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MEASURING THE CLASSIFICATION DIFFICULTY OF COUNTABLE TORSION-FREE ABELIAN GROUPS

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1. The problem.

Question. Can we hope to classify countable torsion-free abelian groups?

Already a few remarks should be made about this question.

First of all the word "classify" is somewhat plastic in its meaning. Someone might for instance take the question to mean whether there is any sense at all in which we can understand countable torsion-free abelian groups, and I am sure "classification" takes on different hues across different guilds and mathematical specialties.

I will take the word "classify" to mean "completely classify by some class of invariants." Here I have in mind something like the Ulm invariants for countable abelian p-groups or Baer's classification for the rank one case.

Secondly one might wonder about the restriction to this particular class of groups. Here I would respond by saying that we cannot hope to classify everything, and some restrictions probably are inevitable. Abelian groups represent the topic of this conference and should be easier and more hopeful than general groups; and the choice of torsion-free further removes potentially distracting details. As for confining ourselves to the countable case, cardinality $\aleph_0$, I would mention the kinds of set theoretical complexities which can arise when one considers uncountable discrete structures. Frequently one is led into independent results and, considering subtle combinatorial properties such as the behavior of the nonstationary ideal, and even classification schemes which would be virtually perfect in the countable case, such as Ulm invariants, may begin to fail when we pass to $\aleph_1$.

Even granting these restrictions, we may want to take a skeptical stance. After all, if a classification scheme was going to be found, then

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surely it would have made itself known already.

So perhaps it would be better to ask:

Question. What would establish that there is no way to completely classify countable torsion-free abelian groups?

Some possible answers:

Answer 1 (An appeal to empirical evidence). No classification scheme has been forthcoming. We have waited long enough. It is safe to assume that nothing here is possible.

For some people this will already be enough. And, if this is your position, then the remainder of the talk is unlikely to hold much interest.

Personally I am inclined to at least look for a deeper explanation of this empirical fact, so let me push on:

Answer 2. We may be able to reduce some other truly horrendous classification problem to that of countable torsion-free abelian groups. For instance, perhaps there is a way we can show them to be as hard to classify as general countable groups.

Still this is a little bit question-begging since we might then want some confirmation of the intuition that general countable groups are unclassifiable.

Answer 3. Perhaps we can develop an abstract theory of invariants and show that there is no *reasonable* way to assign certain classes of objects such as the Ulm invariants or Baer's invariants for rank one, as complete invariants.

In fact it turns out, following work of Friedman, Kechris, Louveau, Stanley and others, that such a theory has been developed. In this talk I want to discuss how that theory bears on the classification problem for countable torsion-free abelian groups.

The meaning of *reasonable* is subject to some negotiation. I will begin by considering the explication which takes a *reasonable reduction* to be one that is Borel in some appropriate Borel structure.

Other explications are possible. For instance, absolutely $\underset{\sim}{\Sigma}_1^{HC}$, as discussed several sections below. Or using reductions in $L(\mathbf{R})$. And there are various other exotic classes that logicians find natural to consider. The Borel category has the advantage of being one which is widely used mathematically and does indeed include most commonly accepted classification schemes.

However the reader would not go too far wrong to simply think of a *reasonable classification* as being one which does not make an egregious appeal to the existence of a well ordering of $\mathbf{R}$ in assigning its invariants.

2. Spaces of abelian groups.

Definition. Let AbGrp $= \{H = (+^H, -(\cdot)^H) \in \mathbf{N}^{\mathbf{N}\times\mathbf{N}} \times \mathbf{N}^{\mathbf{N}} | H$ defines an abelian group structure on $\mathbf{N}\}$.

In the discrete topology, $\mathbf{N}$ is a separable completely metrizable space. Thus $\mathbf{N}^{\mathbf{N}\times\mathbf{N}}$ and $\mathbf{N}^{\mathbf{N}}$ are separable completely metrizable spaces in the product spaces, as is $\mathbf{N}^{\mathbf{N}\times\mathbf{N}} \times \mathbf{N}^{\mathbf{N}}$. We call this kind of space a *Polish space.*

AbGrp is a closed subset of a Polish space, and hence again Polish in the subspace topology.

Definition. TFA $= \{H \in$ AbGrp $: H$ is torsion free$\}$ and $\mathrm{TFA}_n = \{H \in \mathrm{TFA} : H$ has rank $\leq n\}$.

Again TFA is a closed subset of AbGrp, and hence again Polish. TFA_n is not closed, but instead G_δ, that is to say, defined by a countable intersection of open sets, and hence Polish. (For a proof of the general fact that a G_δ subset of a Polish space is again Polish one can see [**9**]. This is also a good reference for other general facts about Polish spaces and Borel sets.)

There are other ways in which we can model these objects. For instance, in Simon Thomas's papers, such as [**14**], [**15**], he takes the space of subgroups of $\mathbf{Q}^n$ to provide a Borel structure on the torsion-free abelian groups of rank $\leq n$. It turns out that, from the point of view of the kinds of questions we will be considering, such choices are immaterial. All the known ways of providing a Borel structure give the same results.

Definition. For X, Y Polish, a function $f : X \to Y$ is Borel if for any open set O we have $f^{-1}[O]$ is Borel, that is to say, in the σ-algebra generated by the open sets.

3. A first approximation: smoothness.

Definition. An equivalence relation E on a Polish space X is *smooth* [**11**] or *tame* or *concretely classifiable* (Kechris) if there is a Borel function

$$f : X \longrightarrow Y,$$

for some Polish space Y, such that for all $x_1, x_2 \in X$,

$$x_1 E x_2 \quad \text{if and only if } f(x_1) = f(x_2).$$

For example. If we take $Y = \mathbf{R}$, then this would correspond to assigning real numbers as complete invariants.

Alas. Almost no real life equivalence relations are smooth.[1]

For example. $\cong |_{\mathrm{TFA}_1}$ (isomorphism of rank 1 torsion-free abelian groups) is *not* smooth.[2]

4. A better approximation: Borel reducibility.

Definition. For E, F equivalence relations on Polish X, Y, we write

$$E \leq_B F$$

if there is a Borel function $f : X \to Y$ such that for all $x_1, x_2 \in X$,

$$x_1 E x_2 \iff f(x_1) F f(x_2).$$

In other words, for any $x \in X$, $[f(x)]_F$ (the F-equivalence class of $f(x)$) is a complete invariant for $[x]_E$.

We can then write $E <_B F$ if $E \leq_B F$ holds by $F \leq_B E$ fails.

For example. Consider $\{0,1\}^{\mathbf{N}} =_{df} 2^{\mathbf{N}}$, the space of infinite binary sequences in the product topology. For $\vec{x} = (x_0, x_1, \dots), \vec{y} = (y_0, y_1, \dots)$, set

$$\vec{x} E_0 \vec{y}$$

if there exists

$$N \quad \text{for all } n > N (x_n = y_n).$$

So this is the equivalence relation of eventual agreement and, under the natural identification of $2^{\mathbf{N}}$ with $\mathcal{P}(\mathbf{N})$ (the power set of $\mathbf{N}$), one has

$$2^{\mathbf{N}}/E_0 \sim \mathcal{P}(\mathbf{N})/\text{Finite}.$$

Sets considered up to finite difference are not totally unreasonable objects to try to assign as complete invariants and, indeed, there is the following classical theorem:

Theorem. (in effect, Baer from [**1**]). $\cong|_{\mathrm{TFA}_1} \leq_B E_0$.

Indeed, this is precise. One can show $E_0 \leq_B \cong|_{\mathrm{TFA}_1}$. And indeed it was shown by Harrington, Kechris and Louveau [**4**] that E_0 corresponds to the *next* level of classification difficulty after smoothness.

For a few years it was open whether Baer's result can be extended to rank 2. This was ultimately shown to be false.

Theorem (Hjorth [**7**]). $\cong|_{\mathrm{TFA}_2} \not\leq E_0$.

Here I should mention as an aside that Simon Thomas has recently obtained a much stronger result:

Theorem (Thomas, [**15**]). *At every n,*

$$\cong|_{\mathrm{TFA}_n} <_B \cong|_{\mathrm{TFA}_{n+1}}.$$

I would have been inclined to consider this the final word on the abstract question of the classification difficulty of the finite rank TFA groups, but after the talk someone pointed out a further issue which is unresolved. We do not know whether $\cong|_{\mathrm{TFA}_2}$ lies *directly after* E_0 in this hierarchy of classification difficulties, that is to say, if $E <_G \cong|_{\mathrm{TFA}_2}$, must it be the case that $E \leq_B E_0$?

Returning to the subject of general torsion-free abelian groups, we already obtain that the rank two torsion-free abelian are strictly more complicated than the rank ones. Thus we would have to look well beyond E_0 in order to find complete invariants for $\cong|_{\mathrm{TFA}}$, the isomorphism relation in the *infinite* dimensional case.

Definition. For $x, y \in 2^{n\times n}$, set ${}_xF_2y$, if

$$\{x(n,\cdot) \mid n \in \mathbf{N}\} = \{y(n,\cdot) \mid n \in \mathbf{N}\};$$

that is to say, for all n_1 there exists n_2, n_3 for all m,

$$\begin{aligned} x(n_1, m) &= y(n_2, m), \\ x(n_3, m) &= y(n_1, m). \end{aligned}$$

Thus F_2-equivalence classes correspond to something like countable sets of reals.[3]

Fact. At each n, $\quad \cong|_{\mathrm{TFA}_n} \leq_B F_2$.

This is proved as so: For each H a rank n torsion-free abelian group, and $\vec{g} = g_1, \ldots, g_n \in H$, we let $\theta_0(H, \vec{g}) = \{(k_1, \ldots, k_n, l) \in \mathbf{Z}^{n+1}\colon l$ divides $k_1 \cdot g_1 + k_2 \cdot g_2 + \cdots + k_n \cdot g_n\}$. Then

$$\theta(H) = \{\theta_0(H, \vec{g}) : \vec{g} \in H^n\}$$

gives us an element of $\mathcal{P}_{\aleph_0}(\mathbf{N}^{n+1})$ as a complete invariant. It is not a big step to turn these invariants into elements of $\mathcal{P}_{\aleph_0}(\mathbf{N})$, and from there to pass to a Borel reduction to the equivalence relation F_2.

The general rule of thumb is that any class of countable structures which are in some sense *finitely generated* or in some sense have *finite rank* are reducible to F_2 by a Borel function. (For a discussion of results in this direction, as well as general schemes for classifying countable structures, see [**8**].)

So we might at least hope that something like *countable sets of reals* can stand as complete invariants for $\cong|_{\text{TFA}}$.

Definition. For $x, y \in 2^{\mathbf{N}^{n+1}}$, set

$$xF_{n+1}y$$

if

$$\{[x(m,\cdot,\dots)]_{F_n} \mid m \in \mathbf{N}\} = \{[y(m,\cdot,\dots)]_{F_n} \mid m \in \mathbf{N}\}.$$

In other words, F_2-equivalent classes correspond to something like elements of $\mathcal{P}_{\aleph_0}(\mathbf{N})$, F_3-equivalence classes correspond to elements of $\mathcal{P}_{\aleph_0}(\mathcal{P}_{\aleph_0}(\mathbf{N}))$ and so on.

In fact, as given by Friedman and Stanley [**3**], one can iterate this definition out through the countable ordinals and define F_α for each $\alpha < \omega_1$. They also observe:

Theorem (Friedman-Stanley). *For each n,*

$$F_n <_B F_{n+1}.$$

At this point we can finally give the chief negative result regarding the isomorphism relation on countable torsion-free abelian groups.

Theorem (Hjorth, [**5**]).

$$F_n \leq_B \cong|_{\text{TFA}}.$$

Combining this with Friedman-Stanley:

Corollary. *At every n,*

$$\cong|_{\mathrm{TFA}} \not\leq_B F_n.$$

(And, as might be expected, this goes out through the ordinals. At every countable α, $F_\alpha <_B \cong|_{\mathrm{TFA}}$.)

5. What we would ultimately want to prove.

Definition. For $\mathcal{L}$ a countable language, with relations $R_1, R_2, \ldots$, having arities $a(1), a(2), \ldots$, and function symbols $F_1, F_2, \ldots$ having arities $b(1), b(2), \ldots$, we let $\mathrm{Mod}\,(\mathcal{L})$ be the space

$$\prod_i 2^{\mathbf{N}^{\alpha(i)}} \times \prod_j \mathbf{N}^{\mathbf{N}^{b(j)}}.$$

We can define the isomorphism relation, $\cong|_{\mathrm{Mod}(\mathcal{L})}$, on this space in the obvious way.

We then say an isomorphism invariant $K \subset \mathrm{Mod}\,(\mathcal{L})$ is said to be *universal* if, given any other countable $\mathcal{L}'$, we have

$$\cong|_{\mathrm{Mod}(\mathcal{L}')} \leq_B \cong|_K.$$

Example 1. Graphs on $\mathbf{N}$ can be viewed as a subset of $2^{\mathbf{N}^2}$. It is a folklore result that the isomorphism relation on this class of countable structures is universal.

Example 2 (Friedman-Stanley, [**3**]). Countable linear orderings are universal in this sense.

Example 3 (Folklore?). Isomorphism on countable groups is universal.

Example 4 (Camerlo-Gao [**2**]). Countable Boolean algebras are universal.

So,

Question. Are countable torsion-free abelian groups universal?

Here I have to admit, with some great shame and hanging of the head, that I had previously announced in print, at [**7**], a positive solution to this problem. The proof was flawed, though it turns out that the result above regarding the F_αs is coming close, since they play a special role in the general investigation of isomorphism of countable structures.

Theorem (Dana Scott, see [**13**], [**16**], [**10**]). *For each countable language $\mathcal{L}$ there are Borel sets $(A_\alpha)_{\alpha\in\aleph_1}$ such that*

(a) *the space of $\mathcal{L}$-structures with underlying set* $\mathbf{N}$ *equals*

$$\bigcup_{\alpha\in\aleph_1} A_\alpha;$$

(b) *at each α,*

$$\cong|_{\cup_{\beta\leq\alpha}A_\alpha} \leq_B F_\alpha.$$

There are various consequences of his result which can probably be considered folklore. For instance, a Borel set of countable structures has a Borel isomorphism relation if and only if it is Borel-reducible to some F_α, if and only if it is included in some $\cup_{\beta<\alpha}A_\beta$, some countable α.

Question (Friedman-Stanley). Let $\mathcal{L}$ be a countable language and let $K \subset \mathrm{Mod}(\mathcal{L})$ be an isomorphism invariant Borel subset. Suppose at each $\alpha < \aleph_1$ we have

$$F_\alpha \leq_B \cong|_K.$$

Must $\cong|_K$ be universal?

Therefore, at the very least, we can say that either countable torsion-free abelian groups are universal, or their failure to be represents an entirely new phenomenon.

6. Details.

Definition. A function F is *absolutely* $\underset{\sim}{\Sigma}_1^{HC}$ if

(a) there is $x \in 2^{\mathbf{N}}$ and ψ in the language of set theory such that

$$F(a) = b$$

if and only if there is a countable transitive structure $\mathcal{M}$ containing a, b, x and satisfying

$$\mathcal{M} | = \psi(a, b, x);$$

(b) (this part is more technical) the formulation of (a) continues to define a total function through all generic extension.

Following Gödel's work we know that (a) alone is not sufficient to guarantee a function is *nicely behaved.* For instance, a function satisfying (a) alone from $\mathbf{R}$ to $\mathbf{R}$ may fail to be Lebesgue-measurable.

It is a kind of folklore result that if we add (b) in addition, then we do indeed obtain all the nice properties we could hope for, such as being universally measurable.[4] For various purposes these kinds of functions actually give a kind of better fit to the notion of *reasonable reduction* or *reasonable schema of classification.*

Example 1. For p a prime, $TA_p = \{H \in \text{AbGrp} \mid H \text{ is a } p\text{-group}\}$, there is an absolutely $\underset{\sim}{\Sigma}_1^{HC}$

$$U : TA_p \longrightarrow 2^{<\omega_1}$$

such that

$$H_1 \cong H_2 \Longleftrightarrow U(H_1) \cong U(H_2).$$

This comes out of the Ulm classification of p-groups.

Example 2. For any countable language $\mathcal{L}$, there is an absolutely $\underset{\sim}{\Sigma}_1^{HC}$

$$S : \text{Mod}\,(\mathcal{L}) \longrightarrow HC \quad \text{(the hereditarily countable sets)}$$

such that

$$\mathcal{M}_1 \cong \mathcal{M}_2 \Longleftrightarrow S(\mathcal{M}_1) = S(\mathcal{M}_2).$$

(D. Scott) Moreover we can write

$$HC = \bigcup_{\alpha < \aleph_1} V_\alpha \cap HC$$

and think of $V_\alpha \cap HC$ as being a subset of the F_α-equivalence classes.

Here I would be more optimistic about a limited conjecture:

Conjecture. $\cong |_{\mathrm{TFA}}$ *is universal with respect to absolutely* $\underset{\sim}{\Sigma}_1^{\mathrm{HC}}$ *functions. That is to say, for any countable language* $\mathcal{L}$*, we may reduce* $\cong |_{\mathrm{Mod}\,(\mathcal{L})}$ *to* $\cong |_{\mathrm{TFA}}$ *by the use of an absolutely* $\underset{\sim}{\Sigma}_1^{\mathrm{HC}}$.

7. Something about the proofs. The argument that $F_2 \leq B \cong |_{\mathrm{TFA}}$ at least is very simple. Indeed somewhat misleadingly so. F_2-equivalence classes can be coded up in countable structures just using unary predicates, and the model theory without relations or functions is extremely simple. For F_3 and beyond relations are necessary, and the proof of $F_3 \leq_B \cong |_{TFA}$ is more involved than the sketch below.

I will also further simplify this sketch by skipping over any argument that the reduction is Borel.

Let $(q_n)_{n \in \mathbf{N}}, (p_n)_{n \in \mathbf{N}}$ be sequences of distinct primes. For $x \in 2^{n \times \mathbf{N}}$ for which we can assume $(x(n, \cdot))n \in \mathbf{N}$ is one-to-one, we define an abelian group $\mathcal{G}_x$ as follows: At each $l \in \mathbf{N}$, we set

$$g_{x,l} \in \mathcal{G}_x$$

so that $g_{x,l}$ is divisible by all powers of p_n if $x(l, n) = 1$ and divisible by all powers of q_n if $x(l, n) = 0$. We then let $\mathcal{G}_x$ be the abelian group generated by these $\{g_{x,l} : l \in \mathbf{N}\}$ and all the divisors we have just insisted on.

The isomorphism type of $\mathcal{G}_x$ encodes $[x]_{F_2} \sim \{x(l, \cdot) : l \in \mathbf{N}\}$. We can reconstruct the latter from the former.

Here goes.

Say that $g \in \mathcal{G}_x$ is *good* if for all n either g is divisible by all powers of p_n, or it is divisible by all powers of q_n. Then, for $g \in \mathcal{G}_x$ good, we let

$$d(g) \in 2^{\mathbf{N}}$$

be defined by

$$(d(g))(n) = 1$$

if and only if g is divisible by all powers of p_n.

And then one indeed obtains

$$\{x(l, \cdot) \mid l \in \mathbf{N}\} = \{d(g) \mid g \in \mathcal{G}_x \text{ good}\}.$$

The other details are routine, and thus it is shown that

$$F_2 \leq_B \cong|_{\mathrm{TFA}}.$$

Here we can get some insight by recalling a general fact (see [**3**] for a proof of this and other basic facts in this region):

Fact. There is no absolutely $\underset{\sim}{\Sigma}{}_1^{HC}$ function

$$\theta : 2^{n \times \mathbf{N}} \longrightarrow 2^{<\omega_1}$$

such that

$$xF_2y \Longrightarrow \theta(x) = \theta(y).$$

In particular, this gives a result first obtained by Garvin Melles using very different means:

Corollary (Melles, [**12**]). *We cannot classify countable torsion-free abelian groups by elements of $2^{<\omega_1}$ using absolutely $\underset{\sim}{\Sigma}{}_1^{HC}$ functions.*

ENDNOTES

1. One of the few exceptions to this lament, and something very much in the mind of George Mackey, is given by the irreducible unitary representations of countable finite by abelian groups considered up to isomorphism. They are smooth.

2. An elementary proof of this fact can be found in the exercises at the end of Section 3.1 of [**6**].

A skeptical reader might be concerned that the inability to assign reals or points in some other concrete space as complete invariants for rank 1 TFA groups is purely an artifact of our decision to work in the Borel category. This would be a very reasonable concern.

It turns out not to be the case. The same obstruction reappears even considering a much more generous class of functions, but a full discussion of this would require an excursion into foundational issues.

3. Here and beyond I am somewhat lazily assuming that the sequence $(x(n,\cdot))_{n\in\mathbf{N}}$ has no repetitions, and thus xF_2y if and only if $\{x(n,\cdot) \mid n\in\mathbf{N}\}=\{y(n,\cdot) \mid n\in\mathbf{N}\}$; in fact, it can be argued that we lose no generality if we restrict our attention to x for which the sequence $n \mapsto x(n,\cdot)$ is one-to-one.

4. See, for instance, [**6**, Section 9.1].

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