r"] & \mathbf{qsaug} \arrow[r, "c"] & q\,\mathbf{SAULim} \\
 & q\mathbf{Unif} \arrow[u, "r", "c"'] \arrow[r, "r"] & \mathrm{CCTH}(q\mathbf{Unif}) \arrow[u, "r", "c"'] \arrow[r, "r"] & \mathbf{qsug} \arrow[u, "r", "c"'] \arrow[r, "c"] & q\,\mathbf{SULim} \arrow[u, "r", "c"'] \\
\mathbf{AUnif} \arrow[r, "r"] \arrow[uur, "r", "c"'] & \mathrm{CCTH}(\mathbf{AUnif}) \arrow[r, "r"] \arrow[uur, "r", "c"'] & \mathbf{saug} \arrow[r, "c"] \arrow[uur, "r", "c"'] & \mathbf{SAULim} \arrow[uur, "r", "c"'] & \\
\mathbf{Unif} \arrow[u, "r", "c"'] \arrow[r, "r"] \arrow[uur, "r", "c"'] & \mathrm{CCTH}(\mathbf{Unif}) \arrow[u, "r", "c"'] \arrow[r, "r"] \arrow[uur, "r", "c"'] & \mathbf{sug} \arrow[u, "r", "c"'] \arrow[r, "c"] \arrow[uur, "r", "c"'] & \mathbf{SULim} \arrow[u, "r", "c"'] \arrow[uur, "r", "c"'] &
\end{tikzcd}
$$

the latest statement is equivalent to $\mathrm{cl}^u_X(\mathbf{L}\cap\mathbf{S}_q(X))=\mathbf{L}\cap\mathbf{S}_q(X)$ (as it also follows from Proposition 3.9 that $\mathrm{cl}^{u,a}_X(\{\eta=0\}\cap\mathbf{S}_q(X))=\mathrm{cl}^u_X(\{\eta=0\}\cap\mathbf{S}_q(X))$). $\square$

Proposition 4.5. CCTH(**Unif**) *is bireflective and bicoreflective in* CCTH(**AUnif**).

Proof. The bireflectiveness, i.e., initial closedness, follows from the bireflectiveness of CCTH(**AUnif**)=**EpiAUnif** in **saug**(Proposition 3.17), Proposition 2.6 and the previous one.

As for the latter claim, it will suffice to show that the **SULim**-bicoreflection (X,η_0) of $(X,\eta)\in$ **EpiAUnif** belongs to CCTH(**Unif**) $\subset$ **SULim**. Hence, by the previous proposition, it only needs to be shown that $(X,\eta_0)\in$ **EpiAUnif** (and note that Proposition 2.6 already implies that $(X,\eta_0)\in\mathbf{sug}\subset\mathbf{saug}$).

To this end, observe for any $\infty>K\geq 0$ that $\{\eta_0\leq K\}\cap\mathbf{S}_q(X)=\{\eta\leq 0\}\cap\mathbf{S}_q(X)$, which is $\mathrm{cl}^{u,a}_{(X,\eta)}$-closed. Furthermore, as $1_X:(X,\eta_0)\to(X,\eta)$ is a uniform contraction, it follows from Proposition 3.16 that $\{\eta_0\leq K\}\cap\mathbf{S}_q(X)$ is $\mathrm{cl}^{u,a}_{(X,\eta_0)}$-closed. Since this latter claim is also evident if $K=\infty$, the bicoreflectiveness has been shown. $\square$

Note. Analogous results can be obtained in a quasi setting, albeit restricted in that case to the "primary characterizations". Additionally, it can (easily) be shown that CCTH(**AUnif**) = CCTH(q**AUnif**) $\cap$**SAULim** and that CCTH(**AUnif**) is bireflectively and bicoreflectively embedded in CCTH(q**AUnif**), which allows us to conclude by summarizing some nice relations in the diagram.

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