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COMPARISON THEOREMS AND STRONG OSCILLATION IN THE HALF-LINEAR DISCRETE OSCILLATION THEORY

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ABSTRACT. Consider the second order half-linear difference equation

$$\triangle(r_k|\triangle y_k|^{\alpha-1}\operatorname{sgn}\triangle y_k) + p_k|y_{k+1}|^{\alpha-1}\operatorname{sgn} y_{k+1} = 0, \quad \alpha > 1. \tag{HL}$$

In the first part we give various types of comparison theorems for this equation, including the so-called telescoping principle, and also for the associated generalized Riccati difference equation. In the second part, we present criteria for strong (non)-oscillation of (HL) and related results. The paper is finished by an example where oscillatory properties of a generalized discrete Euler equation are investigated.

1. Introduction. This paper is a further demonstration of the fact that one can extend (in a sense of the "half-linear generalization" and of the discretization) the most results of the oscillation theory of the Sturm-Liouville linear differential equation

$$(r(t)y')' + p(t)y = 0$$

to the half-linear difference equation

$$\triangle(r_k\Phi(\triangle y_k)) + p_k\Phi(y_{k+1}) = 0, \tag{1}$$

where r_k, p_k are real-valued sequences defined on $\mathbf{N}$ with $r_k \neq 0$ and $\Phi(y) := |y|^{\alpha-1}\operatorname{sgn} y = |y|^{\alpha-2}y$ with $\alpha > 1$. Since the Sturm type

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separation theorem extends to (1), this equation can be classified as oscillatory or nonoscillatory. The Sturm comparison theorem can be extended to (1) as well. Recall that those results follow from the half-linear discrete version of the so-called *Roundabout theorem*, see [**12**], which also provides tools for the investigation of oscillatory properties of (1), namely, the Riccati and the variational techniques that we will use later.

The aim of this contribution is, among others, to present comparison theorems for equation (1) of other types than that classical of the Sturm type. The paper is organized as follows. In the next section, we recall basic concepts of the oscillation theory of (1) established in [**12**]. The main results can be itemized in the following way. In Section 3, we will examine a preservation of (non)oscillation of equation (1) if the sequence p_k in (1) is multiplied by a certain real number. In Section 4, we will see that under certain assumptions, the inequality $p_k \le P_k$ in the classical Sturm type comparison result, see Proposition 2 below, can be replaced by the weaker condition $\sum_{j=k}^{\infty} p_j \le \sum_{j=k}^{\infty} P_j$ and, moreover, a suitable change of the number α also preserves a desired property of equation (1). A generalization of the so-called telescoping principle introduced in [**6**] for the linear difference equation

$$\triangle(r_k \triangle y_k) + p_k y_{k+1} = 0 \tag{2}$$

to the half-linear equation (1) is contained in Section 5. Note that equation (2) is the special case of equation (1) for $\alpha = 2$. A comparison theorem for the generalized Riccati difference equation that is associated to equation (1) is proved in Section 6. Section 7 contains results concerning a strong (non)oscillation and a conditional oscillation of equation (1). The paper is concluded by Section 8, where some oscillatory properties of a generalized discrete Euler equation are investigated.

2. Basic concepts of oscillation theory of equation (1). In this section, we give the basic concepts of oscillation theory of equation (1) that were established in [**12**].

First we recall some definitions.

Definition 1. An interval $(m, m+1]$ is said to contain the *generalized zero* of a solution y of (1) if $y_m \neq 0$ and $r_m y_m y_{m+1} \le 0$. Equation (1) is said to be *disconjugate* on the discrete interval $[m, n]$ provided any solution of this equation has at most one generalized zero on $(m, n+1]$ and the solution $\tilde{y}$ satisfying $\tilde{y}_m = 0$ has no generalized zeros on $(m, n+1]$. Equation (1) is said to be *nonoscillatory* if there exists $m \in \mathbf{N}$ such that this equation is disconjugate on $[m, n]$ for every $n > m$. In the opposite case (1) is said to be *oscillatory*. Oscillation of (1) may be equivalently defined as follows. A nontrivial solution of (1) is called *oscillatory* if it has infinitely many generalized zeros. As we have mentioned in the introductory section, due to the separation theorem for (1), we have the following equivalence: One solution of (1) is oscillatory if and only if every solution of (1) is oscillatory. Hence we can speak about *oscillation* or *nonoscillation of equation* (1).

Define a class U of the so-called *admissible sequences* by

$$U = \{\xi \mid \xi : [m, n+2] \longrightarrow \mathbf{R} \text{ such that } \xi_m = \xi_{n+1} = 0\}.$$

Define an "α-degree" functional $\mathcal{F}$ on U by

$$\mathcal{F}(\xi; m, n) = \sum_{k=m}^{n} [r_k |\triangle \xi_k|^\alpha - p_k |\xi_{k+1}|^\alpha].$$

We say that $\mathcal{F}$ is *positive definite* on U provided $\mathcal{F}(\xi) \geq 0$ for all $\xi \in U$ and $\mathcal{F}(\xi) = 0$ if and only if $\xi = 0$.

The basic oscillatory properties of solutions of (1) are described by the so-called Roundabout theorem.

Proposition 1 [**12**, Theorem 1]. *The following statements are equivalent*:

(i) *Equation* (1) *is disconjugate on* $[m, n]$.

(ii) *Equation* (1) *has a solution* y *without generalized zeros on* $[m, n+1]$.

(iii) *The generalized Riccati difference equation*

$$\triangle w_k + p_k + S(w_k, r_k) = 0, \tag{3}$$

where

$$S(w_k, r_k) = w_k \left(1 - \frac{r_k}{\Phi(\Phi^{-1}(r_k) + \Phi^{-1}(w_k))}\right)$$

and the function Φ^{-1} *is the inverse of* Φ, *i.e.*, $\Phi^{-1}(x) = |x|^{\beta-1}\operatorname{sgn} x$, β *being the conjugate number of* α, *has a solution* w_k *on* $[m, n]$ *with* $r_k + w_k > 0$.

(iv) $\mathcal{F}$ *is positive definite on* U.

Along with equation (1) consider the equation

$$\triangle(R_k\Phi(\triangle x_k)) + P_k\Phi(x_{k+1}) = 0, \tag{4}$$

where P_k and R_k are defined on $\mathbf{N}$ with $R_k \neq 0$. In addition to the Sturm type separation theorem, which follows from the implication (ii) $\Rightarrow$ (i) of Proposition 1, the Roundabout theorem shows that the classical Sturm type comparison result extends to (1). Indeed, we have

$$\begin{aligned}\mathcal{F}(\xi) &= \sum_{k=m}^{n}[r_k|\triangle\xi_k|^\alpha - p_k|\xi_{k+1}|^\alpha] \\ &\leq \sum_{k=m}^{n}[R_k|\triangle_k\xi_k|^\alpha - P_k|\xi_{k+1}|^\alpha],\end{aligned}$$

provided the inequalities from the next proposition hold. From this and Proposition 1 one can easily get the following result.

Proposition 2 [**12**, Theorem 2]. *Suppose that* $R_k \geq r_k$ *and* $P_k \leq p_k$ *for large* k. *If equation* (1) *is nonoscillatory, then so is equation* (4).

The Riccati and the variational techniques mentioned in the introductory section are essentially based on the use of the equivalences (i) $\Leftrightarrow$ (iii) and (i) $\Leftrightarrow$ (iv) of Proposition 1, respectively, to investigate some properties of solutions of (1) such as, e.g., oscillation and nonoscillation.

At the end of this section, we give a list of the properties of the function

$$S(x, y) = S(x, y, \alpha) = x\left(1 - \frac{y}{\Phi(\Phi^{-1}(x) + \Phi^{-1}(y))}\right), \tag{5}$$

appearing in (3) that we will need later.

Lemma 1 (Properties of the function S). *Suppose that* $\alpha > 1$. *The function* $S(x,y,\alpha)$ *has the following properties*:

(i) *Let* $y > 0$. *Then* $xS_x(x,y,\alpha) \geq 0$ *for* $x+y>0$ *where* $S_x(x,y,\alpha)=0$ *if and only if* $x=0$.

(ii) $S(x,y,\alpha) \geq 0$ *for* $x+y>0$ *where the equality holds if and only if* $x=0$.

(iii) *Let* $\overline{S}(x,y,\alpha) = x - S(x,y,\alpha)$. *Then* $\overline{S}(x,y,\alpha) = \overline{S}(y,x,\alpha)$ *for* $x+y \neq 0$ *and* $\overline{S}_x(x,y,\alpha) \geq 0$ *for* $x+y>0$ *where the equality holds if and only if* $y=0$.

(iv) *Let* $x,y>0$. *Then* $S_\alpha(x,y,\alpha) \geq 0$.

Proof. The parts (i), (ii) and (iii) are proved in [**13**, Lemma 1]. To prove (iv) note that if $x,y>0$ then the function S can be rewritten as

$$S(x,y,\alpha) = x\left[1-(1+(x/y)^{\beta-1})^{1-\alpha}\right].$$

Now it is easy to see that

$$\begin{aligned} S_\alpha(x,y,\alpha) &= \frac{x\left[(1+(x/y)^{\beta-1})\ln(1+(x/y)^{\beta-1})-(x/y)^{\beta-1}\ln(x/y)^{\beta-1}\right]}{(1+(x/y)^{\beta-1})^{\alpha}} \\ &\geq 0. \end{aligned}$$

From here we have the statement. $\square$

3. λp_k**-type comparison theorems.** We start this section with the auxiliary statement

Lemma 2 [**3**, Lemma 2]. *If there exists a sequence* u_k *such that* $r_k u_k u_{k+1} > 0$ *and*

$$u_{k+1}[\triangle(r_k\Phi(\triangle u_k)) + p_k\Phi(u_{k+1})] \leq 0$$

for large k, *then* (1) *is nonoscillatory.*

Along with equation (1), consider the equation

$$\triangle(R_k\Phi(\triangle x_k)) + \lambda p_k\Phi(x_{k+1}) = 0, \tag{6}$$

where $R_k > 0$ and $\lambda \in \mathbf{R}$.

Theorem 1. *Suppose that* $0 < r_k \le R_k$ *and* $\lambda \in [0,1]$. *If* (1) *is nonoscillatory, then so is equation* (6).

Proof. Suppose that (1) is nonoscillatory. Let y be its solution such that $y_k > 0$, $k \ge m$ for some $m \in \mathbf{N}$. Set $x_k = y_k^\nu$ where $\nu = \Phi^{-1}(\lambda)$. Then $\triangle x_k \le \nu y_k^{\nu-1} \triangle y_k$ and $\triangle x_k \ge \nu y_{k+1}^{\nu-1} \triangle y_k$ by the Lagrange mean value theorem since $\triangle y_k \ge 0$ if and only if $y_{k+1}^{\nu-1} \le y_k^{\nu-1}$ and $\triangle y_k \le 0$ if and only if $y_{k+1}^{\nu-1} \ge y_k^{\nu-1}$. Further,

$$\begin{aligned}\triangle(r_k\Phi(\triangle x_k)) &= r_{k+1}\Phi(\triangle x_{k+1}) - r_k\Phi(\triangle x_k)\\ &\le r_{k+1}\Phi(\nu y_{k+1}^{\nu-1}\triangle y_{k+1}) - r_k\Phi(\nu y_{k+1}^{\nu-1}\triangle y_k)\\ &= \Phi(\nu y_{k+1}^{\nu-1})\triangle(r_k\Phi(\triangle y_k))\\ &= -\Phi(\nu y_{k+1}^{\nu-1})p_k\Phi(y_{k+1})\\ &= -\Phi(\nu)p_k\Phi(y_{k+1}^\nu)\\ &= -\lambda p_k\Phi(x_{k+1}).\end{aligned}$$

From here, equation (6) is nonoscillatory by Lemma 2 and Proposition 2. □

Remark 1. The "oscillatory counterpart" to Theorem 1 is immediate. Suppose that $r_k > 0$. Let $r_k \ge R_k$ and $\lambda \in \mathbf{R}$ be such that $\lambda \ge 1$. If (1) is oscillatory, then so is equation (6).

4. $\sum p_j \le \sum P_j$-type comparison theorem. Before we present the main result of this section, let us give two auxiliary statements.

Lemma 3 [**4**, Lemma 2.1]. *Suppose that there exists an integer* n_0 *such that,*

$$\liminf_{s\to\infty}\sum_{j=k}^{s} p_j \ge 0 \quad \textit{and} \quad \not\equiv 0 \tag{7}$$

for all $k \ge n_0$. *Then there exists* $m \ge n_0$ *such that* $\sum_{j=m}^{k} p_j \ge 0$ *for all* $k \ge m$.

Lemma 4 [**15**, Lemma 3]. *Assume that* (7) *holds. Further, suppose that* $r_k > 0$ *for* $k \geq m$, $\sum_{j=m}^{\infty} r_j^{1-\beta} = \infty$ *and* $\sum_{j=m}^{\infty} p_j$ *is convergent. Let* y *be a nonoscillatory solution of* (1) *such that* $y_k > 0$ *for all* $k \geq m$. *Then there exists* $n \geq m$ *such that*

$$w_k \geq \sum_{j=k}^{\infty} p_j + \sum_{j=k}^{\infty} S(w_j, r_j) \tag{8}$$

for $k \geq n$, *where* $w_k = r_k \Phi(\triangle y_k)/\Phi(y_k) > 0$ *and the function* S *is defined by* (5).

In what follows, we partially use an idea from [**10**], where equation (1) is considered under stronger assumptions. A similar result (however, without the change of the constant α) for the continuous case, i.e., for the case of the equation

$$(\Phi(y'))' + p(t)\Phi(y) = 0, \tag{9}$$

can be found [**7**], where also somewhat stronger additional conditions are required by comparison with our result.

Along with equation (1) consider the equation

$$\triangle(R_k \Phi_{\bar{\alpha}}(\triangle x_k)) + P_k \Phi_{\bar{\alpha}}(x_{k+1}) = 0, \tag{10}$$

where $\Phi_{\bar{\alpha}}(x) = |x|^{\bar{\alpha}-1} \operatorname{sgn} x$, $\bar{\alpha} > 1$.

Theorem 2. *Assume that the sequences* p_k *and* P_k *satisfy* (7). *Let* $\sum_{j=m}^{\infty} p_j$ *and* $\sum_{j=m}^{\infty} P_j$ *be convergent and*

$$\sum_{j=k}^{\infty} p_j \leq \sum_{j=k}^{\infty} P_j \quad \text{for all large } k. \tag{11}$$

Further, suppose that $0 < R_k \leq r_k$, $\sum_{j=1}^{\infty} R_j^{1-\beta} = \infty$ *and* $1 < \alpha \leq \bar{\alpha}$. *If equation* (10) *is nonoscillatory, then so is equation* (1).

Proof. By Lemma 4, the nonoscillation of (10) implies the existence of $m_1 \in \mathbf{N}$ such that

$$z_k \geq \sum_{j=k}^{\infty} P_j + \sum_{j=k}^{\infty} S(z_j, R_j, \bar{\alpha}) =: Z_k \tag{12}$$

for $k \ge m_1$ (clearly, with $z_k + R_k > 0$). Let $m_2 \in \mathbf{N}$ be such that (11) holds and $\sum_{j=k}^{\infty} p_j \ge 0$ for $k \ge m_2$. Set $m = \max\{m_1, m_2\}$ and define the set Ω and the mapping $\mathcal{T}$ by

$$\Omega = \{w \in l^\infty,\ 0 \le w_k \le Z_k \text{ for } k \ge m\}$$

and

$$(\mathcal{T}w)_k = \sum_{j=k}^{\infty} p_j + \sum_{j=k}^{\infty} S(w_j, R_j, \alpha), \quad k \ge m,\ w \in \Omega.$$

We show that $\mathcal{T}$ has a fixed point in Ω. We must verify that

1) Ω is a bounded, closed and convex subset of l^∞,

2) $\mathcal{T}$ maps Ω into itself,

3) $\mathcal{T}\Omega$ is relatively compact,

4) $\mathcal{T}$ is continuous in Ω.

(i) Clearly, Ω is bounded and convex. Let $x^t = \{x_k^t\}$, $t = 1, 2, \dots,$ be any sequence in Ω such that x^t approaches x (in the sup norm). From our assumptions, for any $\varepsilon > 0$, there exists $n \in \mathbf{N}$ such that $\sup_{k \ge m} |x_k^t - x_k| < \varepsilon$ for all $t \ge n$. Thus, for any fixed k, we have $\lim_{t\to\infty} x_k^t = x_k$. Since $0 \le x_k^t \le Z_k$ for all t, then $0 \le x_k \le Z_k$. We have $k \ge m$ arbitrary and hence x belongs to Ω.

(ii) Suppose that $w \in \Omega$ and define $x_k = (\mathcal{T}w)_k$, $k \ge m$. Obviously, $x_k \ge 0$ for $k \ge m$. We must show that $x_k \le Z_k$, $k \le m$. We have

$$\begin{aligned} x_k &= \sum_{j=k}^{\infty} p_j + \sum_{j=k}^{\infty} S(w_j, R_j, \alpha) \le \sum_{j=k}^{\infty} P_j + \sum_{j=k}^{\infty} S(w_j, R_j, \alpha) \\ &\le \sum_{j=k}^{\infty} P_j + \sum_{j=k}^{\infty} S(w_j, R_j, \bar{\alpha}) \le \sum_{j=k}^{\infty} P_j + \sum_{j=k}^{\infty} S(z_j, R_j, \bar{\alpha}) \end{aligned}$$

by the assumptions of the theorem and by Lemma 1 (i), (iv). Hence, $\mathcal{T}\Omega \subset \Omega$.

(iii) According to [**1**, Theorem 3.3] it suffices to show that $\mathcal{T}\Omega$ is uniformly Cauchy since $\mathcal{T}\Omega$ is bounded. Let $\varepsilon > 0$ be given. We show that there exists $N \in \mathbf{N}$ such that for any $k, l > N$ we have $|(\mathcal{T}x)_k - (\mathcal{T}x)_l| < \varepsilon$ for any $x \in \Omega$. Without loss of generality, suppose

$k < l$. Then we have

$$|(\mathcal{T}x)_k - (\mathcal{T}x)_l| = \left|\sum_{j=k}^{l-1} p_j + \sum_{j=k}^{l-1} S(x_j, R_j, \alpha)\right| \tag{13}$$

$$= \sum_{j=k}^{l-1} p_j + \sum_{j=k}^{l-1} S(x_j, R_j, \alpha) \tag{14}$$

for large k by Lemma 3. Taking into account the properties of p_k and $S(x_k, R_k, \alpha)$ for any $\varepsilon > 0$, one can find $N \in \mathbf{N}$ such that

$$\sum_{j=k}^{l-1} p_j < \frac{\varepsilon}{2}$$

and

$$\sum_{j=k}^{l-1} S(x_j, R_j, \alpha) < \frac{\varepsilon}{2} \quad \text{for } l > k > N.$$

From this and (13), $|(\mathcal{T}x)_k - (\mathcal{T}x)_l| < \varepsilon$, hence $\mathcal{T}\Omega$ is relatively compact.

(iv) Let $x^t = \{x_k^t\}$, $k \geq m$, be a sequence in Ω converging to x. We must show that $\mathcal{T}x^t$ converges to $\mathcal{T}x$. Clearly, $\mathcal{T}x^t \in \mathcal{T}\Omega \subset \Omega$ for any t and also $\mathcal{T}x \in \mathcal{T}\Omega \subset \Omega$. For any $\varepsilon > 0$ one can choose $M \geq m$ such that $(\mathcal{T}x^t)_k < \varepsilon/2$ and $(\mathcal{T}x)_k < \varepsilon/2$ for $k > M$ and for each $t \in \mathbf{N}$. Define

$$(\tilde{\mathcal{T}}x)_{k,l} = \sum_{j=k}^{l} p_j + \sum_{j=k}^{l} S(x_j, R_j, \alpha) \quad \text{for } l \geq k \geq m$$

and $(\tilde{\mathcal{T}}x)_{k,l} = 0$ for $l < k$. Mapping $\tilde{\mathcal{T}}$ is obviously continuous. Therefore, for given $\varepsilon/2 > 0$, there exists $N \in \mathbf{N}$ such that

$$|(\tilde{\mathcal{T}}x^t)_{k,l} - (\tilde{\mathcal{T}}x)_{k,l}| < \frac{\varepsilon}{2} \quad \text{for } t \geq N \quad \text{and } k \geq m.$$

Now, having chosen such M, N as above, the following estimates hold for any $k \geq m$:

$$\begin{aligned}
|(\mathcal{T}x^t)_k - (\mathcal{T}x)_k| &= |(\tilde{\mathcal{T}}x^t)_{k,M} + (\mathcal{T}x^t)_{M+1} - (\tilde{\mathcal{T}}x)_{k,M} - (\mathcal{T}x)_{M+1}| \\
&\leq |(\tilde{\mathcal{T}}x^t)_{k,M} - (\tilde{\mathcal{T}}x)_{k,M}| + |(\mathcal{T}x^t)_{M+1} - (\mathcal{T}x)_{M+1}| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Since $k \geq m$ is arbitrary, we have $\mathcal{T}x^t$ converges to $\mathcal{T}x$.

Therefore, it follows from the Schauder fixed point theorem that there exists an element $w \in \Omega$ such that $w = \mathcal{T}w$. In view of the definition of $\mathcal{T}$, this (positive) sequence w satisfies the equation

$$w_k = \sum_{j=k}^{\infty} p_j + \sum_{j=k}^{\infty} S(w_j, R_j, \alpha), \quad k \geq m,$$

and hence also the equation (3). Consequently, the sequence y given by

$$y_m = m_0 \neq 0$$

and

$$y_{k+1} = (1 + (w_k/R_k)^{\beta-1}) y_k, \quad k \geq m,$$

is a nonoscillatory solution of

$$\triangle(R_k \Phi(\triangle y_k)) + p_k \Phi(y_{k+1}) = 0$$

and hence this equation is nonoscillatory. The statement now follows from Proposition 2. $\square$

Remark 2. A closer examination of the above proof shows that the necessary condition for nonoscillation of equation (1) in Lemma 4 is also sufficient.

5. Telescoping principle. The result presented in this section is a half-linear extension of the so-called telescoping principle which was introduced in [**6**] for the second order linear difference equation (2). In fact, it was introduced firstly for second order linear differential equations in [**8**]. Note that in [**6**] the authors consider equation (2) only under the assumption $r_k > 0$ and hereby our result with $r_k \neq 0$ is new even in the linear case (in spite of the fact that an idea of the proof remains quite the same).

Before we present the main result, let us introduce some concepts and assumptions. Denote by $\mathcal{S}$ the set of all real sequences $y = \{y_k, k \in \mathbf{N}\}$. Assume

$$I = \bigcup_{i=1}^{j} I_i, \quad I_i = (m_i, n_i], \quad i = 1, \dots, j, \quad j \leq \infty, \tag{15}$$

where $m_i, n_i \in \mathbf{N}$, $i = 1, \dots, j$, are such that $m_i < n_i < m_{i+1}$ and $\operatorname{card}(\mathbf{N} \setminus I) = \infty$. Based on the set I, we define an interval shrinking transformation $\tau = \tau_I : \mathbf{N} \to \mathbf{N}$ as follows:

$$K = \tau(k) = \operatorname{card}\big([1, k] \cap I^C\big),$$

where $I^C = \mathbf{N} \setminus I$. Let $M_i = \tau(m_i)$. Then $M_i = \tau(k)$ for $k \in [m_i, n_i]$, $i = 1, \dots, j$. This transformation τ induces a transformation $T = T_I : \mathcal{S} \to \mathcal{S}$ defined as follows:

For $y \in \mathcal{S}$

$$Ty = Y = \{Y_K, K \in \mathbf{N}\} \quad \text{with } Y_K = y_k \quad \text{when } \tau(k) = K.$$

Theorem 3. *Let $r_k \neq 0$, $k \in \mathbf{N}$, and assume that* (15) *holds. Let $R = Tr$ and $P = Tp$ for $T = T_I$. Assume*

$$\sum_{k=m_i+1}^{n_i} p_k \geq 0, \quad i = 1, \dots, j. \tag{16}$$

Suppose $X = \{X_K, K \in \mathbf{N}\}$ is a solution of the equation

$$\triangle(R_K \Phi(\triangle X_K)) + P_K \Phi(X_{K+1}) = 0, \tag{17}$$

such that $R_K X_K X_{K+1} > 0$ for $K < N$ and $R_N X_N X_{N+1} \leq 0$. If the sequence y is a solution of equation (1) *such that $y_1 \neq 0$ and $r_1 \Phi(\triangle y_1)/\Phi(y_1) \leq R_1 \Phi(\triangle X_1)/\Phi(X_1)$, then there exists $l \leq n$ such that $r_l y_l y_{l+1} \leq 0$ where $N = \tau(n)$. More precisely, if $N \leq M_i$, then there exists $l \leq m_i$ such that $r_l y_l y_{l+1} \leq 0$, $i = 1, 2, \dots, j$.*

Proof. In this proof, by $y \not< X$ we mean either $y \geq X$ or y does not exist. The proof is by induction. Assume that the conclusion is not true. Then $w_k = -r_k \Phi(\triangle y_k)/\Phi(y_k)$ satisfies

$$\triangle w_k = p_k + w_k \left(\frac{r_k}{(\Phi^{-1}(r_k) - \Phi^{-1}(w_k))^{\alpha-1}} - 1 \right) \tag{18}$$

or, equivalently,

$$w_{k+1} = p_k + \tilde{S}(w_k, r_k), \quad k = 1, \dots, n, \tag{19}$$

and $w_k < r_k$, $k = 1, \dots, n$, where

$$\tilde{S}(w_k, r_k) = \frac{w_k r_k}{(\Phi^{-1}(r_k) - \Phi^{-1}(w_k))^{\alpha-1}}.$$

Observe that the behavior of the function $\tilde{S}$ is similar to the behavior of the function $\overline{S}$ from Lemma 1. In particular, $\tilde{S}(w_k, r_k)$ is nondecreasing with respect to the first variable for $r_k > w_k$. Let $V_K = -R_K \Phi(\triangle X_K)/\Phi(X_K)$. Then

$$V_{K+1} = P_K + \tilde{S}(V_K, R_K), \quad K = 1, \dots, N-1, \tag{20}$$

$V_K < R_K$, $K = 1, \dots, N-1$ and $V_N \not< R_N$.

If $N \le M_1 = m_1$ then, for $k = 1, \dots, N$, $K = k$ and hence $R_K = r_k$, $P_K = p_k$ and equation (20) is the same as (19). By the hypothesis $w_1 \ge V_1$, comparing (19) and (20) step by step (using the above property of $\tilde{S}$), we find that $w_{k+1} \ge V_{k+1}$, $k = 1, \dots, N-1$. In particular,

$$w_n = w_N \ge V_N \not< R_N = r_n.$$

This implies that $w_n \not< r_n$, contradicting the assumption.

If $M_1 < N \le M_2$, then arguing as above we can state that $w_{m_1+1} = w_{M_1+1} \ge V_{M_1+1}$. Adding (18) for k from m_1+1 to n_1 and using (16), we obtain

$$w_{n_1+1} - w_{m_1+1} = \sum_{k=m_1+1}^{n_1} p_k + \sum_{k=m_1+1}^{n_1} \tilde{S}(w_k, r_k) \ge 0,$$

hence $w_{n_1+1} \ge w_{m_1+1} \ge V_{m_1+1}$. Noting that $\tau(n_1+1) = N_1$, we see that w_k, V_K satisfy the same generalized Riccati equation for $n_1 + \le k \le n$ and $M_1 + 1 \le K \le N$, respectively. As before, we see that $w_n \ge V_N \not< R_N = r_n$ and, again, this implies that $w_n \not< r_n$, contradicting the assumption. The proof of inductive step from i to $i+1$ is similar and hence is omitted. $\square$

Theorem 4 (Telescoping principle). *Under the conditions and with the notation of Theorem* 3*, if equation* (17) *is oscillatory, then equation* (1) *is oscillatory.*

Proof. Let X_k be a solution of (17) with $X_1 \neq 0$. Let y_k be a solution of (1) satisfying $y_1 \neq 0$, $r_1\Phi(\triangle y_1)/\Phi(y_1) \leq R_1\Phi(\triangle X_1)/\Phi(X_1)$. By Theorem 3, there exists $l_1 > 0$ such that $r_{l_1} y_{l_1} y_{l_1+1} \leq 0$. Now, working on the solution for $k \geq l_1 + 1$ instead of $k \geq 1$ and proceeding as before, we show that there exists $l_2 \geq l_1 + 1$ such that $r_{l_2} y_{l_2} y_{l_2+1} \geq 0$. Continuing this process leads to the conclusion that y is oscillatory, hence (1) is oscillatory. □

Via this principle, one can get many new examples of oscillatory half-linear difference equations. We use a process that is the reverse of the construction in Theorem 3. Start with any known oscillatory equation (17). Choose a sequence of integers $M_i \to \infty$. Cut the plane at each vertical line $k = M_i$ and pull the two half-planes apart to form a gap of arbitrary finite length. Now fill the gap with an arbitrary nonzero r_i and any p_i whose sum over the length of the gap is nonnegative. Do this at each M_i and denote the new coefficient sequences thus constructed by r, p. Then equation (1) is oscillatory.

6. Comparison theorem for generalized Riccati difference equations. Along with equation (1) consider equation (4). Associated generalized Riccati difference equations for (1) and (4) are equations (3) and

$$\triangle v_k + P_k + S(v_k, R_k) = 0, \tag{21}$$

respectively.

Theorem 5. *Suppose that $R_k \geq r_k$, $P_k \leq p_k$ for $k \in [m,n]$, and let w_k, v_k be solutions of equations* (3) *and* (21), *respectively, defined on $[m,n]$. If $r_k + w_k \geq 0$ on $[m,n]$ and $v_m \geq w_m$, then $v_k \geq w_k$ and $R_k + v_k > 0$ for $k \in [m,n]$.*

Proof. Let w_k and v_k be solutions of equations (3) and (21), respectively, such that $v_m \geq w_m$ (then $R_m + v_m > 0$) and $w_k + r_k > 0$ for $k \in [m,n]$. One can rewrite these equations as

$$w_{k+1} = -\,p_k + \overline{S}(w_k, r_k)$$

and

$$v_{k+1} = -\,P_k + \overline{S}(v_k, R_k),$$

respectively, where the function $\overline{S}$ is defined in Lemma 1. According to this lemma, $\overline{S}(x,y) \le \overline{S}(X,Y)$ for $x \le X$ and $y \le Y$ with $x+y>0$. Hence,

$$w_{m+1} = -p_m + \overline{S}(w_m, r_m) \le -P_m + \overline{S}(v_m, R_m) = v_{m+1}$$

and $R_{m+1} + v_{m+1} > 0$. Continuing this process step by step we find that $v_k \ge w_k$ and $R_k + v_k > 0$ for $k \in [m,n]$. The theorem is proved. □

Remark 3. Note that the other possibility (that is more difficult) to prove the above theorem is to use the generalized Picone identity established in [**12**].

7. Strong oscillation and nonoscillation. This section is concerned with an extension of the so-called strong oscillation to equation (1) and some further related concepts introduced in [**11**] for the equation

$$y'' + p(t)y = 0.$$

The class of equations (1) can be divided according to the following definition.

Definition 2. (i) Equation (1) is said to be *strongly oscillatory* if the equation

$$\triangle(r_k \Phi(\triangle y_k)) + \lambda p_k \Phi(y_{k+1}) = 0 \tag{22}$$

is oscillatory for all $\lambda > 0$.

(ii) Equation (1) is said to be *strongly nonoscillatory* if equation (22) is nonoscillatory for all $\lambda > 0$.

(iii) Equation (1) is said to be *conditionally oscillatory* if (22) is oscillatory for some $\lambda > 0$ and nonoscillatory for some other $\lambda > 0$. By Proposition 2 it follows that in this case there must exist a positive number $\gamma(p)$ such that (1) is oscillatory for $\lambda > \gamma(p)$ and nonoscillatory for $\lambda < \gamma(p)$ provided p_k is nonnegative. This number $\gamma(p)$ is called the *oscillation constant* of the sequence p_k.

Now we can present strongly oscillation and nonoscillation criteria. Note that here we consider only the case when $\sum^{\infty} p_j$ is convergent since if $\sum^{\infty} p_j = \infty$ (and $\sum^{\infty} r_j^{1-\beta} = \infty$ with $r_k > 0$), then (1) is oscillatory by [**12**, Theorem 4] and obviously also strong oscillatory.

Theorem 6. *Assume that* $r_k > 0$, $\sum^{\infty} p_j$ *is convergent and* $\sum^{\infty} r_j^{1-\beta} = \infty$. *Then the following statements hold*:

(i) *Suppose in addition that*

$$\lim_{k\to\infty} \frac{r_k^{1-\beta}}{\sum_{j=m}^{k-1} r_j^{1-\beta}} = 0 \tag{23}$$

and $\sum^{\infty} p_j \geq 0$. *If* (1) *is strongly oscillatory, then*

$$\limsup_{k\to\infty} \left(\sum_{j=m}^{k-1} r_j^{1-\beta}\right)^{\alpha-1} \sum_{j=k}^{\infty} p_j = \infty. \tag{24}$$

(ii) *Suppose in addition that* $p_k \geq 0$ (*eventually nontrivial*). *If* (24) *is fulfilled, then* (1) *is strongly oscillatory.*

(iii) *Suppose in addition that* $p_k \geq 0$ (*eventually nontrivial*). *If* (1) *is strongly nonoscillatory, then*

$$\lim_{k\to\infty} \left(\sum_{j=m}^{k-1} r_j^{1-\beta}\right)^{\alpha-1} \sum_{j=k}^{\infty} p_j = 0. \tag{25}$$

(iv) *Suppose in addition that* (23) *holds. If* (25) *is fulfilled, then* (1) *is strongly nonoscillatory.*

The proof of this theorem (and also of the next theorem) is essentially the same as in the half-linear continuous case, see [**7**], and hence is omitted, in spite of the fact that in some cases we do not require the sequence p_k to be nonnegative. We refer also to the paper [**9**], where a result similar to the above theorem is presented for a somewhat more special form of equation (1). Note that, to prove our statements, we use the following criteria.

Proposition 3. *Assume that* $r_k > 0$.

(i) *If*

$$\liminf_{k\to\infty} k^{\alpha-1}\sum_{j=k}^{\infty} p_j > \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{26}$$

then (28) *is oscillatory.*

(ii) *Suppose that* $p_k \ge 0$ (*eventually nontrivial*), $\sum^{\infty} p_j < \infty$ *and* $\sum^{\infty} r_j^{1-\beta} = \infty$. *If*

$$\limsup_{k\to\infty}\left(\sum_{j=m}^{k-1} r_j^{1-\beta}\right)^{\alpha-1}\sum_{j=k}^{\infty} p_j > 1,$$

then (1) *is oscillatory.*

(iii) *Suppose that* $\sum^{\infty} r_j^{1-\beta} = \infty$ *and* (23) *holds. If*

$$\limsup_{k\to\infty}\left(\sum^{k-1} r_j^{1-\beta}\right)^{\alpha-1}\left(\sum_{j=k}^{\infty} p_j\right) < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} \tag{27}$$

and

$$\liminf_{k\to\infty}\left(\sum^{k-1} r_j^{1-\beta}\right)^{\alpha-1}\left(\sum_{j=k}^{\infty} p_j\right) > -\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1},$$

then (1) *is nonoscillatory.*

Proof. The proof of criterion (i) can be found in [**14**, Corollary 3] or in [**15**, Corollary 1]. Criterion (ii) is proved in [**15**, Theorem 2], while the proof of (iii) is given in [**3**, Theorem 3]. ◻

The following theorem provides information about the oscillation constant of conditionally oscillatory equation of the form

$$\triangle(\Phi(\triangle y_k)) + p_k\Phi(y_{k+1}) = 0. \tag{28}$$

Denote

$$\mathcal{Q}_* = \liminf_{k\to\infty} k^{\alpha-1}\sum_{j=k}^{\infty} p_j \quad \text{and} \quad \mathcal{Q}^* = \limsup_{k\to\infty} k^{\alpha-1}\sum_{j=k}^{\infty} p_j.$$

Theorem 7. *Suppose that* $0 < \mathcal{Q}_* \le \mathcal{Q}^* < \infty$ *and* $p_k \ge 0$ (*eventually nontrivial*). *Then the oscillation constant* $\gamma(p)$ *of equation* (28) *satisfies*

$$\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\frac{1}{\mathcal{Q}^*} \le \gamma(p) \le \min\left\{\frac{1}{\mathcal{Q}^*}, \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\frac{1}{\mathcal{Q}_*}\right\}.$$

In particular, if $\mathcal{Q}_* = \mathcal{Q}^*$, *then*

$$\gamma(p) = \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\frac{1}{Q^*}.$$

The proofs of the following two theorems are also omitted since they are similar to the linear discrete case, see [**2**]. To prove our statements we again use Proposition 3.

Theorem 8. *Let* p_k *and* $\tilde{p}_k$, $k \in \mathbf{N}$, *be two nonnegative and eventually nontrivial sequences. Further, let* $\gamma(\tilde{p})$, $0 < \gamma(\tilde{p}) < \infty$, *be the oscillation constant of* $\tilde{p}_k$. *If*

$$\Psi := \liminf_{k\to\infty} \frac{\sum_{j=k}^{\infty} p_j}{\sum_{j=k}^{\infty} \tilde{p}_j} > \gamma(\tilde{p}), \tag{29}$$

then (28) *is oscillatory.*

Theorem 9. *Let* p_k *and* $\tilde{p}_k$, $k \in \mathbf{N}$, *be two nonnegative and eventually nontrivial sequences. Further, let* $\gamma(p)$, $0 < \gamma(p) < \infty$, *and* $\gamma(\tilde{p})$, $0 < \gamma(\tilde{p}) < \infty$, *be the oscillation constants of* p_k *and* $\tilde{p}_k$, *respectively. Then* $\Psi \le \gamma(\tilde{p})/\gamma(p)$ *where* Ψ *is defined by* (29), *and if*

$$\limsup_{k\to\infty} \frac{\sum_{j=k}^{\infty} p_j}{\sum_{j=k}^{\infty} \tilde{p}_j} < \gamma(\tilde{p}),$$

then (28) *is nonoscillatory.*

8. Generalized Euler difference equation. In this last section, we investigate some oscillatory properties of the discrete generalized Euler equation

$$\triangle(\Phi(\triangle y_k)) + \frac{\gamma}{(k+1)^\alpha}\,\Phi(y_{k+1}) = 0, \tag{30}$$

where $\gamma \in \mathbf{R}$. Among others, we will need the following auxiliary result, which is a generalization of the well-known Hardy's inequality.

Lemma 5 [**5**]. *If* $\alpha > 1$ *and* $\eta_k \ge 0$ *for* $k \in \mathbf{N}$, *then*

$$\sum_{k=1}^{n} k^{-\alpha}\biggl(\sum_{j=1}^{k}\eta_j\biggr)^{\alpha} < \biggl(\frac{\alpha}{\alpha-1}\biggr)^{\alpha}\sum_{k=1}^{n}\eta_k^{\alpha},$$

unless $\eta_k = 0$ *for* $k = 1, 2, \dots, n$.

To show that (30) is (non)oscillatory, we will distinguish the following four cases:

1) If $\gamma > ((\alpha-1)/\alpha)^\alpha$, then (30) is oscillatory by Proposition 3 (criterion (i)) since

$$\begin{aligned} k^{\alpha-1}\sum_{j=k}^{\infty}\frac{\gamma}{(j+1)^\alpha} &\ge k^{\alpha-1}\gamma\int_{k+1}^{\infty}\frac{1}{x^\alpha}\,dx = \frac{\gamma k^{\alpha-1}}{(\alpha-1)(k+1)^{\alpha-1}} \\ &\ge \frac{1}{\alpha}\biggl(\frac{\alpha-1}{\alpha}\biggr)^{\alpha-1} + \varepsilon, \end{aligned}$$

for suitable $\varepsilon > 0$.

2) If $0 \le \gamma < ((\alpha-1)/\alpha)^\alpha$, then (30) is nonoscillatory by Proposition 3 (criterion (iii)) since

$$\begin{aligned} k^{\alpha-1}\sum_{j=k}^{\infty}\frac{\gamma}{(j+1)^\alpha} &\le k^{\alpha-1}\gamma\int_{k}^{\infty}\frac{1}{x^\alpha}\,dx = \frac{\gamma k^{\alpha-1}}{(\alpha-1)k^{\alpha-1}} \\ &\le \frac{1}{\alpha}\biggl(\frac{\alpha-1}{\alpha}\biggr)^{\alpha-1} - \varepsilon, \end{aligned}$$

for suitable $\varepsilon > 0$.

3) If $\gamma < 0$, then (30) is nonoscillatory by the Sturm type comparison theorem (Proposition 2) and the result in the case 2).

4) If $\gamma = ((\alpha-1)/\alpha)^\alpha$, then (30) is nonoscillatory. Indeed, according to the Roundabout theorem (Proposition 1) it is sufficient to find $n \in \mathbf{N}$ such that for any nontrivial

$$\xi \in \big\{\{\xi_k\}_{k=1}^\infty : \xi_k = 0 \text{ for } k \le n \text{ and } \exists m > n \text{ s.t. } \xi_k = 0 \text{ for } k \ge m\big\}$$

we have

$$\mathcal{F}(\xi; n, \infty) = \sum_{k=n}^{\infty} [r_k |\triangle \xi_k|^\alpha - p_k |\xi_{k+1}|^\alpha] > 0.$$

Put $n = 1$ and $\xi_{k+1} = \sum_{j=1}^k \eta_j$, $k \in \mathbf{N}$, where η is such that ξ is admissible. Clearly, $\triangle \xi_k = \eta_k$. Now there exists $m \in \mathbf{N}$ such that

$$\begin{aligned}\mathcal{F}(\xi; n, \infty) &= \sum_{k=1}^{m} \left[|\eta_k|^\alpha - \frac{\gamma}{(k+1)^\alpha} \Big| \sum_{j=1}^{k} \eta_j \Big|^\alpha \right] \\ &\ge \sum_{k=1}^{m} \left[|\eta_k|^\alpha - \frac{\gamma}{k^\alpha} \bigg(\sum_{j=1}^{k} |\eta_j| \bigg)^\alpha \right] \\ &> 0\end{aligned}$$

by Lemma 5, and hence (30) is nonoscillatory.

Altogether, equation (30) is oscillatory for $\gamma > ((\alpha-1)/\alpha)^\alpha$ and nonoscillatory otherwise. Thus, if we consider (30) as an equation of the form (22), more precisely, if $\lambda = \gamma$ and $p_k = (k+1)^{-\alpha}$, then it is easy to see that the oscillation constant of such sequence p_k is equal to $((\alpha-1)/\alpha)^\alpha$.

On the other hand, the above oscillatory properties of (30) can be used to establish criteria (26) and (27) (with $r_k \equiv 1$) for equation (28) by the comparison of (28) with (30) using Theorem 4 and the above estimates. A similar method was used in [**7**] to prove these types of criteria for (9), however, only under the assumption $p(t) \ge 0$. One can also show, either as a consequence of the more general criteria (26) and (27) or by the comparison of (28) with (30) using Proposition 2, that the condition

$$\left(\frac{\alpha - 1}{\alpha} \right)^\alpha < \liminf_{k \to \infty} (k+1)^\alpha p_k \le \infty$$

implies the oscillation of (28) and the condition

$$-\infty \leq \limsup_{k\to\infty}(k+1)^{\alpha}p_k < \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}$$

implies the nonoscillation of (28).

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