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TIGHT SUBGROUPS OF ALMOST COMPLETELY DECOMPOSABLE GROUPS

ULRICH DITTMANN

ABSTRACT. In this paper we show an extended version of the theorem of Bezout, give a new criterion for the tightness of a completely decomposable subgroup, derive some conditions under which a tight subgroup is regulating and generalize a theorem of Campagna. We give examples of almost completely decomposable groups, all of whose regulating subgroups do not have a quotient with minimal exponent.

1. Integers and rational groups. The following lemma was originally proven by K. Rogers (University of Hawaii).

Lemma 1.1. *Let a, b, c be nonzero integers with* $\gcd(a, b) = 1$. *Then there exists an integer i such that* $\gcd(a + ib, c) = 1$.

Proof. Factor $c = c_1c_2$ such that $\gcd(b, c_2) = 1$ and every prime factor of c_1 is also a factor of b. By the Chinese Remainder theorem the system

$$x \equiv a \pmod{b}, \quad x \equiv 1 \pmod{c_2}$$

has a solution x. Then $x = a + ib$ for some integer i and $\gcd(x, b) = \gcd(a, b) = 1$. Hence $\gcd(x, c_1) = 1$, too. $\square$

Lemma 1.2 (Extended Bezout theorem). *Let a, b, c be three arbitrary integers. Then there exist two integers r, s such that $ar + bs =$* $\gcd(a, b)$ *and* $\gcd(r, c) = 1$.

Proof. Define $g = \gcd(a, b)$, $\bar{a} = a/g$, $\bar{b} = b/g$. Then $\gcd(\bar{a}, \bar{b}) = 1$ and, by the Bezout theorem, there exist two integers r_0, s_0 such that $r_0\bar{a} + s_0\bar{b} = 1$. Note that $\gcd(r_0, \bar{b}) = 1$.

Define $r_i = r_0 + i\bar{b}$ and $s_i = s_0 - i\bar{a}$ for all integers i. Then $r_i\bar{a} + s_i\bar{b} = r_0\bar{a} + s_0\bar{b} = 1$ and $r_ia + s_ib = \gcd(a, b)$.

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By the previous lemma there exists an i such that $\gcd(r_0+i\bar{b}, c) = 1$, because $\gcd(r_0, \bar{b}) = 1$. With $r_i = r_0 + i\bar{b}$, the claim follows. $\square$

The extended Bezout theorem is applied in a different form in this paper: If a and m are two integers, then does there exist an integer b relatively prime to m such that $ab \equiv \gcd(a, m)$ modulo m? The extended Bezout theorem answers this question affirmatively.

The following lemma will be used mostly without reference.

Lemma 1.3. *Let* $\mathbf{Z} \subseteq S \subseteq \mathbf{Q}$ *be a rational group, and let* a, b *be two integers. Then*

a) $(a/b) \in S$ *implies* $(1/b) \in S$ *for relatively prime* a *and* b*, and*

b) $(1/a) \in S$ *and* $(1/b) \in S$ *implies* $1/(\mathrm{lcm}(a, b)) \in S$*, and*

c) *if* $(1/p) \notin S$ *for all primes* $p \mid a$*, then* $S = \langle aS, 1\rangle$ *and* $a^{-1}S = \langle S, a^{-1}\rangle$.

Proof. a) By the Bezout theorem, there exist integers r, s such that $ra + sb = 1$. Then $r(a/b) + s = (1/b)$. With $1 \in S$ the claim follows.

b) By Bezout's theorem, there exist integers r, s such that $ra + sb = \gcd(a, b)$. Then $r(1/b) + s(1/a) = (\gcd(a, b))/ab = 1/(\mathrm{lcm}(a, b))$, because $\gcd(a, b)\mathrm{lcm}(a, b) = ab$.

c) Let $x \in S$. Then $x = (y/z)$ with $y \in \mathbf{Z}$ and $z \in \mathbf{N}$ and $\gcd(y, z) = 1$. Note that $(1/z) \in S$ if and only if $x = (y/z) \in S$ because of part a). So it suffices to show that $(1/z) \in \langle aS, 1\rangle$. Note that $\gcd(a, z) = 1$ because otherwise there would exist a prime p with $(1/p) \in S$ and $p|a$, contradicting the assumption. By the Bezout theorem, there exist integers r, s such that $ra + sz = 1$. Then $r(a/z) + b = (1/z)$. As $(a/z) \in aS$ and $b \in \mathbf{Z}$ we have $(1/z) \in \langle aS, 1\rangle$ and $S = \langle aS, 1\rangle$. The second claim follows simply by division. $\square$

2. Tight criterion.

In [**2**, Proposition 2.7 (2)] Benabdallah, Mader and Ould-Beddi gave a criterion for tightness which required the verification that all rank-1 summands of the subgroup in question were pure. Here we give a different criterion which requires us to check the order of elements in a set derived from the type subgroups.

Lemma 2.1. *Let G be an almost completely decomposable group with completely decomposable subgroup W of finite index. Then W is not tight if and only if there exist a critical type τ and an element $g \in G(\tau) \setminus W_\tau \oplus G^{\#}(\tau)$ of prime order modulo W.*

Proof. "If." It suffices to show that $H := \langle W, g\rangle$ is completely decomposable. Split $g=(1/p)(a+b)$ where $a\in W_\tau$ and $b\in W^{\#}(\tau)$. Note that $\mathrm{ht}_p^W(a)=0$, because otherwise $g\in W_\tau \oplus G^{\#}(\tau)$. As $\mathrm{tp}\,(a)\le \mathrm{tp}\,(b)$ and $\mathrm{ht}_p^W(a) \le \mathrm{ht}_p^W(b)$, there exists a natural number k such that $\chi^W(a)\le\chi^W(kb)$ and $\gcd(k,p)=1$. Then by the Bezout theorem there exist two integers r and s such that $rp+sk=1$. Note that $H=\langle W, g-rb\rangle$ because $rb\in W$. Write $g-rb=(1/p)(a+(1-rp)b)=(1/p)(a+skb)$. As $\chi^W(a)\le\chi^W(skb)$, there exists a homomorphism $\varphi\in\mathrm{Hom}\,(W_\tau, W^{\#}(\tau))$ such that $\varphi(a)=skb$. Hence $p(g-rb)=a+skb=a(1+\varphi)\in W_\tau(1+\varphi)\subseteq W$. But then $W_\tau(1+\varphi)$ is not pure in H. As $|H/W|=p\in\mathbf{P}$, it is clear that $H=(W_\tau(1+\varphi))_*^H \oplus \bigoplus_{\sigma\ne\tau} W_\sigma$. Since $1+\varphi$ induces an isomorphism of W it is clear that $(W_\tau(1+\varphi))_*^H$ is completely decomposable and hence H, too.

"Only if." If W is not tight, then by [**2**, Proposition 2.7 (2)], there exists a rank-1 summand of W which is not pure. Assume then that $W = \bigoplus_j W_j$ where W_1 is not pure in G. Then there exists an element $g \in W_{1_*}^G \setminus W_1$ of prime order over W_1 and W. Set $\tau := \mathrm{tp}\,(g) = \mathrm{tp}\,(W_1)$ and note that $g \in G(\tau)$. Let $m_0 = \exp G/W$. Then $m_0 g \in W_1$ and $g \notin W_1$ imply $g \notin W_1 \oplus G^{\#}(\tau)$. Hence $g \in G(\tau) \setminus W_1 \oplus G^{\#}(\tau)$, as desired. $\square$

From the first half of the above proof we get the following corollary.

Corollary 2.2. *Let W be a completely decomposable group. Suppose that $g \in G(\tau)\setminus(W_\tau\oplus G^{\#}(\tau))$ and $|g+W| = p$ is prime for some critical type τ. Then $\langle W, g\rangle$ is completely decomposable.*

As another corollary we obtain [**2**, Lemma 4.5].

Corollary 2.3. *Let G be an almost completely decomposable group containing a tight subgroup W such that $p(G/W) = 0$ for some* *prime p. Then W is regulating in G.*

Proof. Assume by way of contradiction that W is not regulating. Then there exists a critical type τ such that $G(\tau) \neq W_\tau \oplus G^{\#}(\tau)$. Let $g \in G(\tau) \setminus W_\tau \oplus G^{\#}(\tau)$. Then $g \notin W$ and by $p(G/W) = 0$ we have that g has prime order over W. By the previous lemma W is not tight, a contradiction. $\square$

We extend the previous corollary to a more general case. The proof is shortened significantly by an idea of Otto Mutzbauer.

Corollary 2.4. *Let G be an almost completely decomposable group containing a tight subgroup W such that $k(G/W) = 0$ for some square free integer k. Then W is regulating in G.*

Proof. Assume by way of contradiction that W is not regulating. Then there exists a critical type τ such that $G(\tau) \neq W_\tau \oplus G^{\#}(\tau)$. Let $g \in G(\tau) \setminus W_\tau \oplus G^{\#}(\tau)$ and m be the order of g modulo W. Then m must be square free. Let $p_1, \dots, p_k$ be the prime divisors of m. Then $\gcd[(m/p_1), \dots, (m/p_k)] = 1$ as m is square free. Hence $m = \sum_{i=1}^{k} \mu_i(m/p_i)$ for suitable integers μ_i. Define $g_i := g(m/p_i)$ and note that g_i has order p_i modulo W. As W is tight, we know that $g_i \in W_\tau \oplus g^{\#}(\tau)$ for all i, since all g_i have prime order. But $g = (\sum_i \mu_i(m/p_i))g = \sum_i \mu_i g_i$ and hence $g \in W_\tau \oplus G^{\#}(\tau)$, contradicting our assumption. $\square$

The following corollary generalizes [**2**, Corollary 4.6].

Corollary 2.5. *Let G be an almost completely decomposable group containing a completely decomposable subgroup W such that $k(G/W) = 0$ for some square-free integer k. Then W is contained in a regulating subgroup V of G such that $k(G/V) = 0$.*

Proof. The completely, decomposable group W is contained in some tight subgroup V of G. Note that $k(G/W) = 0$ implies $kG \subseteq W \subseteq V$ and hence $k(G/V) = 0$. So V is regulating by the previous corollary. $\square$

3. Transitions to regulating subgroups. In [**4**, Theorem 2.5], Campagna has shown that the existence of a cyclic quotient implies the existence of a cyclic regulating quotient. We generalize this to the case of more than one generator and thus answer a question posed by Benabdallah, Mader and Ould-Beddi in [**2**, Question 4.1 (2)].

Theorem 3.1. *Let $G = \langle W, g_1, \dots, g_k\rangle$ be an almost completely decomposable group containing the completely decomposable group W of finite index. Then there exists a regulating subgroup V with $G = \langle V, g_1, \dots, g_k\rangle$.*

Proof. Let W not be regulating. We will show that there exists a completely decomposable subgroup W' with $G = \langle W', g_1, \dots, g_k\rangle$ and $|G/W| > |G/W'|$. The claim then follows by induction, as $|G/W|$ is finite.

We will first construct a completely decomposable subgroup W', then we show that $G = \langle W', g_1, \dots, g_k\rangle$. In the last section we show that W' has smaller index in G than W.

Let $W = \bigoplus_{\sigma\in T_{cr}} W_\sigma$ be a homogeneous decomposition of W. As W is not regulating, there exists a critical type τ such that $G(\tau) \neq W_\tau \oplus G^{\#}(\tau)$. Hence there exists a $g \in G(\tau) \setminus (W_\tau \oplus G^{\#}(\tau))$. Note also that $g + W_\tau \subseteq G(\tau) \neq W_\tau \oplus G^{\#}(\tau)$. So every element in $g + W_\tau$ witnesses that W is not regulating.

For later purposes we are now selecting a special element from $g+W_\tau$. Define $\overline{W} = \bigoplus_{\sigma\neq\tau} W_\sigma$. Then $W = W_\tau \oplus \overline{W}$. As $\{g_i + W\}_{i=1}^k$ are generators of G/W, we know that there exist integers $\lambda_1, \dots, \lambda_k$ such that $g + W = \sum_{i=1}^k \lambda_i g_i + W$. Then $g - \sum_{i=1}^k \lambda_i g_i = \sum_{\sigma\in T_{\mathrm{cr}}} v_\sigma \in W$ with $v_\sigma \in W_\sigma$. Our special element $h \in g + W_\tau$ is defined as follows: $h := g - v_\tau \in G(\tau) \setminus (W_\tau \oplus G^{\#}(\tau))$. As $g - (\sum_{i=1}^k \lambda_i g_i) - v_\tau \in \overline{W}$ we obtain

$$h \in \langle \overline{W}, g_1, \dots, g_k\rangle.$$

The particular property of h is that the τ-component of h is solely a linear combination of the τ-components of the generators $\{g_i\}_i$. No element of W_τ is needed for that. This is helpful because W_τ will be replaced later and we do not want the replacement to affect h.

Now we split h in two components, a and b, which we will use as building bricks for the construction of a new τ-homogeneous component of W. Define $\varphi = |h + W|$, and let $h = (1/\varphi)(\sum_{\sigma \in T_{\mathrm{cr}}} w_\sigma)$ with $w_\sigma \in W_\sigma$. As $h \in G(\tau)$ we know that $w_\sigma = 0$ for all $\sigma \not\geq \tau$. So we can set $a = w_\tau$, $b = \sum_{\sigma > \tau} w_\sigma$ and obtain $h = (1/\varphi)(a + b)$. As $\mathrm{tp}\,(a) \leq \mathrm{tp}\,(b)$ and as φ is finite, there exists an integer l such that

$$l^{-1}a \in W, \quad \chi^W(l^{-1}a) \leq \chi^W(b), \quad \mathrm{ht}_p^W(l^{-1}a) = 0 \quad \text{for all } p \mid \varphi.$$

Note that l is not a multiple of φ because then we would have $(1/\varphi)a \in W_\tau$ and $h = (1/\varphi)a + (1/\varphi)b \in W_\tau \oplus G^{\#}(\tau)$, which cannot be.

Define $a' := l^{-1}a$ and $R := \{r \in \mathbf{Q} \mid ra' \in W_\tau\}$. Then obviously $Ra' \subseteq W_\tau$ and $(1/p) \notin R$ for all $p \mid \varphi$, because $\mathrm{ht}_p^W(a') = 0$ for all $p \mid \varphi$. Note that Ra' is a pure subgroup of the homogeneous group W_τ, so by [**3**, 86.8] we know that $W_\tau = Ra' \oplus U$ for some complement $U \subset W_\tau$. By [**3**, 86.7] we know that U is completely decomposable.

Define $\varphi^* := \gcd\,(\varphi, l)$. As l is not a multiple of φ, we know that φ^* is a proper divisor of φ. Note that $(1/\varphi^*)a \in W_\tau$, as $\varphi^* \mid l$. Together with $(1/\varphi^*)(a + b) = (\varphi/\varphi^*)h \in G$ we get that $(1/\varphi^*)b \in G$. By the extended Bezout theorem (Lemma 1.2) there exist two integers r and s such that

$$\varphi^* = rl + s\varphi \quad \text{and} \quad \gcd\,(s, \varphi) = 1.$$

Define $w := r(1/\varphi)(a + b) + sa'$. Then $\varphi w = ra + rb + s\varphi a' = (rl + s\varphi)a' + rb = \varphi^* a' + rb$ and

$$w = \frac{1}{\varphi} rb + \frac{\varphi^*}{\varphi} a' = \frac{\varphi^*}{\varphi}\left(a' + r\frac{1}{\varphi^*} b\right).$$

The last term shows our intent. The summand Ra' of W is to be rotated $(a' \mapsto a' + r(1/\varphi^*)b)$ and shifted (φ^*/φ). We want to show that Ra' can be replaced by Rw to obtain a new completely decomposable subgroup of G.

We get $lw = lr(1/\varphi)(a + b) + lsa' = (\varphi^* - \varphi s)(1/\varphi)(a + b) + as = (\varphi^*/\varphi)(a + b) - s(a + b) + as = (\varphi^*/\varphi)(a + b) - sb$ and $(1/\varphi^*)w = (1/\varphi)(a + b) - s(1/\varphi^*)b$. So we can write

$$\frac{1}{\varphi}(a + b) = \frac{l}{\varphi^*} w + s\frac{1}{\varphi^*} b.$$

Note that for all primes $p \mid \varphi$ we have $(1/p) \notin R$ and hence $\mathrm{ht}_p^W(a') = 0 \leq \mathrm{ht}_p^G[(1/\varphi^*)b]$. This implies $\chi^W(a') \leq \chi^G[(1/\varphi^*)b]$ and hence

$R(1/\varphi^*)b \subseteq G$. We have $(\varphi/\varphi^*)wR = [a' + r(1/\varphi^*)b]R \subseteq Ra' + Rr(1/\varphi^*)b \subseteq G(\tau)$. Also we know that $w \in G(\tau)$. By Lemma 1.3 c) we find $Rw \subseteq G(\tau)$. Now we define

$$W' := Rw \oplus U \oplus \overline{W},$$
$$G' := \langle W', \frac{1}{\varphi^*}b, g_1, \dots, g_k\rangle,$$
$$H := \langle W', g_1, \dots, g_k\rangle.$$

Then $W' \subseteq G$ and hence $G', H \subseteq G$.

We want to show that $H = G$. This is done in two steps; first we show $G' = G$ and then $H = G'$.

Note that $Rb \subseteq W^{\#}(\tau) \subseteq W' \subseteq G'$. By definition we have $(1/\varphi^*)b \in G'$. Again by Lemma 1.3 c) we obtain $R(1/\varphi^*)b \subseteq G'$. We know that $Rw \subseteq W' \subseteq G'$ and that $(\varphi/\varphi^*) \in \mathbf{Z}$. Together we obtain $(\varphi/\varphi^*)Rw \subseteq G'$ and hence $R[a' + r(1/\varphi^*)b] \subseteq G'$. With $r(1/\varphi^*)b \subseteq G'$ we get $Ra' \subseteq G'$. But then $W = Ra' \oplus U \oplus \overline{W} \subseteq G'$ and thus $G' = G$.

Remember that we chose h such that $h \in \langle \overline{W}, g_1, \dots, g_k\rangle \subseteq H$. Hence $h = (1/\varphi)(a+b) = (l/\varphi^*)w + s(1/\varphi^*)b \in H$. Then $(l/\varphi^*)w \in \mathbf{Z}w \subseteq W' \subseteq H$ implies $s(1/\varphi^*)b \in H$. As $\gcd(s, \varphi) = 1$ we know that $\gcd(s, \varphi^*) = 1$ and, by Lemma 1.3 b), we get that $(1/\varphi^*)b \in H$. Hence $H = G' = G = \langle W', g_1, \dots, g_k\rangle$.

It now remains to show that the index of W' in G is smaller than the index of W. We will show this by defining a subgroup X that is contained in both W and W' and then calculating the index of X in W and W'.

Define $X = \varphi^* a' R \oplus U \oplus \overline{W}$. Obviously we have $X \subseteq W = Ra' \oplus U \oplus \overline{W}$. Note that $\varphi^* a' R = (\varphi w - rb)R$. Then $X = (\varphi w - rb)R \oplus U \oplus \overline{W}$. As $Rb \subseteq W^{\#}(\tau) \subseteq \overline{W}$ we have that $X = \varphi w R \oplus U \oplus \overline{W}$. Then obviously $X \subseteq W' = Rw \oplus U \oplus \overline{W}$. So $X \subseteq W \cap W'$. Now we calculate $|W : X|$ and $|W' : X|$. As $(1/p) \notin R$ for all $p \mid \varphi^*$ we have $\varphi^* = |R : \varphi^* R| = |W : X|$. Since $(1/p) \notin R$ for all $p \mid \varphi$ we have $\varphi = |R : \varphi R| = |W' : X|$. With $|G : W||W : X| = |G : W'||W' : X|$ we get $|G : W'| = (\varphi^*/\varphi)|G : W|$ and hence $|G : W'| < |G : W|$. $\square$

4. Examples. In [**2**] there was also the question of whether the existence of a tight subgroup with a quotient of a given exponent implies the existence of a regulating quotient with an equal or smaller exponent (Question 4.1 (1)). We answer the question with the next two examples.

Example 4.1. Let p, q, r, s, t, u be different primes. Let

$$W = \mathbf{Q}^{(q)}x_1 \oplus \mathbf{Q}^{(q,r)}x_2 \oplus \mathbf{Q}^{(q,s)}x_3 \oplus \mathbf{Q}^{(t)}x_4 \oplus \mathbf{Q}^{(u)}x_5,$$

$$g_1 = \frac{1}{p^3}(px_1 + x_2 + x_3 + p^2x_4), \quad g_2 = \frac{1}{p^3}(x_1 + x_5), \quad G = \langle W, g_1, g_2\rangle.$$

We claim that W is tight in G with $G/W \cong \mathbf{Z}_{p^3} \oplus \mathbf{Z}_{p^3}$ and that $G/V \cong \mathbf{Z}_{p^4} \oplus \mathbf{Z}_p$ for every regulating subgroup V of G. In particular, we have that g_1 and g_2 generate G modulo W, modulo any regulating subgroup and modulo the regulator.

Hence the regulating quotients do not have minimal exponent, although they have minimal order.

What makes this example remarkable is the fact that it shows that regulating subgroups may have the minimal property with respect to index, but not necessarily with respect to exponent. Intersecting all tight subgroups with minimal index (that is, regulating subgroups) yields the (index)-regulator. Similarly one could ask about the intersection of all tight subgroups with minimal exponent, which we call the *exponent-regulator*. Except for the obvious fact that the exponent-regulator is a characteristic subgroup, many properties are still open to research.

Proof. Let $V = \mathbf{Q}^{(q)}(1/p^2)(px_1 + x_2 + x_3) \oplus \mathbf{Q}^{(q,r)}x_2 \oplus \mathbf{Q}^{(q,s)}x_3 \oplus \mathbf{Q}^{(t)}x_4 \oplus \mathbf{Q}^{(u)}x_5$. We first verify that V is contained in G. Note that $\mathbf{Q}^{(q)}(px_1 + x_2 + x_3) \subseteq W \subseteq G$ and $\mathrm{ht}_p^G[(1/p^2)(px_1 + x_2 + x_3)] = 0$ implies $\mathbf{Q}^{(q)}(1/p^2)(px_1 + x_2 + x_3) \subseteq G$ by Lemma 1.3 c) and hence $V \subseteq G$.

By the same lemma we conclude that $\mathbf{Q}^{(q)}px_1 \subseteq V \subseteq \langle V, x_1\rangle$ and $x_1 \subseteq \langle V, x_1\rangle$ imply $\mathbf{Q}^{(q)}x_1 \in \langle V, x_1\rangle$. Hence $W \subseteq \langle V, x_1\rangle$ and $G = \langle V, g_1, g_2, x_1\rangle$. It is straightforward to verify that g_1 and g_2 are linearly independent modulo V and that $p^3g_2 \equiv x_1$ modulo V. So

$$G = \langle V, g_1, g_2\rangle \quad \text{and} \quad G/V = \langle g_1 + V\rangle \oplus \langle g_2 + V\rangle.$$

To rewrite the first component of V, set $x_1' = (1/p^2)(px_1 + x_2 + x_3)$. Then $x_1 = (1/p)(p^2x_1' - x_2 - x_3)$ and $g_1 = (1/p)(x_1' + x_4)$ and

$g_2 = (1/p^4)(p^2x_1' - x_2 - x_3 + px_5)$ and $V = \mathbf{Q}^{(q)}x_1' \oplus \mathbf{Q}^{(q,r)}x_2 \oplus \mathbf{Q}^{(q,s)}x_3 \oplus \mathbf{Q}^{(t)}x_4 \oplus \mathbf{Q}^{(u)}x_5$. Obviously $\exp(G/V) = p^4$ and $|G/V| = p^5$.

It is routine to verify that V is regulating. So the regulating index of G is p^5.

By [**1**, Proposition 4.1.10], we know that all other regulating subgroups are of the form $\bigoplus_\tau V_\tau(1+\phi_\tau)$ where $\phi_\tau \in \operatorname{Hom}(V_\tau, G^{\#}(\tau))$. As $G^{\#}(\tau) = 0$ for all $\tau \neq \tau_1$ and as $G^{\#}(\tau_1) = \langle V^{\#}(\tau_1), (1/p)(x_2 + x_3)\rangle$, we find that $V_k = \mathbf{Q}^{(q)}(x_1' + k(1/p)(x_2 + x_3)) \oplus \mathbf{Q}^{(q,r)}x_2 \oplus \mathbf{Q}^{(q,s)}x_3 \oplus \mathbf{Q}^{(t)}x_4 \oplus \mathbf{Q}^{(u)}x_5$ with $k = 0, \dots, p-1$ are all regulating subgroups. Intersecting any two of them yields the regulator

$$R(G) = \mathbf{Q}^{(q)}px_1' \oplus \mathbf{Q}^{(q,r)}x_2 \oplus \mathbf{Q}^{(q,s)}x_3 \oplus \mathbf{Q}^{(t)}x_4 \oplus \mathbf{Q}^{(u)}x_5$$

where $V = \langle R(G), x_1'\rangle$ by Lemma 1.3 c). Hence $G = \langle R(G), g_1, g_2, x_1'\rangle$. As $x_1' \equiv pg_1$ modulo $R(G)$ and $V_k \supseteq R(G)$ we get for all k that $G = \langle V_k, g_1, g_2\rangle$. Note that $|g_2 + V_k| = p^4$ and that G/V_k is not cyclic. So $\exp G/V_k = p^4$ and $G/V_k \cong \mathbf{Z}_{p^4} \oplus \mathbf{Z}_p$.

It remains to show that W is tight. One can verify that g_1 and g_2 are linearly independent modulo W. Hence G/W has order p^6 and exponent p^3. Assume, by way of contradiction, that there was a completely decomposable group $U \supset W$. Then $\exp G/U$ is a divisor of $\exp G/W = p^3$ and $|G/U|$ is a proper divisor of $|G/W| = p^6$. Hence $|G/U| = p^5$ and U is regulating. But then $\exp G/U = p^4$, a contradiction. $\square$

The ad hoc argument in the last paragraph is due to Adolf Mader and has saved us a hard proof via Lemma 2.1.

Next we give a more general example. As the argument resembles the previous examples' proof, we have omitted many of the details.

Example 4.2. Let $G = \langle W, g\rangle$ with $W = \mathbf{Q}^{(q)}a \oplus \mathbf{Q}^{(q,r)}b \oplus \mathbf{Q}^{(q,s)}c \oplus \mathbf{Q}^{(t)}d$ and $g = (1/p^k)(p^2a + b + c + pd)$. Then W contains a tight subgroup of exponent p^{k-1} while all regulating quotients are cyclic of order p^k.

Proof. Set $Y = \mathbf{Q}^{(q,r)}b \oplus \mathbf{Q}^{(q,s)}c \oplus \mathbf{Q}^{(t)}d$. Then $W = \mathbf{Q}^{(p)}a \oplus Y$. Note that W is regulating as $G^{\#}(\tau_a) = \langle W^{\#}(\tau_a), (1/p)(b + c)\rangle$ and

$G(\tau_a) = \langle W(\tau_a), (1/p)(b+c)\rangle$. There are exactly p regulating subgroups

$$V_i = \mathbf{Q}^{(p)}\left(a + i\frac{1}{p}(b+c)\right) \oplus Y$$

for $i = 0, \dots, p-1$. Note that

$$g = \frac{1}{p^k}\Big(p^2\Big[a + \frac{i}{p}(b+c)\Big] + (1-pi)(b+c) + pd\Big)$$

and that $p \nmid (1-pi)$. Hence $|g + V_i| = p^k$ and $G/V_i \cong \mathbf{Z}_{p^k}$ for all i. Set $U = \mathbf{Q}^{(q)}p^2a \oplus Y = \mathbf{Q}^{(q)}(p^2a + b + c) \oplus Y$ and $X = \mathbf{Q}^{(q)}(pa + (1/p)(b+c)) \oplus Y$. As $pa + (1/p)(b+c) \in G$ and $U \subseteq G$, we also get $X \subset G$. Note that $W \subseteq \langle U, a\rangle \subseteq \langle X, a\rangle$ and hence $G = \langle X, g, a\rangle$. We have

$$g = \frac{1}{p^k}\Big(p\Big[pa + \frac{1}{p}(b+c)\Big] + pd\Big) = \frac{1}{p^{k-1}}\Big(\Big[pa + \frac{1}{p}(b+c)\Big] + d\Big)$$

and thus $|g + X| = p^{k-1}$. Also we get $a = (1/p)[pa + (1/p)(b+c)] - (1/p^2)(b+c)$ and hence $|a + X| = p^2$. Hence $G/X \cong \mathbf{Z}_{p^{k-1}} \oplus \mathbf{Z}_{p^2}$. The fact that X is tight comes with the same ad hoc argument as in the previous example. $\square$

REFERENCES

1. Adolf Mader, *Almost completely decomposable abelian groups*, Algebra Logic Appl. **13**, Gordon and Breach Sci. Publ., Amsterdam, 2000.

2. Benabdallah, Mader and Ould-Beddi, *Tight subgroups in almost completely decomposable groups*, J. Algebra **225** (2000), 501–516.

3. Laszlo Fuchs, *Infinite abelian groups*, vols. I and II, Academic Press, New York, 1970, 1973.

4. Matthew Campagna, *Single relation almost completely decomposable groups*, Comm. Algebra **28** (2000), 83–92.

Mathematisches Institut, Universität Würzburg, Am Hubland, 97074 Würzburg, Germany
E-mail address: Ulrich.Dittmann@Allianz.de