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QUASI-PURIFIABLE SUBGROUPS AND HEIGHT-MATRICES

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ABSTRACT. Let G be an arbitrary abelian group. A subgroup A of G is said to be quasi-purifiable in G if a pure subgroup H of G exists containing A such that A is almost-dense in H and H/A is torsion. Such a subgroup H is called a quasi-pure hull of A in G. First we prove that a torsion-free rank-one subgroup A of G is quasi-purifiable in G if and only if, for every prime p and every $a \in A$, $h_p(a) \geqq \omega$ implies $h_p(a) = \infty$. Next we use the result to compute the height-matrix of the torsion-free element a of an abelian group whose torsion part $T(G)$ is torsion-complete, then all torsion-free rank-one subgroups of G are quasi-purifiable in G and hence the height-matrices of the torsion-free elements of the group G can be computed.

1. Introduction. Let p be a prime. A subgroup A of an arbitrary abelian group G is said to be *p-purifiable* (*purifiable*) in G if a p-pure (pure) subgroup H of G containing A which is minimal among the p-pure (pure) subgroups of G that contain A. Such a subgroup H is said to be a *p-pure hull* (*pure hull*) of A in G.

Hill and Megibben [**7**] established properties of pure hulls of p-groups and characterized the p-groups for which all subgroups are purifiable.

Later, Benabdallah and Irwin [**2**] introduced the concept of almost-dense subgroups of p-groups and used it to determine the structure of pure hulls of purifiable subgroups of p-groups.

Furthermore, Benabdallah and Okuyama [**3**] introduce new invariants, the so-called nth *overhangs* of a subgroup of a p-group, which are related to the nth relative Ulm-Kaplansky invariants. Using them, they obtained a necessary condition for subgroups of p-groups to be purifiable.

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Benabdallah, Charles and Mader [**1**] introduced the concept of maximal vertical subgroups supported by a given subsocle of a p-group and characterized the p-groups for which the necessary condition given in [**3**] is also sufficient.

Several results on isomorphy of pure hulls in p-groups are contained in [**12**] and [**13**]. Other results about purifiable subgroups of p-groups are contained in [**4**], [**5**], [**8**], [**11**], [**12**] and [**13**].

Recently, in [**14**], we extended the concept of almost-dense subgroups from p-groups to arbitrary abelian groups (see Definition 2.1) and began to study purifiable subgroups of arbitrary abelian groups. We characterized the groups for which all subgroups are purifiable in [**14**] and characterized in [**15**] the purifiable torsion-free rank-one subgroups in arbitrary abelian groups. However, the characterization of purifiable subgroups in arbitrary abelian groups is an open problem even if the subgroup is torsion-free.

In [**14**, Theorem 1.11], we characterized a p-pure (pure) hull H of a purifiable subgroup A in an arbitrary abelian group as follows:

1. A is p-almost-dense (almost-dense) in H;
2. H/A is a p-group (torsion);
3. (for every prime p), a nonnegative integer m_p exists such that

$$p^{m_p}H[p] \subseteq A.$$

Quasi-p-purifiable (quasi-purifiable) in arbitrary abelian groups is defined as follows.

Definition 1.1. Let p be a prime. A subgroup A of an arbitrary abelian group G is said to be quasi-p-purifiable (quasi-purifiable) in G if a p-pure (pure) subgroups H of G exists containing A such that

1. A is p-almost-dense (almost-dense) in H and
2. H/A is a p-group (torsion).

Such a subgroup H is called a quasi-p-pure hull (quasi-pure hull) of A in G.

In [**12**], we studied quasi-purifiable subgroups of p-groups. In this note we consider quasi-purifiable subgroups in arbitrary abelian groups.

In the rest of this introduction, let G be an arbitrary abelian group and A a subgroup of G.

In Section 2 we recall basic definitions and properties and prove that A is quasi-purifiable in G if and only if, for every prime p, A is quasi-p-purifiable in G. This plays an important role in studying quasi-purifiable subgroups of G.

In Section 3 we present an example of a torsion-free rank-one subgroup that has a quasi-pure hull but no pure hull.

In Section 4 we establish a necessary and sufficient condition for a torsion-free rank-one subgroup to be quasi-purifiable. In fact, if A is torsion-free rank-one, then A is quasi-purifiable in G if and only if, for every $a \in A$ and every prime p,

$$h_p(a) \geqq \omega \quad \text{implies} \quad h_p(a) = \infty \tag{1.2}$$

where, by definition, $h_p(a) = \infty$ if and only if a is in the maximal p-divisible subgroup of G.

We use the result to prove that, if G is an abelian group whose torsion part $T(G)$ is torsion-complete, then all torsion-free rank-one subgroups of G are quasi-purifiable in G and, hence, for every $a \in G \setminus T(G)$ and every prime p, a satisfies the condition (1.2) as Corollary 4.14.

In Section 5 we show how to compute the height-matrix of $a \in G \setminus T(G)$ if a satisfies (1.2) for every prime p. If G is an abelian group whose torsion part $T(G)$ is torsion-complete, then the height-matrices of all torsion-free elements of the group G can be computed in this way. Finally, we compute the height-matrices of some torsion-free elements of the group G in Section 3. Example 3.1 shows that the converse of Corollary 4.14 is not true, namely, even if all torsion-free rank-one subgroups of an abelian group G are quasi-purifiable G, the torsion part of G is not necessarily torsion-complete.

Height-matrices are important. For example, combining results in Rotman [**16**], Megibben [**9**] and Myshkin [**10**], countable mixed groups of torsion-free rank-one are classified in [**6**, Theorem 104.3]. In fact, the countable mixed groups H and K of torsion-free rank-one are isomorphic if and only if $T(H) \cong T(K)$ and the height-matrices $\mathbf{H}(H)$ and $\mathbf{H}(K)$ are equivalent.

All groups considered are arbitrary abelian groups. The terminologies and notations not expressly introduced here follow the usage of [**6**].

Throughout this note, $\mathbf{P}$ denotes the set of all prime integers, p an element of $\mathbf{P}$, G_p the p-primary subgroup and T the maximal torsion subgroup of the arbitrary abelian group G.

2. Notation and basics. We recall definitions and properties mentioned in [**14**]. We frequently use them in this note. Throughout this section let G be an arbitrary abelian group and A a subgroup of G.

Definition 2.1. A is said to be p-almost-dense in G if, for every p-pure subgroup K of G containing A, the torsion part of G/K is p-divisible. Moreover, A is said to be almost-dense in G if A is p-almost-dense in G for every $p \in \mathbf{P}$.

Proposition 2.2 [**14**, Proposition 1.3, Proposition 1.4]. *The following properties are equivalent*:

1. *A is p-almost dense* (*almost-dense*) *in G*;
2. *for all integers $n \geqq 0$* (*and all $p \in \mathbf{P}$*), *$p^n G[p] \subseteq A + p^{n+1}G$.*

Definition 2.3. A is said to be p-purifiable (purifiable) in G if, among the p-pure (pure) subgroups of G containing A, a minimal one exists. Such a minimal p-pure (pure) subgroup is called a p-pure (pure) hull of A.

Proposition 2.4 [**14**, Theorem 1.8, Theorem 1.11]. *Suppose that A is p-purifiable* (*purifiable*) *in G. Then a p-pure* (*pure*) *subgroup H of G containing A is a p-pure* (*pure*) *hull of A in G if and only if the following three conditions are satisfied*:

1. *A is p-almost-dense* (*almost-dense*) *in H*;
2. *H/A is a p-primary* (*torsion*);
3. (*for every $p \in \mathbf{P}$*), *a nonnegative integer m_p exists such that*

$$p^{m_p} H[p] \subseteq A.$$

Comparing the definition (see Definition 1.1) of quasi-purifiable subgroups of abelian groups G with Proposition 2.4, we can see easily that the condition for a subgroup of G to be quasi-purifiable is weaker than the condition for it to be purifiable.

Definition 2.5. For every nonnegative integer n, we define the nth p-overhang of A in G to be the vector space

$$V_{p,n}(G,A) = \frac{(A+p^{n+1}G)\cap p^nG[p]}{(A\cap p^nG)[p]+p^{n+1}G[p]}.$$

Moreover, a set $\{t_i\}$ of nonnegative integers is a p-overhang set of A in G if $V_{p,t_i}(G,A) \neq 0$ for all $i \geqq 1$ and $V_{p,t}(G,A) = 0$ otherwise. A is said to be eventually p-vertical in G if the set $\{t_i\}$ is finite and A is said to be p-vertical in G if the set $\{t_i\}$ is empty.

It is convenient to use the following notations for the numerator and the denominator of $V_{p,n}(G,A)$:

$$A^n_G(p) = (A+p^{n+1}G)\cap p^nG[p] = ((A\cap p^nG)+p^{n+1}G)[p]$$

and

$$A^G_n(p) = (A\cap p^nG)[p]+p^{n+1}G[p].$$

Note that, for any $x \in A^n_G(p)\backslash A^G_n(p)$, we have $h_p(x) = n$. If $x \in A^G_n(p)$, then $h_p^{G/A}(x+A) > n$. If $x \notin A^G_n(p)$, then $a \in A$ and $g \in G$ exist such that $x = a+p^{n+1}g$. Hence $h_p^{G/A}(x+A) > n$. If A is p-almost-dense in G, then $A+p^{n+1}G \supseteq p^nG[p]$, so $A^n_G(p) = p^nG[p]$. If A is torsion-free, then $A^G_n(p) = p^{n+1}G[p]$. Thus, if A is torsion-free and p-almost-dense in G, then

$$V_{p,n}(G,A) = \frac{p^nG[p]}{p^{n+1}G[p]},$$

the nth Ulm-Kaplansky invariant of G_p.

Proposition 2.6 [**14**, Proposition 2.2]. *For every p-pure subgroup K of G containing A,*

$$V_{p,n}(G,A) \cong V_{p,n}(K,A)$$

for all $n \geqq 0$.

Next we can characterize quasi-pure hulls of a quasi-purifiable subgroup in arbitrary abelian groups as follows:

Proposition 2.7. *If A is quasi-purifiable in G, then the following hold.*

1. *If H is a quasi-pure hull of A in G then, for every $p \in \mathbf{P}$, $H^{(p)}$ is a quasi-p-pure hull of A in G where $H^{(p)}$ is defined by $H^{(p)}/A = (H/A)_p$.*

2. *If, for every $p \in \mathbf{P}$, $K^{(p)}$ is a quasi-p-pure hull of A in G, then $\sum_p K^{(p)}$ is a quasi-pure hull of A in G.*

Proof. (1) By hypothesis, H/A is torsion. Let $H^{(p)}/A = (H/A)_p$. We prove that $H^{(p)}$ is p-pure in G. Suppose that $p^n g \in H^{(p)} \subseteq H$ with $g \in G$ and $n \in \mathbf{Z}$. Then $p^n g = p^n h$ for some $h \in H$, so $p^n(g-h) = 0$. Hence $g = x+y+z$ such that $x \in H^{(p)}$, $y \in \sum_{q \neq p} H(q)$ and $z \in G[p^n]$. $p^n g = p^n x + p^n y$ so $p^n y \in H^{(p)} \cap \sum_{q \neq p} K^{(q)} = A$ and it follows that $y \in A$ and $p^n g = p^N(x+y) \in p^n H^{(p)}$. Hence $H^{(p)}$ is p-pure in G. Since A is almost-dense in H, A is p-almost-dense in $H^{(p)}$ and so A is quasi-p-purifiable in G.

(2) For every $p \in \mathbf{P}$, let $K^{(p)}$ be a quasi-p-pure hull of A in G. Let $K = \sum_p K^{(p)}$. We show that K is pure in G. Let $p^m g \in K$ with $g \in G$ and $m \in \mathbf{Z}$. Then we can write $p^m g = u + v$ for some $u \in K^{(p)}$ and $v \in \sum_{q \neq p} K^{(q)}$. Since $(\sum_{q \neq p} K^q)/A$ is p-divisible, $v' \in \sum_{q \neq p} K^{(q)}$ and $a \in A$ exist such that $v = p^m v' + a$. Since $p^m(g - v') = u + a \in K^{(p)} \cap p^m G = p^m K^{(p)}$, K is $p$pure in G. Hence K is pure in G. It is immediate that A is almost-dense in K and K/A is torsion. Hence A is quasi-purifiable in G. $\square$

In view of Proposition 2.7 we can show a relationship between quasi-p-purifiability and quasi-purifiability.

Corollary 2.8. *A subgroup A is quasi-purifiable in G if and only if, for every $p \in \mathbf{P}$, A is quasi-p-purifiable in G.*

3. An example. In this section we present an example of a quasi-purifiable subgroup of an abelian group that is not purifiable.

Example 3.1. For every $p \in \mathbf{P}$, let

$$T_p = \bigoplus_{i=1}^{\infty} \langle y_{p_i} \rangle$$

where $o(y_{pi}) = p^{2i}$ and, for every $p \in \mathbf{P}$ and $i = 1, 2, \dots$, define

$$\mathbf{b}_{pi} = (0, \dots, 0, y_{pi}, py_{pi+1}, p^2 y_{pi+2}, \dots) \in \prod_{i=1}^{\infty} \langle y_{pi} \rangle.$$

Moreover, define

$$a = (\mathbf{b}_{21}, \mathbf{b}_{31}, \dots, \mathbf{b}_{p1}, \dots) \in \prod_p \left(\prod_{i=1}^{\infty} \langle y_{pi} \rangle \right)$$

and

$$g_{pj} = (\mathbf{b}_{21}^{j-1}, \mathbf{b}_{31}^{j-1}, \dots, \mathbf{b}_{q1}^{j-1}, \dots, \mathbf{b}_{pj}, \dots, \mathbf{b}_{r1}^{j-1}, \dots) \in \prod_p \left(\prod_{i=1}^{\infty} \langle y_{pi} \rangle \right)$$

where $q, r \in \mathbf{P}$ with $q \neq p \neq r$, $\mathbf{b}_{q1}^0 = \mathbf{b}_{q1}$ and $p\mathbf{b}_{q1}^{j-1} = \mathbf{b}_{q1}^{j-2}$ for every $q \neq p$ and $j = 2, 3, 4, \dots$. Note that $a = g_{p1}$ for all $p \in \mathbf{P}$. Let $T = \oplus_p T_p$ and

$$G = \langle T, g_{pj} \mid p \in \mathbf{P}, j = 1, 2, \dots \rangle.$$

For convenience, we write y_{pi} instead of $(0, \dots, 0, y_{pi}, 0, \dots)$. Then we have the following properties.

Property 3.2. *For every $p \in \mathbf{P}$ and all integers $i \geqq 1$,*

$$y_{pi} = g_{pi} - pg_{pi1}.$$

Hence

$$G = \langle g_{pj} \mid p \in \mathbf{P}, j = 1, 2, \dots \rangle.$$

Proof. This follows from the definition. $\square$

Property 3.3. *For every prime p and all integers $i \geqq 1$, we have*

$$p^{2i-1}y_{pi} = p^i a - p^{2i}g_{pi+1} \tag{3.4}$$

and

$$(G/\langle a\rangle) = \bigoplus_p \Big(\bigoplus_{j=1}^{\infty}\langle g_{pj} + \langle a\rangle\rangle\Big) \quad \textit{and} \quad o(g_{pj} + \langle a\rangle) = p^{2j-1} \tag{3.5}$$

for $j \geqq 2$.

Proof. By an easy induction we have for $i \geqq 1$, $p^{2i-1}g_{pi} = p^i a$. Hence, by Property 3.2 we have (3.4). For every $p \in \mathbf{P}$, let $G^{(p)} = \langle g_{pj} \mid j = 1, 2, \dots\rangle$. By (3.4), we have $o(g_{pj} + \langle a\rangle\rangle) = p^{2j-1}$ for $j \geqq 2$ and, hence,

$$G^{(p)}/\langle a\rangle = \sum_{j=1}^{\infty}\langle g_{pj} + \langle a\rangle\rangle.$$

By [**6**, Theorem 33.1], we have

$$G^{(p)}/\langle a\rangle = \bigoplus_{j=1}^{\infty}\langle g_{pj} + \langle a\rangle\rangle.$$

By Property 3.2 we have (3.5). □

Property 3.6. $T = T(G)$.

Proof. By [**6**, Theorem 33.1], T_p is pure in G_p. It suffices to prove that $G[p] \subseteq T[p]$. Let $g \in G[p]$. By (3.5), we can write

$$g + \langle a\rangle = \sum_{i=2}^{n}\alpha_i p^{2i-2}g_{pi} + \langle a\rangle,$$

where every α_i is an integer for $1 \leqq i \leqq n$. By (3.4), we have

$$g + \sum_{i=2}^{n}\alpha_i p^{2i-3}y_{pi-1} \in T \cap \langle a\rangle = 0.$$

Hence $T = T(G)$. $\square$

Property 3.7. *$\langle a \rangle$ is quasi-purifiable in G and G is a quasi-pure hull of $\langle a \rangle$.*

Proof. By (3.4) and Property 3.6, $\langle a \rangle$ is almost-dense in G. Since $G/\langle a \rangle$ is torsion, G is a quasi-pure hull of $\langle a \rangle$. $\square$

Property 3.8. *$\langle a \rangle$ is not purifiable in G.*

Proof. For every $p \in \mathbf{P}$, the p-indicator of a is

$$(0, 1, 3, 5, \dots, 2n-1, \dots).$$

By [**15**, Theorem 3.2], $\langle a \rangle$ is not purifiable in G. $\square$

4. Quasi-purifiable torsion-free rank-one subgroups. The goal of this section is to provide a necessary and sufficient condition for a torsion-free rank-one subgroup of an arbitrary abelian group to be quasi-purifiable. First we give an important lemma.

Lemma 4.1. *Let G be an abelian group and A a torsion-free rank-one subgroup of G. Let $x, y \in G[p]$ such that $h_p(y) < h_p^{G/A}(y + A)$ and $h_p(x) < h_p(y)$. Then $h_p^{G/A}(x + A) < h_p(y)$.*

Proof. Suppose that $h_p^{G/A}(x + A) \geqq h_p(y)$. Let $s = h_p(x)$ and $t = h_p(y)$. By hypothesis, $x = a + p^t g$ and $y = b + p^{t+1} h$ for some $a, b \in A$ and $g, h \in G$. Since $r(A) = 1$, integers α, β exist such that $(\alpha, \beta) = 1$ and $\alpha a + \beta b = 0$. Hence,

$$\alpha x + \beta y = \alpha p^t g + \beta p^{t+1} h.$$

Then p divides α, $(\beta, p) = 1$ and $\beta y \in p^{t+1} G$. This contradicts the choice of y. Hence $h_p^{G/A}(x + A) < h_p(y)$. $\square$

Lemma 4.2. *Let G be an abelian group and A a torsion-free rank-one subgroup of G. If $V_{p,s}(G, A) \neq 0$ and $V_{p,t}(G, A) \neq 0$ for some integers $s < t$, then $s < h_p^{G/A}(x + A) < t$ for every $x \in A_G^s(p) \setminus A_s^G(p)$.*

Proof. Let $x \in A_G^s(p) \setminus A_s^G(p)$ and $y \in A_G^t(p) \setminus A_t^G(p)$. By the comment after Definition 2.5, $h_p(x) = s$, $h_p^{G/A}(x + A) > s$, $h_p(y) = t$ and $h_p^{G/A}(y + A) > t$. By Lemma 4.1 $s < h_p^{G/A}(x + A) < t$. □

Definition 4.3. Let G be an abelian group, A a torsion-free rank-one subgroup of G and $\{t_i\}$ the p-overhang set of A in G. Define

$$c_i = \max\{h_p^{G/A}(y + A) \mid y \in A_G^{t_i}(p) \setminus A_{t_i}^G(p)\}$$

if this exists.

Lemma 4.4. *Let G be an abelian group, A a torsion-free rank-one subgroup of G and $\{t_i\}$ the p-overhang set of A in G. Suppose that A is not eventually p-vertical in G. Then, for every $i \geqq 1$, c_i (see Definition 4.3) exists and $t_i < c_i < t_{i+1}$. Setting*

$$f_k = \sum_{j=1}^{k} (t_{j+1} - c_j)$$

for all $k \geqq 1$, $a \in A$ exists such that

$$h_p(p^n a) = \begin{cases} t_1 & \text{for } n = 0, \\ c_1 + n & \text{for } 1 \leqq n \leqq f_1, \\ c_1 + n + \sum_{i=2}^{k+1} (c_i - t_i) & \text{for } f_k < n \leqq f_{k+1},\ k \geqq 1. \end{cases}$$

Proof. By Lemma 4.2, $\{h_p^{G/A}(y + A) \mid y \in A_G^{t_i}(p) \setminus A_{t_i}^G(p)\}$ is bounded and hence c_i exists. Then, for every $i \geqq 1$, $t_i < c_i < t_{i+1}$.

For every $i \geqq 1$, $x_i \in A_G^{t_i}(p) \setminus A_{t_i}^G(p)$, $a_i \in A$ and $g_i \in G$ exist such that

$$x_i = a_i + p^{c_i} g_i. \tag{4.5}$$

Then $h_p(x_i) = h_p(a_i) = t_i < c_i = h_p^{G/A}(x_i + A) = h_p^{G/A}(p^{c_i} g_i + A)$ and $h_p(p^j g_i) = j$ for $0 \leqq j \leqq c_i$. If $h_p(p^{c_i+1} g_i) > c_i + 1$, then $pa_i = p^{c_i+2} g$ for some $g \in G$. Let $y = a_i - p^{c_i+1} g$. Then $h_p(y) = t_i$ and $0 \neq y \in A_G^{t_i}(p) \setminus A_{t_i}^G(p)$. This contradicts the maximality of

$h_p^{G/A}(x_i + A)$. Hence $h_p(p^{c_i+1}g_i) = c_i + 1$. Suppose by induction that $h_p(p^{c_i+k}g_i) = c_i + k$ for $1 \leqq k < t_{i=1} - c_i$. If $h_p(p^{c_i+k+1}g_i) > c_i + k + 1$, then $g' \in G$ exists such that $-p^{k+1}a_i = p^{c_i+k+1}g_i = p^{c_i+k+2}g'$. Since $-p^k a_i = p^{c_i+k}g_i$, by induction hypothesis, we have $h_p(p^k a_i) = c_i + k$. Then $p^k a_i + p^{c_i+k+1}g' \in A_G^{c_i+k}(p) = A_{c_i+k}^G(p) = p^{c_i+k+1}G[p]$. This is a contradiction. Therefore,

$$h_p(p^j g_i) = j \tag{4.6}$$

for $0 \leqq j \leqq t_{i+1}$.

By (4.5) and (4.6) for all $i \geqq 1$, $h_p(a_i) = t_i$ and $h_p(p^n a_i) = h_p(p^{n+c_i}g_i) = c_i + n$ for $1 \leqq n \leqq t_{i+1} - c_i$. Set $a = a_1$. Then $h_p(a) = t_1$ and $h_p(p^n a) = c_1 + n$ for $1 \leqq n \leqq f_1$. Since $h_p(p^{f_1}a) = t_2 = h_p(a_2)$, it is easily seen that $h_p(p^n a) = c_1 + n + c_2 - t_2$ for $f_1 < n \leqq f_2$. Suppose by induction that $h_p(p^n a) = c_1 + n + \sum_{i=2}^{k+1}(c_i - t_i)$ for $f_k < n \leqq f_{k+1}$. As in the previous paragraph, we obtain $h_p(p^n a) = c_1 + n + \sum_{i=2}^{k+2}(c_i - t_i)$ for $f_{k=1} < n \leqq f_{k+2}$. $\square$

We give a useful lemma and use it to prove Lemma 4.8.

Lemma 4.7. *Let G be an abelian group and A a subgroup of G. Suppose that $A \cap p^m G$ is quasi-p-purifiable in $p^m G$ for some $m \geqq 0$. Then A is quasi-p-purifiable in G.*

Proof. Let H be a quasi-p-pure hull of $A \cap p^m G$ in $p^m G$. Since $(A + H) \cap p^m G = H$, by [**14**, Lemma 4.4], $A + H$ can be extended to a p-pure subgroup K of G such that $K \cap p^m G = H$. Therefore, $p^m K = H$. Since $A \cap p^m G$ is almost-dense in H, we have $p^{m+i}K[p] = p^i H[p] \subseteq (A \cap p^m G) + p^{i+1}H = (A + p^{i+1}H) \cap p^m G \subseteq A + p^{i+1}H = A + p^{m+i+1}K$ for all $i \geqq 0$.

For every p-pure subgroup R of K containing A, define

$$E(R) = \{t \geqq 1 \mid A + p^t R \not\supseteq p^{t-1}R[p]\}$$

and set

$$\mathbf{m}(R) = 0 \text{ if } E(R) \neq 0, \quad \text{and } \mathbf{m}(R) = \max\{x \in E(R)\} \text{ if } E(R) \neq \varnothing.$$

Note that $\mathbf{m}(R) \leqq m+1$ and hence a p-pure subgroup L of G containing A exists such that $\mathbf{m}(L)$ is minimal. By [**14**, Lemma 1.2], we see that $\mathbf{m}(L) = 0$. hence a p-pure subgroup L of G containing A exists such that A is p-almost-dense in L. Since $p^m(K/A) \cong p^m K/(A \cap p^m G) = H/(A \cap p^m G)$ is a p-group, K/A is a p-group. Hence, L/A is a p-group and L is a quasi-p-pure hull of A in G. $\square$

Lemma 4.8. *Let G be an abelian group and A a torsion-free rank-one subgroup of G. Suppose that A is not eventually p-vertical in G. Then A is quasi-p-purifiable in G.*

Proof. For all $i \geqq 1$, let t_i, x_i, a_i and g_i be as in the proof of Lemma 4.4, and let c_i be as in Definition 4.3. Specifically, by (4.5) and (4.6),

$$x_i \in A_G^{t_i}(p) \setminus A_{t_i}^G(p), px_i = 0, x_i = a_i + p^{c_i} g_i, a_i \in A, g_i \in G, \tag{4.9}$$

$$h_p(x_i) = h_p(a_i) = t_i < c_i = h_p^{G/A}(x_i + A) = h_p^{G/A}(p^{c_i} g_i + A), \tag{4.10}$$

$$h_p(p^j g_i) = j \quad \text{for } 0 \leqq j \leqq t_{i+1}. \tag{4.11}$$

Let $H = \langle p^{t_1} g_i, A \cap p^{t_1} G \mid i \geqq 1 \rangle$. By (4.9), $o(p^{t_1} g_i + (A \cap p^{t_1} G)) = p^{c_i - t_1 + 1}$. We show that H is p-pure in $p^{t_1} G$. By [**6**, Theorem 33.1], $H/(A \cap p^{t_1} G)$ is p-pure in $p^{t_1} G/(A \cap p^{t_1} G)$.

Let $p^n g \in H$ with $g \in p^{t_1} G$ and $n \in \mathbf{Z}$. Since $p^n g + (A \cap p^{t_1} G) \in H/(A \cap p^{t_1} G) \cap p^n(p^{t_1} G/(A \cap p^{t_1} G)) = p^n(H/(A \cap p^{t_1} G))$, $b \in A \cap p^{t_1} G$ and $h \in H$ exist such that $p^n g = b + p^n h$.

Suppose that $0 < n \leqq t_2 - t_1$. Since $r(A) = 1$, α_1 and β_1 exist such that $(\alpha_1, \beta_1) = 1$ and $\alpha_1 a_1 = \beta_1 b$. Since $h_p(a_1) = t_1$ and $h_p(b) \geqq n + t_1 > t_1$, p divides α_1 and $(\beta_1, p) = 1$. Let $\alpha_1 = p\alpha_1'$ where α_1' is an integer. By (4.9), we have $\alpha_1 a_1 = \alpha_1' p a_1 = -\alpha_1' p^{c_1+1} g_1 = -\alpha_1' p^{c_1 - t_1 + 1} p^{t_1} g_1$. Since $h_p(b) \geqq t_1 + n$, by (4.11) we have $\alpha_1 a_1 = \alpha_1'' p^n p^{t_1} g_1$ for some integer α_1''. Hence $\beta_1 p^n g = \alpha_1'' p^n p^{t_1} g_1 + \beta_1 p^n h \in p^n H$.

Suppose that $t_i - t_1 < n \leqq t_{i+1} - t_1$ for $i \geqq 2$. Since $r(A) = 1$, α_i and β_i exist such that $(\alpha_i, \beta_i) = 1$ and $\alpha_i a_i = \beta_i b$. Since $h_p(a_i) = t_i$ and $h_p(b) \geqq n + t_1 > t_i - t_1 + t_1 = t_i$, p divides α_i and $(\beta_i, p) = 1$. Let $\alpha_i = p\alpha_i'$ where α_i' is an integer. By (4.9) we have

$\alpha_i a_i = \alpha_i' p a_i = -\alpha_i' p^{c_i+1} g_i = -\alpha_i' p^{c_i-t_1+1} p^{t_1} g_i$. Since $h_p(b) \geqq t_1 + n$, by (4.11) we have $\alpha_i a_i = \alpha_i'' p^n p^{t_1} g_i$ for some integer α_i''. Hence $\beta_i p^n g = \alpha_i'' p^n p^{t_1} g_i + \beta_i p^n h \in p^n H$. Hence H is p-pure in $p^{t_1}G$.

For every $i \geqq 1$ let $z_i \in G$ such that $x_i = p^{t_i} z_i$ and $h_p(z_i) = 0$. Since

$$x_i = p^{t_i} z_i = p^{t_i - t_1} p^{t_1} z_i \in H \cap p^{t_i - t_1}(p^{t_1}G) = p^{t_i - t_1} H,$$

for every $i \geqq 1$, $y_i \in H$ exists such that $x_i = p^{t_i - t_1} y_i$, $h_p(y_i) = t_1$, $h_p^H(y_i) = 0$. Let $U = \oplus_{i=1}^{\infty} \langle y_i \rangle$.

We prove that $U = H_p$. By [**6**, Theorem 33.1], U is pure in H_p. It suffices to prove that $U[p] \supset H[p]$. Note that

$$H/(A \cap p^{t_1}G) = \bigoplus_{i=1}^{\infty} (p^{t_1} g_i + (A \cap p^{t_1}G)).$$

Let $y \in H[p]$. Then, by (4.9), we can write

$$y + (A \cap p^{t_1}G) = \sum_{i=1}^{n} \gamma_i p^{c_i} g_i + (A \cap p^{t_1}G) = \sum_{i=1}^{n} \gamma_i x_i + (A \cap p^{t_1}G)$$

where n and every γ_i is an integer for $1 \leqq i \leqq n$. Thus

$$y - \sum_{i=1}^{n} \gamma_i x_i \in H_p \cap (A \cap p^{t_1}G) = 0.$$

Hence $U = H_p$. By (4.9), $A \cap p^{t_1}G$ is p-almost-dense in H. By Lemma 4.7, A is quasi-p-purifiable in G. $\square$

Theorem 4.12. *Let G be an abelian group and A a torsion-free rank-one subgroup of G. Then A is quasi-p-purifiable in G if and only if, for every $a \in A$,*

$$h_p(a) \geqq \omega \quad \textit{implies} \quad h_p(a) = \infty.$$

Proof. ($\Rightarrow$). If A is not eventually p-vertical in G, then, by Lemma 4.4, $a \in A$ exists such that $h_p(p^n a) < \omega$ for all integers $n \geqq 0$.

Since $r(A) = 1$, $h_p(p^n b) < \omega$ for all integers $n \geqq 0$ and every $b \in A$. Hence, without loss of generality, we may assume that A is eventually p-vertical in G. Let H be a quasi-p-pure hull of A in G. By Proposition 2.6, A is eventually p-vertical in H. Since A is p-almost-dense in H, A is p-purifiable in H by [**14**, Theorem 4.7]. By [**15**, Theorem 3.2], $h_p(a) \geqq \omega$ implies $h_p(a) = \infty$.

($\Leftarrow$). If A is not eventually p-vertical in G then, by Lemma 4.8, A is quasi-p-purifiable in G. If A is eventually p-vertical in G, then by hypothesis and [**15**, Theorem 3.2], A is p-purifiable in G and hence A is quasi-p-purifiable in G. $\square$

Corollary 2.8 and Theorem 4.12 combined lead to the following result.

Corollary 4.13. *Let G be an abelian group and A a torsion-free rank-one subgroup of G. Then A is quasi-purifiable in G if and only if, for every $a \in A$ and every $p \in \mathbf{P}$,*

$$h_p(a) \geqq \omega \quad \textit{implies} \quad h_p(a) = \infty.$$

Corollary 2.8, Corollary 4.13 and [**14**, Theorem 4.8] combined lead to the following result.

Corollary 2.14. *Let G be an abelian group whose torsion part T is torsion-complete. Then all torsion-free rank-one subgroups of G are quasi-purifiable in G.*

Proof. Let A be a torsion-free rank-one subgroup of G. If A is not eventually p-vertical in G then, by Lemma 4.8, A is quasi-p-purifiable in G. If A is eventually p-vertical in G, then, by [**14**, Theorem 4.8], A is p-purifiable and hence quasi-p-purifiable in G. Hence by Corollary 2.8, A is quasi-purifiable in G.

5. The height-matrices of torsion-free elements. In this section we use the previous results to compute the height-matrix of the torsion-free element a of an abelian group G which satisfies the condition that, for every $p \in \mathbf{P}$ and every integer $n \geqq 0$, $h_p(p^n a) \geqq \omega$

implies $h_p(p^n a) = \infty$ which is equivalent to saying that, for every $p \in \mathbf{P}$ and every integer $n \geqq 0$, either $h_p(p^n a) < \omega$ or $h_p(p^n a) = \infty$. Before doing that, we need the following lemma.

Lemma 5.1. *Let G be an abelian group, A a torsion-free rank-one subgroup of G and $\{t_i\}$ the p-overhang set of A in G. Suppose that A is not p-vertical in G but eventually p-vertical in G and $|\{t_i\}| = r$. Then, for every $1 \leqq i \leqq r-1$, c_i (see Definition 4.3) exists and $t_i < c_i < t_{i+1}$ and one of the following conditions is satisfied*:

1. $\sup\{h_p^{G/A}(y+A) \mid y \in A_G^{t_r}(p) \setminus A_{t_r}^G(p)\} < \omega$;
2. *there exists $a \in A$ such that $h_p(a) = t_r$ and $h_p(pa) \geqq \omega$.*

Proof. By Lemma 4.2 $\{h_p^{G/A}(y+A) \mid y \in A_G^{t_i}(p) \setminus A_{t_i}^G(p)\}$ is bounded and hence c_i exists and, for every $1 \leqq i \leqq r-1$, $t_i < c_i < t_{i+1}$.

Suppose that the first condition is not satisfied. Then $h_p^{G/A}(y + A) \geqq \omega$ for some $y \in A_G^{t_r}(p) \setminus A_{t_r}^G(p)$ or $h_p^{G/A}(y + A) < \omega$ for all $y \in A_G^{t_r}(p) \setminus A_{t_r}^G(p)$. In the first case, since

$$p^\omega(G/A)[p] \cap (G[p] + A)/A \subseteq ((p^\omega G + A)/A)[p]$$

by [**15**, Lemma 2.1], $x \in A_G^{t_r}(p) \setminus A_{t_r}^G(p)$ exists such that $x + A \in ((p^\omega G + A)/A)[p]$. Then $a \in A$ and $g \in p^\omega G$ exist such that $x = a + g$ and thus $h_p(a) = t_r$ and $h_p(pa) \geqq \omega$. In the second case

$$\sup\{h_p^{G/A}(y+A) \mid y \in A_G^{t_r}(p) \setminus A_{t_r}^G(p)\} = \omega$$

and $y_j \in A_G^{t_r}(p) \setminus A_{t_r}^G(p)$ exist for $j \geqq 1$ such that $h_p^{G/A}(y_j + A) < \omega$. For every $j \geqq 1$, let $d_j = h_p^{G/A}(y_j + A)$. By the comment after Definition 2.5, $t_r < d_1$. For every $j \geqq 1$, $b_j \in A$ and $h_j \in G$ exist such that $y_j = b_j + p^{d_j} h_j$. Note that $h_p(b_j) = t_r$ for all $j \geqq 1$. Since $r(A) = 1$, β_j, γ_j exist such that $(\beta_j, \gamma_j) = (\beta_j, p) = (\gamma_j, p) = 1$ and $\beta_j b_1 = \gamma_j b_j$. Since $\beta_j p b_1 = \gamma_j p b_j = -\gamma_j p^{c_j+1} h_j$ for all $j \geqq 1$, $\beta_j p b_1 \in p^\omega G$. Therefore, the assertion is clear. $\square$

Now we compute the height-matrix of the torsion-free element a of an abelian group G which satisfies the condition that, for every prime p and every integer $n \geqq 0$, either $h_p(p^n a) < \omega$ or $h_p(p^n a) = \infty$.

Corollary 5.2. *Let G be an abelian group and $a \in G \setminus T$. Suppose that, for every integer $n \geqq 0$, either $h_p(p^n a) < \omega$ or $h_p(p^n a) = \infty$. Let $m = h_p(a)$ and let $\{t_i\}$ be the p-overhang set of $\langle a \rangle$ in G. Define $c_i = \max\{h_p^{G/\langle a\rangle}(y + \langle a \rangle) \mid y \in \langle a\rangle_G^{t_i}(p) \setminus \langle a \rangle_{t_i}^G(p)\}$ if this exists. Then there are three possibilities.*

(1) $|\{t_i\}| = \aleph_0$. *Then*

$$h_p(p^n a) = \begin{cases} m+n & \text{for } 0 \leqq n \leqq e_1 - m, \\ m+n+\sum_{i=1}^k (c_i - t_i) & \text{for } e_k - m < n \leqq e_{k+1} - m,\ k \geqq 1, \end{cases}$$

and

$$e_k = \begin{cases} t_1 & \text{for } k = 1 \\ t_1 + \sum_{i=2}^k (t_i - c_{i-1}) & \text{for } k \geqq 2. \end{cases} \tag{5.3}$$

(2) $|\{t_i\}| = r$ *for some positive integer r. Then*

$$h_p(p^n a) = \begin{cases} m+n & \text{for } 0 \leqq n \leqq e_1 - m, \\ m+n+\sum_{i=1}^k (c_i - t_i) & \begin{cases} \text{for } e_k - m < n \leqq e_{k+1} - m \\ \text{and } 1 \leqq k \leqq r-1, \end{cases} \end{cases}$$

and, for $n > e_r - m$,

$$h_p(p^n a) = \begin{cases} m+n+\sum_{i=1}^r (c_i - t_i) & \text{if } h_p(p^n a) < \omega \text{ for all } n \geqq 1 \\ \infty & \text{if } h_p(p^s a) \geqq \omega \text{ for some integer} \\ & s \geqq 1 \end{cases}$$

where e_k is as in (5.3).

(3) $|\{t_i\}| = 0$. *Then*

$$h_p(p^n a) = m + n$$

for all $n \geqq 0$.

Proof. (1) In this case $\langle a \rangle$ is not eventually p-vertical in G. By Lemma 4.4 the assertion is clear.

(2) In this case $\langle a \rangle$ is not p-vertical in G, but eventually p-vertical in G. For $n \geqq e_r - m$, by Lemma 4.4 the claim holds. By Lemma 5.1 one of the following conditions is satisfied:

1. $\sup\{h^{G/\langle a\rangle}(y+\langle a\rangle) \mid y \in \langle a\rangle_G^{t_r}(p) \setminus \langle a\rangle_{t_r}^G(p)\} < \omega$;
2. $b \in \langle a\rangle$ exists such that $h_p(b) = t_r$ and $h_p(pb) \geqq \omega$.

If the first case holds, then c_r is defined and $h_p(p^{e_r-m+1}a) = c_r$. Since $V_{p,n}(G,\langle a\rangle) = 0$ for all $n > r$, by [**14**, Lemma 4.2], $\langle a\rangle \cap p^{t_r+1}G$ is p-vertical in $p^{t_r+1}G$. By [**14**, Theorem 2.8] and a routine induction,

$$h_p(p^n a) = m + n + \sum_{i=1}^{r}(c_i - t_i)$$

for $n > e_r - m$.

Suppose that the second case holds. By hypothesis, we have $h_p(pb) = \infty$. Note that $h_p(p^{e_r-m}a) = t_r$ by the first case of (2) and $h_p(b) = t_r$. Since $r(A) = 1$ we have $h_p(p^n a) = \infty$ for $n > e_r - m$.

(3) In this case $\langle a\rangle$ is p-vertical in G. If $m \geqq \omega$ then, by hypothesis, $= \infty$. If $m < \omega$, then, by [**14**, Theorem 2.8], the assertion is clear. □

Remark 5.4. By Corollary 4.13 and Corollary 4.14, the height-matrices of all torsion-free elements of an abelian group G whose torsion part T is torsion-complete can be computed by Corollary 5.2.

Finally we reconsider the group G in Example 3.1. By Property 3.2 we can write

$$G = \langle g_{pj} \mid p \in \mathbf{P}, i = 1, 2, \dots\rangle.$$

By Corollary 5.2, for every $p \in \mathbf{P}$, the p-indicator of g_{pi} is

$$(0, 1, 3, 5, \dots, 2n-1, \dots),$$

and hence

$$h_p(p^n g_{pi}) = \begin{cases} n & \text{for } 0 \leqq n \leqq 2i-1, \\ 2n-2i+1 & \text{for } n \geqq 2i. \end{cases}$$

Since the group G is of torsion-free rank-one, all the torsion-free elements of G are equivalent and, hence, for every prime p the p-heights of all the torsion-free elements of G are less than ω. By Corollary 4.13 every torsion-free subgroup of G is quasi-purifiable in G. However, T is not torsion-complete. Therefore, Example 3.1 shows that the converse of Corollary 4.14 is not true.

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