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CLOSURE ALONG AN ADMISSIBLE SUBSET, SEMINORMALITY AND T-CLOSEDNESS

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ABSTRACT. We introduce the closure of an injective ring morphism $A \to B$ along an admissible subset X of $\mathrm{Spec}\,(A)$ (an admissible subset X is the spectral image of a flat epimorphism $A \to E$). Then we give a theory of seminormality and t-closedness along admissible subsets which extends Yanagihara's work on S-seminormality.

0. Introduction. In order to give a unified treatment for p-seminormality of Swan and F-closedness of Asanuma, Yanagihara introduced S-seminormality for rings A with respect to a multiplicative subset S of A. We refer to Yanagihara's paper for more details [**26**]. We studied t-closedness in two papers [**16, 17**]. This last notion is closely linked with seminormality and quasi-normality. Yanagihara's work gave us the idea to introduce S-t-closedness. However, it quickly appeared that the reason the theory works is the existence of the flat epimorphism $A \to A_S$. Thus we decided to extend the theory to any flat epimorphism. Outside localizations, flat epimorphisms appear in many contexts of commutative algebra and algebraic geometry. For instance, affine subsets of a spectrum give rise to flat epimorphisms. Evidently, such an extension brings about many technical problems but provides much more flexibility to handle results. When $A \to B$ is an injective ring morphism, we identify A to a subring of B. If someone prefers, he could consider ring extensions $A \subset B$.

In Section 1, we begin by giving results on flat epimorphisms. Some of them come from papers of Lazard [**10**] and Raynaud [**22**]. Following Raynaud, we say that a subset X of the spectrum of a commutative ring A is admissible if there is a flat epimorphism $e : A \to E$ such that ${}^a e(\mathrm{Spec}\,(E)) = X$. Actually, an admissible subset X determines the flat epimorphism $A \to E$ within an isomorphism. We show that for $X \subset \mathrm{Spec}\,(A)$, there is a smallest admissible subset X_a containing X; for instance, $(V(I))_a = \mathrm{Spec}\,(A_{1+I})$ if I is an ideal of a ring A.

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Furthermore, let $f : A \to B$ be a ring morphism and $X \subset \mathrm{Spec}\,(A)$. If X is admissible, so is ${}^af^{-1}(X)$. If f is pure and ${}^af^{-1}(X)$ is admissible, then X is admissible. Now consider an admissible subset X of $\mathrm{Spec}\,(A)$ associated to the flat epimorphism $A \to E$ and let $A \to B$ be an injective ring morphism. We say that $A \to B$ is an X-isomorphism if $A_P \to B_P$ is an isomorphism for every $P \in X$. Then $f : A \to B$ is an X-isomorphism if and only if $E \to E \otimes_A B$ is an isomorphism. When $A \to B$ is an X-isomorphism, X is canonically homeomorphic to ${}^af^{-1}(X)$ which allows us to identify these sets. To be an X-isomorphism is stable under a base change preserving injectivity.

The aim of Section 2 is to provide for an injective ring morphism $f : A \to B$ and an admissible subset X of $\mathrm{Spec}\,(A)$, a factorization $A \to {}^X_BA \hookrightarrow B$ where ${}^X_BA$ is the largest A-subalgebra C of B such that $A \to C$ is an X-isomorphism. Let $A \to E$ be a flat epimorphism associated to X, then ${}^X_BA$ is the pullback defined by $B \to B \otimes_A E$ and $E \to B \otimes_A E$. We show that ${}^X_BA$ is the set of all elements $b \in B$ such that $X \subset D(I)$ and $Ib \subset A$ for some ideal I of A. We call ${}^X_BA$ the X-closure of A in B and say that $A \to B$ is X-closed when $A = {}^X_BA$. If S is a multiplicative subset of A, we set ${}^S_BA = {}^X_BA$ where $X = \mathrm{Spec}\ (A_S)$. An element b of B belongs to ${}^S_BA$ if and only if there is some $s \in S$ such that $sb \in A$. We thus recover a construction of Yanagihara. We get that ${}^X_BA$ is the smallest A-subalgebra C (with structural morphism g) of B such that C is ${}^ag^{-1}(X)$-closed. We show that X-closures have a good behavior with respect to the usual constructions of commutative algebra as localizations, polynomial extensions and cartesian squares. Moreover, X-closedness is descended by pure morphisms. Surprisingly, after a polynomial extension base change, an X-closure can be considered as an S-closure.

In Section 3, we use X-isomorphisms to define and study infra-integrality and subintegrality along admissible subsets. Recall that an injective integral morphism $A \to B$ is said to be infra-integral if its residual extensions are isomorphisms [**16**] and subintegral if in addition $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ is bijective [**25**]. If $X \subset \mathrm{Spec}\,(A)$ is an admissible subset, we say that $A \to B$ is X-infra-integral, respectively X-subintegral, if $A \to B$ is an infra-integral, respectively subintegral, X-isomorphism. Hence $A \to B$ is X-infra-integral if and only if $A_P \to B_P$ is an isomorphism for every $P \in X$ and $\mathrm{k}(P) \to \mathrm{k}(Q)$ is an isomorphism for every prime ideal Q of B lying over some $P \notin X$. Mimicking [**16**] and [**25**], we define elementary X-infra-integral, respectively X-subintegral, morphisms and establish a convenient theory for these morphisms.

Section 4 is devoted to the heart of the theory. We build two new closures for an injective ring morphism $A \to B$: the X-t-closure and the X-seminormalization of A in B. Recall that for such a morphism, the t-closure ${}_B^t A$, respectively seminormalization ${}_B^+ A$, of A in B can be defined as being the largest A-subalgebra C of B such that $A \to C$ is infra-integral, respectively subintegral. Replacing infra-integrality with X-infra-integrality, we get the X-t-closure ${}_B^{(X,t)} A$ of A in B and similarly the X-seminormalization ${}_B^{(X,+)} A$ of A in B. As a first result, we get that ${}_B^{(X,t)} A = {}_B^X A \cap {}_B^t A$ and ${}_B^{(X,+)} A = {}_B^X A \cap {}_B^+ A$. Most of the results are similar when considering either closures. Then ${}_B^{(X,t)} A$ is the X-closure of A in ${}_B^t A$ and the t-closure of A in ${}_B^X A$. These closures have a good behavior with respect to the already mentioned constructions of commutative algebra. Next we say that $A \to B$ is X-t-closed if $A = {}_B^{(X,t)} A$. It turns out that $A \to B$ is X-t-closed if an element b of B is in A whenever there are some $r \in A$ and an ideal I of A such that $b^2 - rb$, $b^3 - rb^2 \in A$, $Ib \subset A$ and $X \subset D(I)$ while $A \to B$ is X-seminormal if $b \in B$ belongs to A whenever $b^2, b^3 \in A$ and there is some ideal I of A such that $X \subset D(I)$ and $Ib \subset A$. Then X-t-closedness or X-seminormality is preserved by the usual constructions of commutative algebra, is descended by pure morphisms and is ascended in cartesian squares. These notions localize and globalize. In fact, it is enough to consider $\mathrm{Spec}\,(A) \setminus X$ to get X-t-closedness or X-seminormality.

In Section 5, we consider X-seminormal rings with absolutely flat total quotient ring $\mathrm{Tot}\,(A)$ (such rings are considered and studied in our paper [**21**] in which they are called decent and where we define decent schemes). The reason is that the class of decent rings contains all rings used in algebraic geometry. Moreover, their integral closures have good behavior. However, we consider weak Baer rings (see the definition below) as long as X-t-closed rings are concerned. Indeed, we showed in our papers [**16, 17**] that this class of rings is the right one to consider. A weak Baer ring is decent. Then a decent ring, respectively a weak Baer ring, A is said to be X-seminormal, respectively X-t-closed, if A is X-seminormal, respectively X-t-closed, in $\mathrm{Tot}\,(A)$. As expected, A is

X-seminormal if and only if $A \to \mathrm{Tot}\,(A)$ is X-seminormal and similarly for t-closedness. Again these notions have a good behavior, essentially because we consider decent rings. We show that a decent ring is X-seminormal if and only if for any pair $(x, y) \in A^2$ such that $x^3 = y^2$ there is some $t \in A$ such that $x = t^2$, $y = t^3$ whenever there exist $a_1, \dots, a_n$, $z_1, \dots, z_n \in A$ such that $X \subset D(a_1, \dots, a_n)$ and $a_i^2 x = z_i^2$, $a_i^3 y = z_i^3$ for $i = 1, \dots, n$. For X-t-closedness we have to consider $x^3 + rxy - y^2 = 0$ instead of $x^3 = y^2$ and suitable conditions. Actually, these properties are related to the flat epimorphism $A \to E$ associated to X. To see this, we need to define new classes of morphisms. If $\varphi : A \to E$ is a ring morphism, we say that φ is almost t-closed if for any triple $(x, y, r) \in A^3$ such that $x^3 + rxy - y^2 = 0$ and $\varphi(x) = u^2 - \varphi(r)u$, $\varphi(y) = u^3 - \varphi(r)u^2$ for some $u \in E$, there is some $t \in A$ such that $x = t^2 - rt$, $y = t^3 - rt^2$. A similar definition gives almost seminormality. Here is the main result of this section. A weak Baer ring A is X-t-closed if and only if the flat epimorphism associated to X is almost t-closed and, also, if and only if the natural map $A \to \prod_{P \in X} A_P$ is almost t-closed.

In Section 6, we give examples and properties of X-seminormal or X-t-closed rings. Their properties are in general the expected properties of seminormality or t-closedness, although new phenomena appear as X varies. Some examples are given by means of quadratic orders.

In the Appendix, we show that flat epimorphisms can be characterized locally by standard flat epimorphisms analogous to standard etale morphisms (see [**23**]). By using hyper-resultants, we characterize when the morphism $A \to (A[z]/(p(z)))_{q(z)} = B$ is a flat epimorpism ($p(z)$ a monic polynomial such that $p'(z)$ is invertible in B). We also consider rings such that every admissible subset is of the form $\mathrm{Spec}\,(A_S)$, S a multiplicative subset. This happens for rings of real continuous functions on a topological space E. Such rings are always seminormal. This is not the case for t-closedness.

Now we give some notation and conventions. All considered rings are commutative. If A is a ring, then $\mathrm{Tot}\,(A)$ is its total quotient ring and A' its integral closure. A ring A is said to be a weak Baer ring if the annihilator of each element is generated by an idempotent. A weak Baer ring is decent, that is to say $\mathrm{Tot}\,(A)$ is absolutely flat. If P is a prime ideal of a weak Baer ring, then A_P is an integral domain. For more details on weak Baer rings, see for instance [**17**]. The set of all idempotents of a ring A is denoted by $\mathrm{Bool}(A)$. If M is an A-module, we denote by $\mathrm{Ass}_A(M)$ the set of all weak Bourbaki associated prime ideals of M. Any undefined notation comes from the work of Bourbaki.

1. Isomorphisms along admissible subsets of a spectrum. We use results of Lazard and Raynaud about (flat) epimorphisms. They were given in the Samuel's seminar on epimorphisms [**9**] and [**22**]. A ring morphism $f : A \to E$ is said to be an epimorphism if f is an epimorphism in the category of commutative rings. A ring morphism $A \to E$ is an epimorphism if and only if $E \otimes_A E \to E$ is an isomorphism. This property is stable under any base change, as well as flatness of ring morphisms. Moreover, let $\{A \to A_i\}_{i\in I}$ be a direct system of ring morphisms with direct limit $A \to E$; if $A \to A_i$ is an epimorphism, respectively a flat ring morphism, for each $i \in I$, so is $A \to E$. We recall that a faithfully flat epimorphism is an isomorphism and that a finite epimorphism is surjective [**9**].

Let $f : A \to B$ be a ring morphism. In the following, we denote by $\mathcal{X}_A(B)$ (or $\mathcal{X}(B)$) the subset ${}^a f(\mathrm{Spec}(B))$ of $\mathrm{Spec}(A)$. If A is a ring, the closed subsets of the patch topology on $\mathrm{Spec}(A)$ (in French, *topologie constructible* [**8**]) are the subsets $\mathcal{X}(B)$ where $A \to B$ is a ring morphism. A closed subset for the patch topology is called a proconstructible subset and the patch closure of $X \subset \mathrm{Spec}(A)$ is denoted by X^c.

Let A be a ring and S a multiplicative subset of A ($1 \in S$). Then $A \to A_S$ is well known to be a flat epimorphism such that $\mathcal{X}(A_S) = \cap_{s\in S} D(s)$.

Proposition 1.1 [**10, 22**]. *Let $A \to E$ be a ring morphism. The following conditions are equivalent*:

(1) *$A \to E$ is a flat epimorphism.*

(2) *$A_P \to E_P$ is an isomorphism for every prime ideal P in $\mathcal{X}(E)$.*

(3) *$A_P \to E_Q$ is an isomorphism for every prime ideal Q of E lying over P in A and* $\mathrm{Spec}(E) \to \mathrm{Spec}(A)$ *is injective.*

Definition 1.2 [**22**]. Let A be a ring. A subset X of $\mathrm{Spec}(A)$ is termed admissible if there is a flat epimorphism $A \to E$ such that

$X = \mathcal{X}(E)$.

An admissible subset X is quasi-compact and stable under generalizations. Actually, we can say more.

Proposition 1.3 [**22**]. *Let $\mathcal{X}(E)$ be an admissible subset of* $\mathrm{Spec}\,(A)$*. Then the locally ringed space induced over $\mathcal{X}(E)$ by the affine scheme* $\mathrm{Spec}\,(A)$ *is an affine scheme isomorphic to* $\mathrm{Spec}\,(E)$*. In particular,* $\mathrm{Spec}\,(E) \to \mathcal{X}(E)$ *is a homeomorphism.*

Lemma 1.4. *Let A be a ring and Y an admissible subset of* $\mathrm{Spec}\,(A)$ *associated to the flat epimorphism $A \to F$. Let X be a proconstructible subset of* $\mathrm{Spec}\,(A)$ *such that $X = \mathcal{X}(B)$ for some ring morphism $A \to B$. Then we have $X \subset Y$ if and only if there is a ring morphism $F \to B$ such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \\ & F \arrow[ur] & \end{tikzcd}$$

Moreover, such a morphism $F \to B$ is unique.

Proof. Assume that $X \subset Y$. If Q is a prime ideal of B lying over $P \in X$, there is a prime ideal R in F lying over P since $X \subset Y$. By a well-known property of tensor products, there is some prime ideal S in $F \otimes_A B$ lying over Q and R. Thus $B \to F \otimes_A B$ is a faithfully flat epimorphism whence an isomorphism. Therefore, the commutative diagram exists. The converse is obvious. To end, $F \to B$ is unique since $A \to F$ is an epimorphism. $\square$

Proposition 1.5. *Let X be an admissible subset of* $\mathrm{Spec}\,(A)$*. Up to an isomorphism, there is a unique flat epimorphism $A \to E$ such that $X = \mathcal{X}(E)$.*

Proof. Let $A \to E$ and $A \to F$ be two flat epimorphisms such that $\mathcal{X}(E) = \mathcal{X}(F)$. By using 1.4 for $X = \mathcal{X}(E) \subset Y = \mathcal{X}(F)$ and

$X = \mathcal{X}(F) \subset Y = \mathcal{X}(E)$, we get ring morphisms $F \to E$ and $E \to F$ such that $A \to F \to E = A \to E$, $A \to E \to F = A \to F$. Since $A \to E$ and $A \to F$ are epimorphisms, $E \to F$ is an isomorphism (its inverse is $F \to E$). $\square$

The previous results allow us to associate with every admissible subset X of $\mathrm{Spec}\,(A)$ a unique flat epimorphism $e_X : A \to E(X)$ such that $X = \mathcal{X}(E(X))$. Notice that for every $P \in X$ there is a unique $Q \in \mathrm{Spec}\,(E(X))$ such that $e_X^{-1}(Q) = P$ and then $Q = PE(X)$ [**10**].

Definition 1.6. Let A be a ring and $X \subset \mathrm{Spec}\,(A)$. We define S_X to be the set of all elements $a \in A$ such that $X \subset D(a)$. Then S_X is a saturated multiplicative subset of A. We set $X_m = \mathcal{X}(A_{S_X})$ so that $X \subset X_m$.

We recall the following well-known result. Let $A \to B$ and $A \to C$ be two ring morphisms, and set $B \otimes_A C = D$. Then we have $\mathcal{X}(D) = \mathcal{X}(B) \cap \mathcal{X}(C)$.

Definition 1.7. If A is a ring, we define $\mathcal{A}(A)$ to be the set of all admissible subsets X of $\mathrm{Spec}\,(A)$ and a partial ordering $\preceq$ of $\mathcal{A}(A)$ by

$$Y \preceq X \Longleftrightarrow X \subset Y.$$

Then $(\mathcal{A}, \preceq)$ is directed. Indeed, if $A \to E$ and $A \to F$ are two flat epimorphisms so is $A \to E \otimes_A F$. Thus the intersection of two admissible subsets is an admissible subset.

Proposition 1.8. *Let A be a ring and X a subset of* $\mathrm{Spec}\,(A)$. *There is a smallest admissible subset $X_a \supset X$ and we have $X_a = (X^c)_a \subset X_m$.*

Proof. Consider the subset $\mathcal{D}$ of all elements Y in $\mathcal{A}(A)$ such that $X \subset Y$. Then $(\mathcal{D}, \preceq)$ is directed by 1.7 and $\mathcal{D}$ is not empty since X_m belongs to $\mathcal{D}$. If $Y \preceq Z$ in $\mathcal{D}$, there is a unique morphism $f_{Z,Y} : E_Y \to E_Z$ such that $A \to E_Z = A \to E_Y \to E_Z$ by 1.4. Then $\{E_Y\}_{Y \in \mathcal{D}}$ is a direct system because every $A \to E_Y$ is an epimorphism. Let $A \to E$ be its direct limit. Then $A \to E$ is a flat epimorphism.

Moreover, $\mathcal{X}(E) = \cap_{Y \in \mathcal{D}} Y$ [**8**, I.3.4.10]. The result follows because $X \subset Y \Leftrightarrow X^c \subset Y$ for any $Y \in \mathcal{A}(A)$. $\square$

Remark 1.9. The previous result shows that for any ring morphism $A \to B$ there is a largest flat epimorphism $A \to E_A(B)$ factoring $A \to B$, that is to say $A \to E_A(B)$ is factored by any flat epimorphism factoring $A \to B$. Moreover, we have $(\mathcal{X}(B))_a = \mathcal{X}(E_A(B))$. This result has already been proved by Morita when $A \to B$ is injective by a different method [**11**, 3.3]. If B' is the ring of all elements $b \in B$ such that $B = (A :_A b)B$, set $B_1 = B'$. Then, by transfinite induction, define B_α for an ordinal α as follows. Put $B_{\alpha+1} = B'_\alpha$ and $B_\alpha = \cap_{\beta<\alpha} B_\beta$ for a limit ordinal α. There is an ordinal ω such that $B_\omega = B_{\omega+1}$. Then $E_A(B)$ is nothing but B_ω. The factorization $A \to E_A(B) \to B_1$ will be used later. Notice that $B_1 = E_A(B)$ when $A \to B_1$ is a flat epimorphism.

Let X be a subset of $\operatorname{Spec}(A)$ and define $G(X)$ to be the set of all generalizations of elements of X (a prime ideal Q of A belongs to $G(X)$ if and only if there is some $P \in X$ such that $Q \subset P$). Now consider $X = \{P\}$. We have that $X_a = G(P) = \operatorname{Spec}(A_P)$. Indeed, $P \in X_a$ gives $G(P) \subset X_a$ since X_a is stable under generalizations. We get the converse by $X_a \subset X_m = G(P)$. We can also give an explicit construction when X is closed. This follows from an unpublished paper of Ferrand [**4**], so we give a proof:

Proposition 1.10. *Let $X = V(I)$ be a closed subset of* $\operatorname{Spec}(A)$*, I an ideal. Then we have $X_a = G(V(I)) = \operatorname{Spec}(A_{1+I})$. It follows that $(V(I) \cap D(r))_a = G(V(I) \cap D(r))$ for any element $r \in A$.*

Proof. First we show that $G(V(I)) = \operatorname{Spec}(A_{1+I})$. If $Q \subset P$ are prime ideals such that $I \subset P$, then $(1+I) \cap Q = \varnothing$. Thus one inclusion is obvious. To show the converse, it is enough to consider a prime ideal P maximal with respect to being disjoint from $1 + I$. Assume that there is some $a \in I \setminus P$. Then there is some $x \in I$ such that $1+x \in P+Aa$ contradicting $(1+I)\cap P = \varnothing$. Therefore, we have $I \subset P$. Since the elements of $1 + I$ are units in A/I, there is a factorization $A \to A_{1+I} \to A/I$. Now, let $A \to B$ be a flat epimorphism factoring

$A \to A/I$ and consider the co-cartesian square

$$\begin{array}{ccc} A & \longrightarrow & A_{1+I} \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & B_{1+I}. \end{array}$$

Since $A \to B_{1+I}$ is flat, its spectral image Y in $\mathrm{Spec}\,(A)$ is stable under generalizations. Moreover, there is a ring morphism $B_{1+I} \to A/I$ because the diagram is co-cartesian whence $V(I) \subset Y$. It follows that $G(V(I)) \subset Y$. Hence, the flat epimorphism $A_{1+I} \to B_{1+I}$ is an isomorphism since its spectral map is surjective. Therefore, $A \to B$ factorizes $A \to A_{1+I}$ and the proof is complete by 1.4. Now observe that $X = V(I) \cap D(r) = \mathcal{X}(A/I \otimes_A A_r)$ and set $J = I_r$, $R = A_r$. Then $A \to R_{1+J}$ is a flat epimorphism factoring $A \to R/J$. Let $A \to B$ be a flat epimorphism factoring $A \to R/J$, then $B \to R/J$ is factored by $B \to B_r$. It follows that $R \to R_{1+J}$ is factored by $R \to B_r$ so that $A \to R_{1+J}$ is factored by $A \to B$. Therefore, we get $X_a = \mathcal{X}(R_{1+J})$. Since $\varphi : A \to R$ is generalizing by flatness, we have $X_a = {}^a\varphi(G(V(J))) = G({}^a\varphi(V(J))) = G(X)$. □

Lemma 1.11. *Let $f : A \to B$ be a morphism and $X \subset \mathrm{Spec}\,(A)$, respectively $Y \subset \mathrm{Spec}\,(B)$, admissible subsets, such that ${}^af(Y) \subset X$. Let $A \to E$ and $B \to F$ be the flat epimorphisms associated to X and Y.*

(1) *${}^af^{-1}(X)$ is an admissible subset of $\mathrm{Spec}\,(B)$ associated to the flat epimorphism $B \to E \otimes_A B$.*

(2) *There is a unique ring morphism $E \to F$ such that the following diagram commutes*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ E & \longrightarrow & F. \end{array}$$

Therefore, there is a morphism of affine schemes $Y \to X$ which identifies with $\mathrm{Spec}\,(F) \to \mathrm{Spec}\,(E)$.

Proof. Observe that $B \to B \otimes_A E$ is a flat epimorphism so that the image Z of $\mathrm{Spec}\,(B \otimes_A E)$ in $\mathrm{Spec}\,(B)$ is admissible. Clearly, we have

${}^a f(Z) \subset X$ whence $Z \subset {}^a f^{-1}(X)$. Now consider $Q \in {}^a f^{-1}(X)$, then $P = {}^a f(Q)$ belongs to X so that there is some prime ideal $R \in \mathrm{Spec}\,(E)$ lying over P. By a classical property of tensor products, there is a prime ideal S in $E \otimes_A B$ lying over Q and R so that $Q \in Z$. Thus we have $Z = {}^a f^{-1}(X)$ and (1) is proved. Next we get a ring morphism $E \otimes_A B \to F$ such that $B \to F = B \to E \otimes_A B \to F$ by 1.4, since $Y \subset {}^a f^{-1}(X)$. Then $E \to F$ is defined in an obvious way and is unique since $A \to E$ is an epimorphism. □

We are now ready to give the main definition of this section.

Definition 1.12. Let $A \to B$ be an injective ring morphism and X an admissible subset of $\mathrm{Spec}\,(A)$. Then $A \to B$ is said to be an X-isomorphism if $A_P \to B_P$ is an isomorphism for every $P \in X$.

Proposition 1.13. *Let $f : A \to B$ be an injective ring morphism and X an admissible subset of* $\mathrm{Spec}\,(A)$. *The following conditions are equivalent*:

(1) *$A \to B$ is an X-isomorphism.*

(2) *$E(X) \to E(X) \bigotimes_A B$ is an isomorphism.*

(3) *The morphism of affine schemes $Y = {}^a f^{-1}(X) \to X$ is an isomorphism. In this case, $Y \to X$ is a homeomorphism and $A_P \to B_Q$ is an isomorphism for every $Q \in Y$ lying over $P \in X$.*

Proof. Let $e : A \to E$ be the flat epimorphism associated with X. Let $P = e^{-1}(Q)$ be an element of X. Setting $D = E \otimes_A B$, we have a co-cartesian square

$$\begin{array}{ccc} A_P & \longrightarrow & B_P \\ \downarrow & & \downarrow \\ E_Q & \longrightarrow & D_Q \end{array}$$

where the vertical arrows are isomorphisms by 1.1. Since $A \to E$ is flat, $E \to D$ is injective. Thus (1) $\Leftrightarrow$ (2) follows and (2) $\Leftrightarrow$ (3) is an easy consequence of 1.11. □

Remark 1.14. Let A be a ring. Examples of admissible subsets are affine open subschemes of $\mathrm{Spec}\,(A)$, subsets $\mathrm{Spec}\,(A_S)$ where S is a multiplicative subset of A.

Let $A \to B$ be an injective ring morphism and S a multiplicative subset of A and set $X = \mathcal{X}(A_S)$. Then $A \to B$ is an X-isomorphism if and only if $A_S \to B_S$ is an isomorphism. In this case, we say that $A \to B$ is an S-isomorphism.

The definition of an X-isomorphism can be extended to the following cases: $X = \{P\}$ or $X = V(I)$. In every case, we have $X_a = G(X)$ so that $A \to B$ is an X-isomorphism if and only if $A \to B$ is an X_a-isomorphism. Indeed, if $A_P \to B_P$ is an isomorphism, so is $A_Q \to B_Q$ for any prime ideal $Q \subset P$.

In the next sections, we apply our general theory to injective integral morphisms. So we need some criteria for such a morphism to be an X-isomorphism.

Lemma 1.15. *Let $f : A \to B$ be an injective ring morphism, $X \subset \mathrm{Spec}\,(A)$ an admissible subset and I an ideal of A. Then we have $X \subset D(I)$ if and only if $E(X) = IE(X)$.*

Proof. Assume that $E(X) \neq IE(X)$. Then there is some prime ideal Q in $E(X)$ such that $IE(X) \subset Q$ so that $X \not\subset D(I)$. Conversely, if $X \not\subset D(I)$, there is some P in X such that $I \subset P$. Now, $PE(X)$ is a prime ideal of $E(X)$ lying over P. From $IE(X) \subset PE(X)$, we get that $E(X) \neq IE(X)$. □

Proposition 1.16. *Let $f : A \to B$ be an injective integral morphism. For any directed family $\{A_\lambda\}$ of finite A-subalgebras of B such that $B = \cup A_\lambda$, we denote by I_λ the conductor of $A \to A_\lambda$. Let X be an admissible subset of* $\mathrm{Spec}\,(A)$. *The following statements are equivalent*:

(1) *$A \to B$ is an X-isomorphism.*

(2) *$A \to A_\lambda$ is an X-isomorphism for each λ.*

(3) *$X \subset D(I_\lambda)$ for each λ.*

(4) *$E(X) = I_\lambda E(X)$ for each λ.*

In particular, when $A \to B$ is finite with conductor I, then $A \to B$ is an X-isomorphism if and only if $X \subset D(I)$.

Proof. Since B is the direct limit of the family $\{A_\lambda\}$, clearly (2) implies (1). Assume that (1) holds, then the map $E(X) \bigotimes_A A_\lambda \to E(X) \bigotimes B$ is injective by flatness of $A \to E(X)$ whence is an isomorphism since $E(X) \to E(X) \bigotimes B$ is bijective by 1.13. It follows that (2) holds. Thus we can assume now that $A \to B$ is finite with conductor I. If (1) is verified, let $P \in X$; from $(B/A)_P = 0$ we deduce that $P \notin \mathrm{Supp}(B/A) = V(I)$ so that $X \subset D(I)$. Thus (1) implies (3). The converse is true since $A_P \to B_P$ is an isomorphism for any P in $D(I)$. Then (3) is equivalent to (4) by 1.15. □

Proposition 1.17. *Let $A \to B$ be an injective morphism and X an admissible subset of* $\mathrm{Spec}\,(A)$*. Let $g : A \to R$ be a ring morphism and set $Y = {}^a g^{-1}(X)$, $S = R \otimes_A B$. If $A \to B$ is an X-isomorphism and $R \to S$ is injective, then $R \to S$ is a Y-isomorphism.*

Proof. For any prime ideal $Q \in Y$ lying over $P \in X$ we have a co-cartesian square

$$\begin{array}{ccc} A_P & \longrightarrow & B_P \\ \downarrow & & \downarrow \\ R_Q & \longrightarrow & S_Q \end{array}$$

The proof follows easily. □

2. The closure with respect to an admissible subset. We are aiming to show that any injective ring morphism $A \to B$ has a closure with respect to any admissible subset of $\mathrm{Spec}\,(A)$.

Theorem 2.1. *Let $A \to B$ be an injective ring morphism and X an admissible subset of* $\mathrm{Spec}\,(A)$ *associated to the flat epimorphism $A \to E$. Define F to be the pullback of the ring morphisms $B \to B \otimes_A E$ and $E \to B \otimes_A E$ so that there is a cartesian and co-cartesian* *commutative square*

$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[d, "\varepsilon"'] & B \arrow[d] \\ E \arrow[r] & B \bigotimes_A E. \end{tikzcd}$$

(1) *There is a factorization* $A \to F \to B$ *by injective morphisms and* F *is the largest* A*-subalgebra* C *of* B *such that* $A \to C$ *is an* X*-isomorphism.*

(2) *There is a factorization* $A \to F \to E$ *where* $F \to E$ *is a flat epimorphism.*

Proof. The two factorizations exist because F is a pullback. Now $E \to B \otimes_A E$ is injective by flatness of $A \to E$. Therefore $F \to B$ is injective. That F is a pullback is equivalent to the exactness of the sequence of F-modules

$$F \xrightarrow{\lambda} B \times E \xrightarrow{\mu} B \bigotimes_A E$$

where $\lambda(x) = (\varphi(x), \varepsilon(x))$ and $\mu(b, e) = b \otimes 1 - 1 \otimes e$. Then $A \to F$, $A \to E$ and $A \to B$ define A-modules. It is easy to check that λ and μ are morphisms of A-modules. Take any P in X and tensorize the exact sequence above by A_P. By flatness we get an exact sequence so that F_P is the pullback associated with $E_P \to B_P \otimes_{A_P} E_P$ and $B_P \to B_P \otimes_{A_P} E_P$. The last morphism is an isomorphism since so is $A_P \to E_P$. Therefore, $F_P \to E_P$ is an isomorphism since F_P is a pullback. It follows that $A_P \to F_P$ is an isomorphism. Thus $A \to F$ is an X-isomorphism. Now let C be an A-subalgebra of B such that $A \to C$ is an X-isomorphism. By 1.13, $E \to E \otimes_A C$ is an isomorphism so that there is a factorization $A \to C \to E$. Thus we get a commutative square

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \\ E \arrow[r] & B \bigotimes_A E. \end{tikzcd}$$

Using one more time that F is a pullback, we get a factorization $A \to C \to F$ by the universal property of such a pullback. To end, observe that $E \to E \otimes_A F$ is an isomorphism and $F \to E \otimes_A F$ a flat epimorphism. Thus $F \to E$ is a flat epimorphism. $\square$

Definition 2.2. Let $A \to B$ be an injective ring morphism and X an admissible subset of $\mathrm{Spec}\,(A)$. We set $F = \overset{X}{\underset{B}{}} A$ where F is the A-algebra defined in 2.1. Then $\overset{X}{\underset{B}{}} A$ is called the X-closure of A in B.

Clearly, $A \to B$ is an X-isomorphism if and only if $B = \overset{X}{\underset{B}{}} A$. We say that $A \to B$ is X-closed if $A = \overset{X}{\underset{B}{}} A$.

Theorem 2.3. *Let $A \to B$ be an injective ring morphism with conductor $\mathfrak{C}$ and X an admissible subset of* $\mathrm{Spec}\,(A)$. *Let b be an element of B. The following statements are equivalent*:

(1) *b lies in $\overset{X}{\underset{B}{}} A$.*

(2) *There exists some ideal, respectively finitely generated ideal, I of A such that $X \subset D(I)$ and $Ib \subset A$.*

(3) *$b/1$ lies in* $\mathrm{im}\,(A_P \to B_P)$ *for any $P \in X$.*

In particular, $A \to B$ is X-closed if and only if an element $b \in B$ lies in A whenever there is some ideal I of A such that $X \subset D(I)$ and $Ib \subset A$. Moreover, $A \to B$ is an X-isomorphism when $X \subset D(\mathfrak{C})$ or $X = \varnothing$ and is X-closed when $X =$ $\mathrm{Spec}\,(A)$.

Proof. To begin with, notice that if $X \subset D(I)$ for some ideal I, there is a finitely generated ideal $J \subset I$ such that $X \subset D(J)$. Indeed, we have $X \subset \cup_{a \in I} D(a)$, X is patch closed and $D(a)$ is open in the patch topology. Moreover, the patch topology is compact. Define C to be the set of all elements $b \in B$ such that $Ib \subset A$ for some ideal I such that $X \subset D(I)$. Obviously, C is an A-subalgebra of B. Let $b \in C$ and consider any prime ideal $P \in X$, there is some $s \in I \setminus P$ such that $sb \in A$. It follows that $A_P \to C_P$ is bijective whence $C \subset \overset{X}{\underset{B}{}} A = F$. Let $A \to E$ be the flat epimorphism associated to X. Now an element $b \in B$ lies in F if and only if there is some $e \in E$ such that $e \otimes 1 = 1 \otimes b$ in $E \otimes_A B$. Since $\varepsilon : A \to E$ is a flat epimorphism, there is a finitely generated ideal J in A such that $JE = E$ and $Je \subset \varepsilon(A)$ by [**11**, 3.1].

Hence for any element $a \in J$ we get $1 \otimes (ab - a\alpha) = 0$ where $a \cdot e = \varepsilon(\alpha)$. By flatness of $A \to E$, there exist a finite family $(a_\lambda) \in A$ and a finite family $(e_\lambda) \in E$ such that $1 = \Sigma\, a_\lambda \cdot e_\lambda$ and $a_\lambda ab = a_\lambda \alpha$ [**3**, Section 2, no. 11, Proposition 13]. If K is the ideal of A generated by the elements a_λ, we get $KE = E$ and $aKb \subset A$. Now set $I = JK$. This ideal is finitely generated and $E = IE$ so that $X \subset D(I)$ by 1.15. Moreover, we have $Ib \subset A$. Therefore, we get ${}^{X}_{B}A \subset C$ which completes the proof of (1) $\Leftrightarrow$ (2). Consider the subring D of all elements $b \in B$ such that (3) holds. Then $A \to D$ is an X-isomorphism by definition of D. Now if $A \to R$ is an X-isomorphism where R is an A-subalgebra of B, then clearly $R \subset D$. Therefore, we have ${}^{X}_{B}A = D$ and (1) $\Leftrightarrow$ (3). $\square$

Example 2.4. Let $f : A \to B$ be an injective ring morphism and S a multiplicative subset of A. Then $X = \cap\,[\,D_A(s)\,;\, s \in S\,] = \mathrm{Spec}\,(A_S)$ is admissible as well as ${}^{a}f^{-1}(X) = \cap\,[\,D_B(s)\,;\, s \in S\,] = \mathrm{Spec}\,(B_S)$. We put ${}^{X}_{B}A = {}^{S}_{B}A$. It follows easily from $E = A_S$ that ${}^{S}_{B}A$ is the set of all elements $b \in B$ such that $sb \in A$ for some $s \in S$. We thus recover Yanagihara's definition [**26**].

Example 2.5. Let $A \to B$ be an injective ring morphism and I an ideal of A. Then for $S = 1 + I$, we get an A-subalgebra $C = {}^{S}_{B}A$ of B (see 1.10 and 1.14). Then C is the largest A-subalgebra C of B such that $A_P \to C_P$ is an isomorphism for any $P \in V(I)$. Using 2.4, we see that b lies in C if and only if there is an ideal J which is comaximal with I and such that $Jb \subset A$.

Definition 2.6. Let I be an ideal of a ring A, $X \subset \mathrm{Spec}\,(A)$ an admissible subset and S a multiplicative subset of A. We denote by

(1) X/I the set of all prime ideals P/I in A/I such that $P \in X$.

(2) X_S the set of all prime ideals P_S in A_S such that $P \in X$.

Then X/I is an admissible subset of $\mathrm{Spec}\,(A/I)$ and X_S is an admissible subset of $\mathrm{Spec}\,(A_S)$. This last statement is an easy consequence of 1.11, (1).

Proposition 2.7. *Let $A \to B$ be an injective ring morphism and X an admissible subset of* $\mathrm{Spec}\,(A)$.

(1) *If S is a multiplicative subset of A, then* ${}^{X_S}_{B_S}A_S = ({}^{X}_{B}A)_S$.

(2) *If I is a common ideal of A and B, then* ${}^{X/I}_{B/I}A/I = ({}^{X}_{B}A)/I$.

Proof. To show (1), we can argue as in the beginning of the proof of 2.1, (1) since $A \to A_S$ is flat and X_S is associated to $A_S \to E_S$ by 1.11, (1). We show (2). We set $F = {}^{X}_{B}A$. Consider the exact sequence $F \to B \times E \to B \otimes_A E$ and tensorize by A/I. We get a 0-sequence $F/IF \to (B/IB) \times (E/IE) \to (B/IB) \otimes_{A/I} (E/IE)$ so that $F/I \subset {}^{X/I}_{B/I}A/I$. Now let C' be any A/I-subalgebra of B/I and $q : B \to B/I$ the canonical morphism. Then $C = q^{-1}(C')$ is an A-subalgebra of B and $C' = C/I$. If $A/I \to C/I$ is an X/I-isomorphism, then $A_P \to C_P$ is an isomorphism for any P/I in X/I that is to say for any $P \in X \cap V(I)$. Now if P lies in $D(I)$, it is well known that $A_P \to C_P$ is an isomorphism. Therefore, we have $C' \subset F/I$ and (2) is proved. $\square$

Corollary 2.8. *Let $A \to B$ be an injective ring morphism and X an admissible subset of* Spec (A).

(1) *$A \to B$ is an X-isomorphism, respectively is X-closed, if and only if $A_P \to B_P$ is an X_P-isomorphism, respectively is X_P-closed, for any prime ideal P in A.*

(2) *If I is a common ideal of A and B, then $A \to B$ is an X-isomorphism, respectively is X-closed, if and only if $A/I \to B/I$ is an X/I-isomorphism, respectively is X/I-closed.*

Proposition 2.9. *Let $g : A \to R$ be a flat morphism, $A \to B$ an injective morphism and X an admissible subset of* Spec (A). *Denote by Y the admissible subset ${}^a g^{-1}(X)$ of* Spec (R). *Then we have*

$$\left({}^{X}_{B}A\right) \bigotimes R = {}^{\quad Y}_{B \bigotimes R}R.$$

In particular, if $A \to B$ is X-closed, then $R \to B \otimes_A R$ is Y-closed.

Proof. We can argue as in the proof of 2.7, (1). $\square$

The above result can be used with $R = A[z]$ where z is an indeterminate over A. We will give a more direct proof and an improvement in a next result.

If $A \to B$ is an X-isomorphism, we will identify X and ${}^a f^{-1}(X)$ since they are homeomorphic by 1.13. *This convention is used in the following results.*

Proposition 2.10. *Let $f : A \to B$, $g : B \to C$ be injective ring morphisms and X an admissible subset of* $\mathrm{Spec}\,(A)$. *Then $A \to C$ is an X-isomorphism if and only if $A \to B$ is an X-isomorphism and $B \to C$ is an X-isomorphism.*

Proof. Straightforward by using 1.11 and 1.13. $\square$

Proposition 2.11. *Let $A \to B$ be an injective ring morphism, X an admissible subset of* $\mathrm{Spec}\,(A)$ *and let $c_X : A \to {}^X_B A$ be the canonical morphism.*

(1) *X is homeomorphic to ${}^a c_X^{-1}(X)$ so that X can be identified with ${}^a c_X^{-1}(X)$.*

(2) *${}^X_B A \to B$ is X-closed.*

(3) *Let C be an A-subalgebra of B, with structural morphism g, such that C is ${}^a g^{-1}(X)$-closed in B, then we have ${}^X_B A \subset C$.*

Proof. (1) follows from 1.13 and (2) from 2.10. We show (3). If $b \in B$ and I is an ideal of A such that $Ib \subset A$ and $X \subset D(I)$, we get ${}^a g^{-1}(X) \subset D(IC)$ and $(IC)b \subset C$ so that $b \in C$ when $C \to B$ is ${}^a g^{-1}(X)$-closed. $\square$

Proposition 2.12. *Let $f : A \to A'$ be a pure morphism, for instance, faithfully flat, and X a subset of* $\mathrm{Spec}\,(A)$ *such that ${}^a f^{-1}(X)$ is admissible.*

(1) *X is admissible.*

(2) *Let $A \to B$ be a ring morphism. If $A' \to B \otimes_A A'$ is ${}^a f^{-1}(X)$-closed and injective, then $A \to B$ is injective and X-closed.*

Proof. (1) is shown in [**15**, 4.8]. Assume that the hypothesis of (2) holds and let there be an element $b \in B$ and an ideal I of A such that $X \subset D(I)$ and $Ib \subset A$. By using ${}^af^{-1}$, we get ${}^af^{-1}(X) \subset {}^af^{-1}(D(I)) = D(IA')$. Moreover, we have $IA'(b \otimes 1) \subset \mathrm{im}\,(A' \to B \otimes_A A')$. It follows that $b \otimes 1 = 1 \otimes a'$ for some a' in A'. Now observe that A is the pullback defined by the morphisms $A' \to B \otimes_A A'$ and $B \to B \otimes_A A'$ by purity of $A \to A'$ [**17**, 2.28]. Thus b lies in A. Hence $A \to B$ is X-closed. □

Definition 2.13. Let $A \to B$ be an injective integral morphism, X an admissible subset of $\mathrm{Spec}\,(A)$ and $b \in B$. Then b is called X-integral if there is some ideal I of A such that $X \subset D(I)$ and $bI \subset A$.

(1) If b is X-integral, then $A \to A[b]$ is said to be an elementary X-isomorphism.

(2) A composite of finitely many elementary X-isomorphisms is said to be a c-elementary X-isomorphism.

The definition makes sense. Indeed, assume that b is a zero of a monic polynomial with degree $n > 0$. The conductor $\mathfrak{C}$ of $A \to A[b]$ is $\cap_{i=1}^{n-1}(A : b^i)$ and X-integrality of b implies that $X \subset D(I) \subset D(\mathfrak{C})$. Since $A \to A[b]$ is finite, we deduce from 1.16 that $A \to A[b]$ is an X-isomorphism. Conversely, if $A \to A[b]$ is an X-isomorphism, then 1.16 shows that $A \to A[b]$ is an elementary X-isomorphism. Clearly, a c-elementary X-isomorphism is an X-isomorphism by 2.10.

Lemma 2.14. *Let $A \to B$ be an injective integral morphism, R and S two A-subalgebras of B. Let X be an admissible subset of* $\mathrm{Spec}\,(A)$*. If $A \to S = A[s]$ is an elementary X-isomorphism and $g : A \to R$ is the structural morphism, then $R \to R[s]$ is an elementary ${}^ag^{-1}(X)$-isomorphism.*

Proof. The relations $(IR)s \subset R$ and ${}^ag^{-1}(X) \subset D(IR)$ follow from $Is \subset A$ and $X \subset D(I)$. □

Corollary 2.15. *Let $A \to B$ be an injective integral morphism and X an admissible subset of* $\mathrm{Spec}\,(A)$*. The set of all A-subalgebras C of*

B such that $A \to C$ is a c-elementary X-isomorphism is directed and ${}^{X}_{B}A$ is the union of all A-subalgebras C of B such that $A \to C$ is a c-elementary X-isomorphism.

Proof. Let $A \to A_n = R \subset B$ and $A \to B_p = S \subset B$ be c-elementary X-isomorphisms and set $B_1 = A[b_1]$. Then by 2.14, $A \to A_n \to A_n[b_1]$ is a c-elementary X-isomorphism and there is a morphism $B_1 \to A_n[b_1]$. Thus the proof follows by induction. There is a c-elementary X-isomorphism $A \to T$ such that R, $S \subset T$. $\square$

Remark 2.16. Let $f : A \to A'$ be a ring morphism and $X \subset \mathrm{Spec}\,(A)$, $X' \subset \mathrm{Spec}\,(A')$ admissible subsets such that ${}^{a}f(X') \subset X$. Assume that there is a commutative square of ring morphisms with injective vertical arrows

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ B \arrow[r] & B'. \end{tikzcd}$$

Then it is easy to show by using 2.3, (2) that there is a commutative square of ring morphisms

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ {}^{X}_{B}A \arrow[r] & {}^{X'}_{B'}A'. \end{tikzcd}$$

Proposition 2.17. *Let $A \to B$ be an injective ring morphism, X an admissible subset of* $\mathrm{Spec}\,(A)$ *and $\mathcal{F} = \{X_\lambda\}$ a family of admissible subsets of* $\mathrm{Spec}\,(A)$.

(1) *If $Y \supset X$ is an admissible subset of* $\mathrm{Spec}\,(A)$, *then ${}^{Y}_{B}A \subset {}^{X}_{B}A$.*

(2) *If $X = \cap X_\lambda$ and $\mathcal{F}$ is stable under finite intersections, then ${}^{X}_{B}A = \cup\, {}^{X_\lambda}_{B}A$.*

(3) *If $X = \cup X_\lambda$, then ${}^{X}_{B}A = \cap\, {}^{X_\lambda}_{B}A$.*

Proof. (1) follows immediately from 2.3, (2). We deduce from (1) that ${}^{X}_{B}A \supset \cup\, {}^{X_\lambda}_{B}A$ under the hypothesis of (2). Next we show the converse.

If $b \in {}^X_B A$, there is some ideal I of A such that $X \subset D(I)$ and $Ib \subset A$. Then the subsets X_λ and $V(I)$ are closed in the patch topology. This topology being compact, by the finite intersection property, there is some index λ such that $X_\lambda \subset D(I)$ so that $b \in {}^{X_\lambda}_B A$. The hypotheses of (3) being granted, we get by (1) that ${}^X_B A \subset \cap {}^{X_\lambda}_B A$. We show the converse. Let b in $\cap {}^{X_\lambda}_B A$, there exist ideals I_λ of A such that $X_\lambda \subset D(I_\lambda)$ and $I_\lambda b \subset A$. Letting $I = \sum I_\lambda$, we get $X \subset D(I)$ and $Ib \subset A$. $\square$

Remark 2.18. If $\mathrm{Spec}\,(A) = \cup X_\lambda$ where every X_λ is admissible, then $A \to B$ is an isomorphism if and only if $A \to B$ is an X_λ-isomorphism for each λ.

Another construction of the X-closure can be given by using a new pullback. We recall some constructions given in [**14**, IV]. Let A be a ring and $X \subset \mathrm{Spec}\,(A)$. The flat topology on $\mathrm{Spec}\,(A)$ assigns to X the closure

$$X^\circ = \cap\,[\,D(I)\,;\; X \subset D(I) \text{ and } D(I) \text{ quasi-compact}\,].$$

Therefore, X is closed for the flat topology if and only if X is quasi-compact and stable under generalizations. Let z be an indeterminate over A. Then the multiplicative subset Σ_X of $A[z]$ is defined to be the set of all polynomials $p(z)$ such that $X \subset D(c(p(z)))$ where $c(p(z))$ is the content ideal of $p(z)$, generated by the coefficients of $p(z)$. Then we set $A_X = A[z]_{\Sigma_X}$. The canonical morphism $A \to A_X$ is flat with spectral image X°.

Now if X is an admissible subset of $\mathrm{Spec}\,(A)$ associated to the flat epimorphism $A \to E$, we have a factorization $A \to E \to A_X$ where $E \to A_X$ is faithfully flat [**14**, IV, Proposition 7].

Proposition 2.19. *Let $f : A \to B$ be an injective integral morphism, $X \subset \mathrm{Spec}\,(A)$ an admissible subset and set $Y = {}^a f^{-1}(X)$. There is a co-cartesian square*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ A_X & \longrightarrow & B_Y. \end{array}$$

Moreover, ${}^{X}_{B}A$ *is the pullback associated to the morphisms* $A_X \to B_Y$ *and* $B \to B_Y$*.*

Proof. Read that the co-cartesian square exists in [**14**, IV, Proposition 3]. Now let $F \subset B$ be the pullback defined above. An element $b \in B$ belongs to F if and only if there is a polynomial $s(z) \in \Sigma_X$ such that $s(z)b \in A[z]$ because $B_Y = B[z]_{\Sigma_X}$. To end, use the definition of an element of Σ_X and 2.3. $\square$

Corollary 2.20. *Let* $A \to B$ *be an injective integral morphism and* $X \subset \mathrm{Spec}\,(A)$ *an admissible subset. The following statements are equivalent*:

(1) $A \to B$ *is an* X*-isomorphism.*

(2) $A_X \to B_Y = B[z]_{\Sigma_X}$ *is an isomorphism.*

(3) *For any polynomial* $b(z) \in B[z]$*, there is some* $s(z) \notin \cup\,[\,P[z]\,;\,P \in X]$ *such that* $s(z)b(z) \in A[z]$*.*

Proof. Let $A \to E$ be the flat epimorphism associated to X and set $F = E \otimes_A B$. Observe that there is a co-cartesian square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & F \arrow[d] \\ A_X \arrow[r] & B_Y \end{tikzcd}$$

where $E \to A_X$ is faithfully flat. Indeed, $A \to E$ is an epimorphism so that the above square is commutative. Then use the following result. If $\boxed{1}$ and $\boxed{2}$ are commutative squares and $\boxed{1}$ and $\boxed{1}\boxed{2}$ are co-cartesian, then $\boxed{2}$ is co-cartesian. Therefore, $E \to F$ is an isomorphism if and only if $A_X \to B_Y$ is an isomorphism. Then we get (1) $\Leftrightarrow$ (2) by 1.13. Now the statement (3) expresses that $A_X \to B_Y = B[z]_{\Sigma_X}$ is surjective because $A \backslash \Sigma_X$ is the union of all prime ideals $P[z]$ in $A[z]$ where $P \in X$ (see [**14**, IV, Lemme 2]). $\square$

Next we compute closures with respect to polynomial extensions.

Theorem 2.21. *Let $A \to B$ be an injective morphism, $X \subset \mathrm{Spec}\,(A)$ an admissible subset and z an indeterminate over A. If $j : A \to A[z]$ is the canonical morphism and $Y = {}^a j^{-1}(X)$, we have*

$$\left(\mathop{}_{B}^{X} A\right)[z] = \mathop{}_{B[z]}^{\Sigma_X} A[z] = \mathop{}_{B[z]}^{Y} A[z].$$

Proof. According to 1.17, $A[z] \to \left(\mathop{}_{B}^{X} A\right)[z]$ is a Y-isomorphism so that we have $\left(\mathop{}_{B}^{X} A\right)[z] \subset \mathop{}_{B[z]}^{Y} A[z]$. Now consider an element $p(z) \in \mathop{}_{B[z]}^{Y} A[z]$. There is a finitely generated ideal J of $A[z]$ such that $Y \subset D(J)$ and $Jp(z) \subset A[z]$. Observe that ${}^a j : \mathrm{Spec}\,(A[z]) \to \mathrm{Spec}\,(A)$ is surjective so that ${}^a j(Y) = X$. From $Y \subset D(J)$, we deduce that $X \subset {}^a j(D(J))$. Denote by $c(J) = I$ the content ideal of J, that is to say the ideal $\sum [\, c(p(z))\,;\, p(z) \in J\,]$ of A. We get ${}^a j(D(J)) = D(I)$. In fact, set $J = (p_1(z), \dots, p_h(z))$, we have $D(J) = D(p_1(z)) \cup \cdots \cup D(p_h(z))$. Then ${}^a j(D(p_i(z))) = D(c(p_i(z)))$ by [**14**, III, Corollaire 3]. It follows that ${}^a j(D(J)) = D(c(p_1(z)) + \cdots + c(p_h(z)))$. Obviously, we have $K = c(p_1(z)) + \cdots + c(p_h(z)) \subset I$. Now, it can be seen that $I = c(J)$ is the set of all elements in A which are a coefficient of at least a polynomial in J (this is not quite obvious). Thus an element $a \in I$ is a coefficient of a polynomial $q(z) = b_1(z)p_1(z) + \cdots + b_h(z)p_h(z)$ so that $a \in c(q(z)) \subset \sum c(b_i(z))c(p_i(z)) \subset K$. To summarize, we have $X \subset D(I)$ and I is a finitely generated ideal since $I = K = (\alpha_1, \dots, \alpha_n)$. Now, we intend to show that $\alpha_i^{k_i} c(p(z)) \subset A$ for some integer k_i. Remember that α_i is a coefficient of a polynomial $q(z) \in J$ so that $q(z)p(z) = a(z) \in A[z]$. The "content formula" expresses that there is some integer n such that $c(q(z)p(z))c(q(z))^n = c(p(z))c(q(z))^{n+1}$. We deduce that $\alpha_i^{n+1} c(p(z)) \subset c(a(z))c(q(z))^n \subset A$ whence $\alpha_i^{k_i} c(p(z)) \subset A$ for some integer k_i. Set $k = \sup(k_i)$, we get $I^{n(k-1)+1} c(p(z)) \subset A$. Setting $H = I^{n(k-1)+1}$, we deduce from $Hc(p(z)) \subset A$ and $X \subset D(H)$ that all the coefficients of $p(z)$ belong to $\mathop{}_{B}^{X} A$. At this stage, we have shown that $\left(\mathop{}_{B}^{X} A\right)[z] = \mathop{}_{B[z]}^{Y} A[z]$.

Next, we claim that $\left(\mathop{}_{B}^{X} A\right)[z] = \mathop{}_{B[z]}^{\Sigma_X} A[z]$. To begin with, let $b \in \mathop{}_{B}^{X} A$. There is a finitely generated ideal $I = (a_0, \dots, a_n)$ of A such that $X \subset D(I)$ and $Ib \subset A$. Letting $s(z) = a_0 + \cdots + a_n z^n$, we have $X \subset D(c(s(z)))$ whence $s(z) \in \Sigma_X$. It follows then that $s(z)(bz^p) \in A[z]$ for any integer p. To conclude, we have gotten $\left({}^X_B A\right)[z] \subset {}^{\Sigma_X}_{B[z]} A[z]$.

Conversely, let $b(z)$ be an element of ${}^{\Sigma_X}_{B[z]} A[z]$. There is some $s(z) \in \Sigma_X$ such that $s(z)b(z) = a(z) \in A[z]$. By using again the content formula, we get for some integer n, $c(s(z)b(z))c(s(z))^n = c(b(z))c(s(z))^{n+1}$. Therefore, letting $I = c(s(z))^{n+1}$, the relation $Ic(b(z)) \subset A$ follows as well as $X \subset D(I)$. Hence the proof is complete since the last relations imply that $b(z) \in \left({}^X_B A\right)[z]$. □

Remark 2.22. A consequence of the preceding result is worth noticing. To some extent, by a faithfully flat base change, the general theory of X-closures can be reduced to the theory of S-closures, S a multiplicative subset.

We end this section by giving some examples of X-closures. The following lemma is crucial in Section 5, when studying X-seminormal or X-t-closed rings.

Lemma 2.23. *Let $A \to B$ be an injective morphism where B is an absolutely flat ring and $X \subset \operatorname{Spec}(A)$ an admissible subset associated to the flat epimorphism $A \to E$. If $C = {}^X_B A$, there exist a common radical pure ideal I of C and B such that $A \to E = A \to C \to C/I = C_{1+I}$, $D_C(I) \cap \mathcal{X}_C(B) \subset \operatorname{Min}(C)$ and a pushout pullback diagram with surjective vertical arrows*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \\ C/I \arrow[r] & B/I. \end{tikzcd}$$

Proof. Denote by $A \to E$ the flat epimorphism associated to X and set $F = E \otimes_A B$. Since the domain of the flat epimorphism $B \to F$ is an absolutely flat ring, $B \to F$ is surjective with kernel I. Then the canonical morphism $C \to E$ is also surjective with kernel I since C is a pullback [**6**, 1.4]. It follows that I is a radical ideal by absolute flatness of B and is pure since $C \to C/I = E$ is a flat surjective map. Moreover, we have $C/I = C_{1+I}$ by 1.10. Therefore, there is a common radical pure ideal I of C and B. Now let P be an element of $D_C(I) \cap \mathcal{X}B)$. There is an isomorphism $C_P \to B_Q$ for some prime ideal Q of B. Then C_P is a field by absolute flatness of B so that P is a minimal prime ideal. $\square$

Proposition 2.24. *Let $A \to B$ be an injective morphism where B is an absolutely flat ring and $X \subset \mathrm{Spec}\,(A)$ an admissible subset. Then $A \to B$ is X-closed if and only if there is a common radical pure ideal I of A and B such that $X = V(I)$.*

Proof. Assume that A is X-closed in B. In view of 2.23, there is a common radical pure ideal of A and B such that $X = V(I)$. Conversely, assume that the preceding statement holds. We have $X = \mathrm{Spec}\,(A_{1+I})$. If b belongs to $\overset{X}{\underset{B}{}}A$, there is some $x \in I$ such that $(1+x)b \in A$. Thus b lies in A since I is an ideal of A and B. Hence A is X-closed in B. $\square$

Proposition 2.25. *Let $A \to B$ be an injective morphism and $X \subset \mathrm{Spec}\,(A)$ an admissible subset associated to the flat epimorphism $A \to E$ such that $\mathcal{X}(B) \subset X$. Then $A \to \overset{X}{\underset{B}{}}A$ can be identified with $A \to E$.*

Proof. According to the hypotheses, a prime ideal Q of B lies over $P \in X$ so that there is some prime ideal R in $E \otimes_A B$ lying over Q. Thus $B \to E \otimes_A B$ is a faithfully flat epimorphism whence an isomorphism. It follows that $\overset{X}{\underset{B}{}}A \to E$ is an isomorphism. $\square$

Remark 2.26. A ring A is called decent when its total quotient ring $\mathrm{Tot}\,(A)$ is absolutely flat [**21**]. A ring A is decent if and only if A is reduced and $\mathrm{Min}\,(A) = \mathcal{X}(\mathrm{Tot}\,(A))$. The above result shows that the X-closure of a decent ring A in its total quotient ring is given by $A \to E$ if $\mathrm{Min}\,(A) \subset X$. Therefore, when $\mathrm{Min}\,(A) \subset X$, a decent ring A is X-closed in its total quotient ring if and only if $X = \mathrm{Spec}\,(A)$.

Remark 2.27. Let $A \to B$ be an injective flat epimorphism and $X = \mathcal{X}(B)$. Then the X-closure of A in B equals B. Indeed, in this case, $\overset{X}{\underset{B}{}}A$ is the dominion of the morphism $A \to B$, that is to say the set of all elements $b \in B$ such that $b \otimes 1 = 1 \otimes b$ in $B \otimes_A B$. But the dominion of an epimorphism $A \to B$ equals B.

Proposition 2.28. *Let A be a ring and $X \subset \operatorname{Spec}(A)$ an admissible subset such that $\operatorname{Ass}(A) \subset X$ (this last property holds when A is an integral domain and $X \neq \varnothing$).*

(1) *The canonical flat morphism $A \to A_X = B$ is injective and $\mathcal{X}(B) = X$.*

(2) *The canonical injective morphism $A \to \overset{X}{\underset{B}{}}A = B'$ is a flat epimorphism such that $\mathcal{X}(B') = X$ and $B' \to A_X$ is faithfully flat.*

Proof. Statement (1) follows from [**10**, II.3.3] and [**14**, IV]. To show (2) we use Morita's method (see 1.9). First observe that $X \subset D(I) \Leftrightarrow IB = B$ for an ideal I of A. Indeed, $IB = B$ is equivalent to $I[z] \cap \Sigma_X \neq \varnothing$. If this last condition holds, we get a polynomial $f(z)$ with coefficients lying in I and such that $X \subset D(c(f(z)))$ whence $X \subset D(I)$. Conversely, from $X \subset D(I)$ and quasi-compactness of X, we get a finitely generated ideal $J \subset I$ such that $X \subset D(J)$. Thus there is a polynomial $f(z) \in I[z]$ such that $X \subset D(c(f(z)))$. Now let B_1 be the subring of all elements $b \in B$ such that $(A :_A b)B = B$. The previous observation shows that $B_1 = \overset{X}{\underset{B}{}}A = B'$. Now we have a factorization $A \to E \to E_A(B) \to B_1$ (where $A \to E$ is a flat epimorphism associated to X) by 1.4 and 1.9. If $P \notin X$ is a prime ideal in A, then $PE = E$ implies $PB_1 = B_1$ while $A_P \to (B_1)_P$ is an isomorphism for $P \in X$. In view of [**10**, IV, 2.4], $A \to B_1$ is a flat epimorphism and, according to 1.9, we have $\mathcal{X}(B_1) = X$. Thanks to 1.5, $E \to B_1$ is an isomorphism. Therefore, $B' \to B$ is faithfully flat [**14**, IV, Proposition 7]. □

Proposition 2.29. *Let $f : A \to B$ be a ring morphism, X an admissible subset of* $\operatorname{Spec}(A)$ *and $Y = {}^a f^{-1}(X)$.*

(1) *If J is an ideal of B, then $Y \subset D(J)$ is equivalent to $X \subset D(f^{-1}(J))$.*

(2) *Assume that $A \to A'$ and $B \to B'$ are injective ring morphisms such that the following diagram is a pullback*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B'. \end{tikzcd}$$

Then we have ${f'}^{-1}({}^{Y}_{B'}B) = {}^{X}_{A'}A$.

Proof. The proof of $X \subset D(f^{-1}(J)) \Rightarrow Y \subset D(J)$ is straightforward. Now assume that $Y \subset D(J)$ and there is a prime ideal $P \in X$ such that $f^{-1}(J) \subset P$. The ring morphism $R = A/f^{-1}(J) \to B/J = S$ is injective. Let P' be the prime ideal in R lying over P and $M' \subset P'$ a minimal prime ideal of R. Since $R \to S$ is injective, there is a minimal prime ideal N' in S lying over M'. Then M' lies over a prime ideal M in A and N' over a prime ideal N in B such that $J \subset N$, $f^{-1}(N) = M \subset P$. Since X is stable under generalizations and $P \in X$, we get $M \in X$. Thus we have $N \in Y \cap V(J)$, a contradiction. Therefore, $X \subset D(f^{-1}(J)) \Leftarrow Y \subset D(J)$ is proved. Assume that we have a cartesian square as in (2). It follows from 2.16 that $f'({}^{X}_{A'}A) \subset {}^{Y}_{B'}B$ whence ${f'}^{-1}({}^{Y}_{B'}B) \supset {}^{X}_{A'}A$. We show the converse inclusion. If a' is in A' and such that $f'(a') \in {}^{Y}_{B'}B$, there is some ideal J of B such that $Jf'(a') \in B$ and $Y \subset D(J)$. By (1) we have $X \subset D(f^{-1}(J))$ and $f(f^{-1}(J)) \subset J$ gives $f'(f^{-1}(J)a') \subset B$. We deduce from this last relation that $f^{-1}(J)a' \subset A$ because the diagram is a pullback so that $a' \in {}^{X}_{A'}A$. Then ${f'}^{-1}({}^{Y}_{B'}B) \subset {}^{X}_{A'}A$ follows. $\square$

Corollary 2.30. *Let $A \to B$ be a ring morphism and $X \subset \operatorname{Spec}(A)$ an admissible subset. Then ${}^{X}_{B}A$ is the inverse image of ${}^{\Sigma_X}_{B[z]}A[z]$ under the canonical morphism $B \to B[z]$.*

3. Infra-integrality and subintegrality along admissible subsets. We are aiming to give a theory for X-seminormality and X-t-closedness where X is an admissible subset. We use mainly Swan's work on seminormality [**25**] and our papers on t-closedness [**16, 17**]. We first need to recall some definitions.

Let $A \to B$ be an injective integral morphism. Then $A \to B$ is said to be infra-integral if its residual extensions are isomorphisms [**16**] and subintegral if $A \to B$ is infra-integral and $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ is bijective [**25**].

Definition 3.1. Let $A \to B$ be an injective morphism and X an admissible subset of $\mathrm{Spec}\,(A)$.

(1) $A \to B$ is said to be X-infra-integral if $A \to B$ is an X-isomorphism and is infra-integral.

(2) $A \to B$ is said to be X-subintegral if $A \to B$ is an X-isomorphism and is subintegral.

Such morphisms are integral.

If $X = \mathcal{X}(A_S)$ where S is a multiplicative subset of A, an X-infra-integral morphism $f : A \to B$ is called an S-infra-integral morphism and similarly for subintegrality. In this case, we have $\mathcal{X}(B_S) = {}^a f^{-1}(\mathcal{X}(A_S))$. Furthermore, f is S-infra-integral if and only if $A_S \to B_S$ is an isomorphism and f is infra-integral.

It follows from this definition that $A \to B$ is X-infra-integral, respectively X-subintegral, if and only if $A \to B$ is infra-integral, respectively subintegral, and $B = {}^X_B A$. Moreover, $A \to B$ is X-infra-integral if and only if for any prime ideal P in A and any prime ideal Q in B lying over P we have

$$\text{if } P \in X, \text{ then } A_P \to B_P \text{ is an isomorphism}$$
$$\text{if } P \notin X, \text{ then } k(P) \to k(Q) \text{ is an isomorphism.}$$

Indeed, the first condition implies that $A_P \to B_Q$ is an isomorphism by 1.13.

Proposition 3.2. *Let $f : A \to B$ be an injective morphism and X an admissible subset of* $\mathrm{Spec}\,(A)$.

(1) *Let S be a multiplicative subset of A. If $A \to B$ is X-infra-integral, respectively X-subintegral, then $A_S \to B_S$ is X_S-infra-integral, respectively X_S-subintegral.*

(2) *$A \to B$ is X-infra-integral, respectively X-subintegral, if and only if $A_P \to B_P$ is X_P-infra-integral, respectively X_P-subintegral, for any prime ideal P in A.*

(3) *If I is a common ideal of A and B, then $A \to B$ is X-infra-integral, respectively X-subintegral, if and only if $A/I \to B/I$ is X/I-infra-integral, respectively X/I-subintegral.*

(4) *If $B \to C$ is an injective morphism, then $A \to C$ is X-infra-integral, respectively X-subintegral, if and only if $A \to B$ is X-infra-integral, respectively X-subintegral, and $B \to C$ is X-infra-integral, respectively X-subintegral.*

Proof. (2) follows from (1). Then (1) and (3) are consequences of 2.7 and [**16**, 1.16]. Now (4) follows from 2.10. □

The following definition uses notions coming from [**25**] and [**16**].

Definition 3.3. Let $A \to A[b] = B$ be an injective ring morphism and X an admissible subset of $\mathrm{Spec}\,(A)$.

(1) $A \to B$ is said to be elementary X-infra-integral if $A \to B$ is an elementary X-isomorphism (an X-isomorphism) and is elementary infra-integral (that is to say there is some $r \in A$ such that $b^2 - rb$, $b^3 - rb^2 \in A$).

(2) $A \to B$ is said to be elementary X-subintegral if $A \to B$ is an elementary X-isomorphism (an X-isomorphism) and is elementary subintegral (that is to say b^2, $b^3 \in A$).

(4) A sequence of finitely many elementary X-infra-integral morphisms is said to be a c-elementary X-infra-integral morphism. A similar definition can be given for subintegrality.

Corollary 3.4. *Let $A \to A[b] = B$ be an injective ring morphism and X an admissible subset of* $\mathrm{Spec}\,(A)$.

(1) *$A \to B$ is elementary X-infra-integral if and only if there is some $r \in A$ such that $b^2 - rb$, $b^3 - rb^2 \in A$ and there is some ideal I in A such that $X \subset D(I)$ and $Ib \subset A$.*

(2) *$A \to B$ is elementary X-subintegral if and only if b^2, $b^3 \in A$ and there is some ideal I in A such that $X \subset D(I)$ and $Ib \subset A$.*

Proof. If $A \to B$ is elementary infra-integral, we have $B = A + Ab$ and the conductor of $A \to B$ is $A : b$. Therefore, $A \to A[b]$ is elementary X-infra-integral if and only if $A \to A[b]$ is elementary infra-integral and $X \subset D(A : b)$ by 1.16. This last condition holds if and only if there is some ideal I of A such that $X \subset D(I)$ and $Ib \subset A$. □

When $X = \mathcal{X}(A_S)$, the conditions $X \subset D(I)$ and $Ib \subset A$ are equivalent to the existence of some $s \in S$ such that $sb \in A$. A similar statement holds for subintegrality.

Proposition 3.5. *Let $A \to B$ be an X-infra-integral morphism and $b \in B$. There is a sequence of elementary X-infra-integral morphisms $A \to A_1 \to \cdots \to A_n \subset B$ such that $b \in A_n$. A similar statement holds for subintegrality.*

Proof. If $b \in B$, by [**16**, 2.5] there is a sequence of infra-integral morphisms $A \to A_1 \to \cdots \to A_n \subset B$ such that $b \in B$. Then use 3.2, (4). □

Remark 3.6. Let $A \to B$ be an injective integral morphism and X an admissible subset of $\operatorname{Spec}(A)$. Let $\{A_\lambda\}$ be the set of all A-subalgebras of finite type of B and denote by I_λ the conductor of $A \to A_\lambda$. Then $A \to B$ is X-infra-integral if and only if $A \to B$ is infra-integral and $X \subset D(I_\lambda)$ for each λ. This follows from 1.16.

4. t-closure and seminormalization along admissible subsets. Let $f : A \to B$ be an injective integral morphism and X an admissible subset of $\operatorname{Spec}(A)$. If P is a prime ideal of A and Q a prime ideal of B lying over P, we denote by $f_P : A_P \to B_P$ and $f_Q : A_P \to B_Q$ the canonical morphisms. The Jacobson radical of a ring C is denoted by $\operatorname{Rad}(C)$.

We define two A-subalgebras ${}^{(X,t)}_{\;\;B}A$ and ${}^{(X,+)}_{\;\;B}A$ of B as follows

$$\overset{(X,+)}{\underset{B}{}}A = \left\{ b \in B;\ \forall P \in \operatorname{Spec}(A) \left\{ \begin{array}{l} \dfrac{b}{1} \in f_P(A_P) + \operatorname{Rad}(B_P) \\ \qquad\qquad\qquad \text{if } P \notin X \\ \dfrac{b}{1} \in f_P(A_P) \text{ if } P \in X \end{array} \right. \right\}$$

$$\overset{(X,t)}{\underset{B}{}} A = \left\{ b \in B; \forall Q \in \mathrm{Spec}\,(B) \begin{cases} \dfrac{b}{1} \in f_Q(A_P) + QB_Q \text{ if } P \notin X \\ \dfrac{b}{1} \in f_P(A_P) \text{ if } P \in X \end{cases} \right\}.$$

When $X = \varnothing$, we recover the classical seminormalization ${}^{+}_{B}A$ of Swan [**25**] and t-closure ${}^{t}_{B}A$ [**16**] of A in B.

If $X = \mathcal{X}(A_S)$, we put ${}^{(X,t)}_{B}A = {}^{(S,t)}_{B}A$ and ${}^{(X,+)}_{B}A = {}^{(S,+)}_{B}A$.

Lemma 4.1. *Let $A \to B$ be an injective integral morphism and X an admissible subset of* Spec (A)*. Then we have*

$$\overset{(X,t)}{\underset{B}{}} A = \overset{X}{\underset{B}{}} A \cap \overset{t}{\underset{B}{}} A \quad \textit{and} \quad \overset{(X,+)}{\underset{B}{}} A = \overset{X}{\underset{B}{}} A \cap \overset{+}{\underset{B}{}} A.$$

Proof. We carry on a proof for t-closedness. Obviously, we have ${}^{(X,t)}_{B}A \subset {}^{X}_{B}A \cap {}^{t}_{B}A$ by 2.3, (3) since $B_P \to B_Q$ exists. The converse is straightforward. $\square$

Definition 4.2. Let $A \to B$ be an injective morphism, X an admissible subset of Spec (A) and $\overline{A}$ the integral closure of A in B. Then ${}^{(X,t)}_{\overline{A}}A$ is called the X-t-closure of A in B and ${}^{(X,+)}_{\overline{A}}A$ the X-seminormalization of A in B. These closures are still denoted by ${}^{(X,t)}_{B}A$ and ${}^{(X,+)}_{B}A$. Notice that $A \to {}^{(X,t)}_{B}A$ and $A \to {}^{(X,+)}_{B}A$ are integral.

Indeed, we can write

$$\overset{(X,t)}{\underset{B}{}} A = \overset{X}{\underset{B}{}} A \cap \overset{t}{\underset{B}{}} A \quad \text{and} \quad \overset{(X,+)}{\underset{B}{}} A = \overset{X}{\underset{B}{}} A \cap \overset{+}{\underset{B}{}} A.$$

This follows from ${}^{t}_{B}A = {}^{t}_{\overline{A}}A \subset \overline{A}$, ${}^{X}_{B}A \cap \overline{A} = {}^{X}_{\overline{A}}A$ (a similar argument can be given for seminormality).

Let $A \to B$ be an injective morphism. We recall that ${}^{t}_{B}A$, respectively ${}^{+}_{B}A$, is the largest A-subalgebra C of B such that $A \to C$ is infra-integral, respectively subintegral, [**25, 16**].

Proposition 4.3. *Let $A \to B$ be an injective morphism and X an admissible subset of* Spec (A)*. Then ${}^{(X,t)}_{B}A$, respectively ${}^{(X,+)}_{B}A$, is* *the greatest A-subalgebra C of B such that $A \to C$ is X-infra-integral, respectively X-subintegral.*

Moreover, we have the canonical factorization

$$A \to {}^{(X,+)}_{\ \ B}A \to {}^{(X,t)}_{\ \ B}A \to B.$$

Proof. Let $f : A \to R$ and $g : R \to S$ be injective morphisms. If $g \circ f$ is infra-integral, respectively an X-isomorphism, so is f by [**16**, 1.14] and 3.2, (4). Hence the factorizations $A \to {}^{(X,t)}_{\ \ B}A \to {}^{X}_{B}A$ and $A \to {}^{(X,t)}_{\ \ B}A \to {}^{t}_{B}A$ show that $A \to {}^{(X,t)}_{\ \ B}A$ is X-infra-integral. Now let $A \to C$ be an X-infra-integral subalgebra of B. Then $A \to C$ is an X-isomorphism so that $C \subset {}^{X}_{B}A$ and is infra-integral so that $C \subset {}^{t}_{B}A$ [**16**, 2.6]. Thus we get $C \subset {}^{(X,t)}_{\ \ B}A$. The proof for subintegrality is similar. $\square$

Theorem 4.4. *Let $A \to B$ be an injective morphism and X an admissible subset of* Spec(A).

(1) *${}^{(X,t)}_{\ \ B}A$ is the t-closure of A in ${}^{X}_{B}A$ and ${}^{(X,+)}_{\ \ B}A$ the seminormalization of A in ${}^{X}_{B}A$.*

(2) *${}^{(X,t)}_{\ \ B}A$ is the X-closure of A in ${}^{t}_{B}A$ and ${}^{(X,+)}_{\ \ B}A$ the X-closure of A in ${}^{+}_{B}A$.*

Proof. Define C to be the t-closure of A in ${}^{X}_{B}A$. Then $A \to C$ is infra-integral and an X-isomorphism by 2.10. Thus we get $C \subset {}^{(X,t)}_{\ \ B}A$. Furthermore, from the factorization $A \to {}^{(X,t)}_{\ \ B}A \to {}^{X}_{B}A$ and infra-integrality of $A \to {}^{(X,t)}_{\ \ B}A$, we deduce that ${}^{(X,t)}_{\ \ B}A \subset C$. Now let D be the X-closure of A in ${}^{t}_{B}A$. Then $A \to D$ is X-infra-integral. Therefore, we have $D \subset {}^{(X,t)}_{\ \ B}A$. In view of 2.1, we deduce from the factorization $A \to {}^{(X,t)}_{\ \ B}A \to {}^{t}_{B}A$ that ${}^{(X,t)}_{\ \ B}A \subset D$ since $A \to {}^{(X,t)}_{\ \ B}A$ is an X-isomorphism. $\square$

Let $A \to B$ be an injective morphism. We recall that $A \to B$ is said to be seminormal, respectively t-closed, if an element $b \in B$ belongs to A whenever b^2, $b^3 \in A$, respectively there is some $r \in A$ such that $b^2 - rb,\ b^3 - rb^2 \in A$.

Then ${}^{+}_{B}A$, respectively ${}^{t}_{B}A$, is the smallest A-subalgebra C of B such that $C \to B$ is seminormal, respectively t-closed, [**25, 16**].

Definition 4.5. Let $A \to B$ be an injective morphism and X an admissible subset of $\operatorname{Spec}(A)$. Denote by $\overline{A}$ the integral closure of A in B.

(1) $A \to B$ is said to be X-t-closed if an element b in B lies in A whenever there exist some $r \in A$ such that $b^2 - rb$, $b^3 - rb^2 \in A$ and some ideal I in A such that $X \subset D(I)$ and $bI \subset A$.

Therefore, $A \to B$ is X-t-closed if and only if $A \to \overline{A}$ is X-t-closed.

(2) $A \to B$ is said to be X-seminormal if an element b in B lies in A whenever b^2, $b^3 \in A$ and there is some ideal I in A such that $X \subset D(I)$ and $bI \subset A$.

Therefore, $A \to B$ is X-seminormal if and only if $A \to \overline{A}$ is X-seminormal.

In particular, we have when $X = \operatorname{Spec}(A_S)$:

(1) $A \to B$ is S-t-closed if and only if an element b in B lies in A whenever there exist some $r \in A$ and $s \in S$ such that $b^2 - rb$, $b^3 - rb^2 \in A$ and $sb \in A$.

(2) $A \to B$ is S-seminormal if and only if an element b in B lies in A whenever b^2, $b^3 \in A$ and there is some s in S such that $sb \in A$.

Theorem 4.6. *Let $A \to B$ be an injective morphism and X an admissible subset of* $\operatorname{Spec}(A)$*. The following statements are equivalent*:

(1) $A \to B$ *is X-t-closed, respectively X-seminormal.*

(2) $A \to {}^{X}_{B}A$ *is t-closed, respectively seminormal.*

(3) $A = {}^{(X,t)}_{B}A$*, respectively* $A = {}^{(X,+)}_{B}A$.

Proof. We give a proof for t-closedness. (1) $\Leftrightarrow$ (2) is straightforward (see Definition 4.5). The condition (2) is equivalent to $A = {}^{t}_{C}A$ where $C = {}^{X}_{B}A$ [**16**, 3.3]. Then we have ${}^{t}_{C}A = {}^{(X,t)}_{B}A$ (see 4.4). $\square$

Theorem 4.7. *Let $A \to B$ be an injective morphism and X an admissible subset of* $\mathrm{Spec}\,(A)$*. Then $A \to B$ is X-t-closed, respectively X-seminormal, if and only if $A \to {}_B^t A$, respectively $A \to {}_B^+ A$, is X-closed.*

Proof. Set $C = {}_B^t A$. We have ${}_C^X A \subset {}_B^X A$ and $C = {}_C^t A = {}_B^t A$ so that ${}_C^X A = {}_C^{(X,t)} A \subset {}_B^{(X,t)} A$. Now, if $A \to B$ is X-t-closed then $A = {}_B^{(X,t)} A$ by 4.6 so that $A = {}_C^X A$ and $A \to {}_B^t A$ is X-closed. Conversely, assume that $A \to {}_B^t A$ is X-closed. Consider an element $b \in B$ such that $b^2 - rb$, $b^3 - rb^2 \in A$ for some $r \in A$ and such that $Ib \subset A$ for some ideal I of A satisfying $X \subset D(I)$. Then $A \to A[b]$ is elementary infra-integral whence $A[b] \subset {}_B^t A$. It follows that b lies in A. Consequently, $A \to B$ is X-t-closed. $\square$

Theorem 4.8. *Let $f : A \to B$ be an injective morphism and X an admissible subset of* $\mathrm{Spec}\,(A)$*. Then ${}_B^{(X,t)} A$, respectively ${}_B^{(X,+)} A$, is the smallest A-subalgebra C of B, with structural morphism $g : A \to C$, such that $g : C \to B$ is ${}^a g^{-1}(X)$-t-closed, respectively ${}^a g^{-1}(X)$-seminormal.*

Proof. We give a proof for t-closedness. Letting $D = {}_B^{(X,t)} A$, we see that $A \to D$ is X-infra-integral as well as $D \to {}_B^{(X,t)} D$ and so is $A \to {}_B^{(X,t)} D$ by 3.2, (4). Thus we get ${}_B^{(X,t)} D \subset {}_B^{(X,t)} A = D$ by 4.3 so that $D \to B$ is X-t-closed. Let C be an A-subalgebra of B such that $C \to B$ is ${}^a g^{-1}(X)$-t-closed and set $Y = {}^a g^{-1}(X)$. It follows that $C = {}_B^{(Y,t)} C = {}_B^t C \cap {}_B^Y C$ according to 4.2 and 4.6. In view of [**16**, 3.5], we have ${}_B^t A \subset {}_B^t C$ since ${}_B^t C \to B$ is t-closed. Moreover, we have also ${}_B^X A \subset {}_B^Y C$. Indeed, if $b \in {}_B^X A$, there is some ideal I of A such that $X \subset D(I)$ and $bI \subset A$. But $bIC \subset C$ and $Y \subset {}^a g^{-1}(D(I)) = D(IC)$ show that b lies in ${}_B^Y C$. Therefore, ${}_B^{(X,t)} A = {}_B^t A \cap {}_B^X A \subset C$ follows. $\square$

Theorem 4.9. *Let $A \to B$ be an injective morphism and X an admissible subset of* $\mathrm{Spec}\,(A)$*. Let C be an A-subalgebra of B such that $A \to C$ is X-infra-integral, respectively X-subintegral, and $C \to B$ is X-t-closed, respectively X-seminormal, then C is the X-t-closure,* *respectively X-seminormalization, of A in B. Moreover, these closures are contained in $\overline{A}$. In particular, $A \to B$ is an isomorphism when $A \to B$ is X-infra-integral and X-t-closed, respectively X-subintegral and X-seminormal.*

Proof. X-infra-integrality of $A \to C$ gives $C \subset {}^{(X,t)}_{B}A$ by 4.3. Moreover, we have ${}^{(X,t)}_{B}A \subset C$ by 4.8. □

Proposition 4.10. *Let $A \to B$ be an injective morphism and X an admissible subset of* Spec (A).

(1) *Let $f : A \to E$ be a flat epimorphism, $F = E \otimes_A B$ and $Y = {}^a f^{-1}(X)$, then we have*

$$ {}^{(Y,t)}_{\;\;F}E = \left({}^{(X,t)}_{\;\;B}A \right) \bigotimes_A E \quad \textit{and} \quad {}^{(Y,+)}_{\;\;F}E = \left({}^{(X,+)}_{\;\;B}A \right) \bigotimes_A E. $$

(2) *If S is any multiplicative subset of A, then we have*

$$ {}^{(X_S,t)}_{\;\;B_S}A_S = \left({}^{(X,t)}_{\;\;B}A \right)_S \quad \textit{and} \quad {}^{(X_S,+)}_{\;\;B_S}A_S = \left({}^{(X,+)}_{\;\;B}A \right)_S. $$

(3) *If I is any common ideal of A and B, then we have*

$$ {}^{(X/I,t)}_{\;\;B/I}A/I = \left({}^{(X,t)}_{\;\;B}A \right)/I \quad \textit{and} \quad {}^{(X/I,+)}_{\;\;B/I}A/I = \left({}^{(X,+)}_{\;\;B}A \right)/I. $$

(4) *If z is an indeterminate, letting $Y = {}^a j^{-1}(X)$ where $j : A \to A[z]$ is the canonical morphism, we have*

$$ \left({}^{(X,t)}_{\;\;B}A \right)[z] = {}^{(Y,t)}_{\;\;B[z]}A[z] = {}^{(\Sigma_X,t)}_{\;\;B[z]}A[z] $$
$$ \left({}^{(X,+)}_{\;\;B}A \right)[z] = {}^{(Y,+)}_{\;\;B[z]}A[z] = {}^{(\Sigma_X,+)}_{\;\;B[z]}A[z]. $$

Proof. In view of 4.2, the X-t-closure is the intersection of the X-closure and the t-closure. Then $\left({}^{X}_{B}A \right) \otimes_A E = {}^{Y}_{F}E$ by 2.9 and

$\left({}^{t}_{B}A\right) \otimes_A E = {}^{t}_{F}E$ by [**21**, 5.5] combine to yield

$$\overset{(Y,t)}{\underset{F}{}}E = \left(\overset{X}{\underset{B}{}}A\right) \bigotimes_A E \bigcap \left(\overset{t}{\underset{B}{}}A\right) \bigotimes_A E = \left(\overset{X}{\underset{B}{}}A \cap \overset{t}{\underset{B}{}}A\right) \bigotimes_A E$$

by flatness of $A \to E$. Thus (1) is obtained and (2) follows. Now (3) and (4) are consequences of 2.7, 2.9, 2.21 and similar properties which hold for t-closure [**16**]. $\square$

Proposition 4.11. *Consider a cartesian square of ring morphisms*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

Let $X \subset$ Spec(A) be an admissible subset and set $Y = {}^a g^{-1}(X)$. When $A' \to B'$ is Y-seminormal, respectively Y-t-closed, $A \to B$ is X-seminormal, respectively X-t-closed.

Proof. If $A' \to B'$ is injective so is $A \to B$. Assume that $A' \to B'$ is Y-t-closed. Let b in B and assume that there is some a in A and some ideal I in A such that $b^2 - ab$, $b^3 - ab^2 \in A$, $bI \subset A$ and $X \subset D(I)$. Then $h(b)^2 - g(a)h(b)$, $h(b)^3 - g(a)h(b)^2$ lie in A', $(I \cdot A')h(b) \subset A'$ and we have $Y \subset D(IA')$. Thus $h(b)$ lies in A' so that b lies in A. Consequently, $A \to B$ is X-t-closed. $\square$

Proposition 4.12. *Let $A \to B$ be an injective morphism and X an admissible subset of* Spec(A). *Let $g : A \to A'$ be a pure morphism and set $Y = {}^a g^{-1}(X)$ and $B' = A' \otimes_A B$. If $A' \to B'$ is Y-t-closed, respectively Y-seminormal, then $A \to B$ is X-t-closed, respectively X-seminormal.*

Proof. According to [**17**, 2.19], we have a cartesian square as in 4.11. The result follows from 4.11. $\square$

Definition 4.13. Let $A \to B$ be an injective ring morphism, X an admissible subset of Spec(A) and $b \in B$. The element b is said to be X-infra-integral, respectively X-subintegral, over A if there is a sequence of elementary X-infra-integral, respectively X-subintegral, morphisms $A \to A_1 \to \cdots \to A_n \subset B$ such that $b \in A_n$.

We can observe that the set of all A-subalgebras C of B such that $A \to C$ is c-elementary X-infra-integral, respectively X-subintegral, is directed. To see this, use 2.14, similar results for t-closedness and seminormality [**25, 16**] and argue as in the proof of 2.15.

Theorem 4.14. *If $A \to B$ is an injective morphism, then ${}^{(X,t)}_{\ \ B}A$, respectively ${}^{(X,+)}_{\ \ B}A$, is the set of all X-infra-integral, respectively X-subintegral, elements of B.*

Proof. Let b be an X-infra-integral element. There is a sequence of elementary X-infra-integral morphisms $A \to A_n \subset B$ such that $b \in A_n$. Then A_n is contained in ${}^{(X,t)}_{\ \ B}A$ by 4.3 and so is b. The converse is obtained from 3.5 since $A \to {}^{(X,t)}_{\ \ B}A$ is X-infra-integral (see 4.3). □

Proposition 4.15. *Let $f : A \to B$ and $g : B \to C$ be injective ring morphisms and $X \subset \mathrm{Spec}\,(A)$ an admissible subset. If $A \to B$ is X-t-closed, respectively X-seminormal, and $B \to C$ is ${}^af^{-1}(X)$-t-closed, respectively ${}^af^{-1}(X)$-seminormal, then $A \to C$ is X-t-closed, respectively X-seminormal. Conversely, if $A \to C$ is X-t-closed, respectively X-seminormal, so is $A \to B$.*

Proof. Straightforward. □

Theorem 4.16. *Let $f : A \to B$ be an injective integral morphism with conductor $\mathfrak{C}$ and $X \subset \mathrm{Spec}\,(A)$ an admissible subset.*

We set $X' = \mathrm{Spec}\,(A) \setminus X$, $Y = {}^af^{-1}(X)$ and $Y' = \mathrm{Spec}\,(B) \setminus Y = {}^af^{-1}(X')$.

(1) *Assume that ${}^{(X,+)}_{\ \ B}A = {}^{(X,t)}_{\ \ B}A$. There exists an injective map*

$$\theta : \mathrm{Min}\,(V_A(\mathfrak{C})) \cap X' \to \mathrm{Min}\,(V_B(\mathfrak{C})) \cap Y'$$

such that ${}^af(\theta(P)) = P$.

(2) *If the map $Y' \to X'$ induced by ${}^a f$ is bijective, then* ${}^{(X,+)}_{\ \ B}A = {}^{(X,t)}_{\ \ B}A$.

Proof. Suppose that ${}^{(X,+)}_{\ \ B}A = {}^{(X,t)}_{\ \ B}A$ and consider $P \in \mathrm{Min}\,(V_A(\mathfrak{C})) \cap X'$. Since $A/\mathfrak{C} \to B/\mathfrak{C}$ is integral and injective, there is a prime ideal $Q \in \mathrm{Min}\,(V_B(\mathfrak{C})) \cap Y'$ lying over P by incomparability in integral extensions. We need only to show that Q is unique to get (1). Indeed $\theta(P) = Q$ fits. In view of 4.10, (3), we can reduce to the case $\mathfrak{C} = 0$ and P is a minimal prime ideal of A. If $A_P \to B_P$ is an isomorphism, we are done. If not, let J be the conductor of the injective integral morphism $A_P \to B_P$. We have $\mathrm{Spec}\,(A_P) = \{PA_P\}$ and $PA_P \in \mathrm{Min}\,(V(J))$. Besides, X_P is empty so that ${}^{X_P}_{B_P}A_P = B_P$. It follows from 4.2 and 4.10, (2) that ${}^{+}_{B_P}A_P = {}^{t}_{B_P}A_P$. Now (1) is a consequence of [**16**, 3.11] because there is a unique prime ideal in B_P lying over PA_P. Next we show (2). The map $Y' \to X'$ is surjective since $A \to B$ is injective and integral. Assume that $Y' \to X'$ is injective and that ${}^{(X,+)}_{\ \ B}A \neq {}^{(X,t)}_{\ \ B}A$. Then there is a prime ideal P in A and two prime ideals in ${}^{(X,t)}_{\ \ B}A$ lying over P. Assume not; then $A \to {}^{(X,t)}_{\ \ B}A$ is spectrally injective whence subintegral so that ${}^{(X,t)}_{\ \ B}A \subset {}^{(X,+)}_{\ \ B}A$, a contradiction. Now observe that ${}^{(X,t)}_{\ \ B}A$ is an A-subalgebra of ${}^{X}_{B}A$. There are again two prime ideals of ${}^{X}_{B}A$ lying over P. It follows that P does not lie in X (if not, $A_P \to \left({}^{X}_{B}A\right)_P$ is an isomorphism). This leads to a contradiction and we have ${}^{(X,+)}_{\ \ B}A = {}^{(X,t)}_{\ \ B}A$. ◻

Corollary 4.17. *Let $f : A \to B$ be a finite injective birational morphism between one-dimensional integral domains and $X \subset \mathrm{Spec}\,(A)$ an admissible subset. Then ${}^{(X,+)}_{\ \ B}A = {}^{(X,t)}_{\ \ B}A$ if and only if $Y' \to X'$ is bijective.*

Proof. We need only to show an implication by 4.16, (2). Assume that ${}^{(X,+)}_{\ \ B}A = {}^{(X,t)}_{\ \ B}A$. In view of the hypotheses on $A \to B$, the conductor $\mathfrak{C}$ is nonzero. Therefore, the prime ideals in A and B, containing $\mathfrak{C}$, belong to $\mathrm{Min}\,(V(\mathfrak{C}))$. According to 4.16, (1), there is a bijection $V_A(\mathfrak{C}) \cap X' \to V_B(\mathfrak{C}) \cap Y'$ so that the restriction $V_B(\mathfrak{C}) \cap Y' \to V_A(\mathfrak{C}) \cap X'$ of ${}^a f$ is injective. Moreover, $D_B(\mathfrak{C}) \to D_A(\mathfrak{C})$ is bijective since $\mathfrak{C}$ is the conductor. It follows that $Y' \to X'$ is bijective. $\square$

Proposition 4.18. *Let $A \to B$ be an injective t-closed, respectively seminormal, morphism and $X \subset \operatorname{Spec}(A)$ an admissible subset. Then ${}_B^X A \to B$ is X-t-closed and t-closed, respectively X-seminormal and seminormal.*

Proof. We give a proof for t-closedness. First, ${}_B^X A \to B$ is X-closed so that this morphism is X-t-closed. Let $x \in B$ such that $x^2 - rx = a$, $x^3 - rx^2 = b \in {}_B^X A$ for some $r \in {}_B^X A$. There exist ideals I, J, K of A such that $X \subset D(I)$, $D(J)$, $D(K)$ and Ia, Jb, $Kr \subset A$. Set $L = IJK$ so that $La, Lb, Lr \subset A$ and $X \subset D(L)$. For any $z \in L$ we get $(zx)^2 - (zr)(zx)$, $(zx)^3 - (zr)(zx)^2 \in A$ so that $zx \in A$ since $A \to B$ is t-closed. It follows that x lies in ${}_B^X A$ because $Lx \subset A$. Thus ${}_B^X A \to B$ is t-closed. $\square$

Proposition 4.19. *Let $A \to B$ be an injective ring morphism with conductor I such that $X \subset D(I)$. Then $A \to B$ is X-t-closed, respectively X-seminormal, if and only if $A \to B$ is t-closed, respectively seminormal.*

Proof. Indeed, when $X \subset D(I)$, we get for any $b \in B$ that $Ib \subset A$. In this case, ${}_B^X A = B$ so that ${}_B^t A = {}_B^{(X,t)} A$. $\square$

Proposition 4.20. *Let $A \to B$ be an injective morphism and $X \subset \operatorname{Spec}(A)$ an admissible subset such that $A \to B$ is X-t-closed, respectively X-seminormal.*

(1) *$A_S \to B_S$ is X_S-t-closed, respectively X_S-seminormal, for any multiplicative subset S of A.*

(2) *$A[z] \to B[z]$ is Σ_X-t-closed, respectively Σ_X-seminormal, for an indeterminate z over A.*

Proof. Use 4.10. $\square$

The following results show that X-t-closedness or X-seminormality can be often reduced to local t-closedness or seminormality.

Theorem 4.21. *Let $A \to B$ be an injective morphism and $X \subset$ Spec (A) an admissible subset.*

(1) *If $A_P \to B_P$ is t-closed, respectively seminormal, for every $P \in$ Spec $(A) \setminus X$, then $A \to B$ is X-t-closed, respectively X-seminormal.*

(2) *The converse is true if A and B are one-dimensional weak Baer rings and $Bool(A) = Bool(B)$.*

Proof. We denote by Z' the localization of an object Z with respect to a prime ideal P of A. Assume that $A' \to B'$ is t-closed for any P in Spec $(A) \setminus X$. It follows that $A' \to {}^{X'}_{B'}A'$ is t-closed. Now if P belongs to X, we have $A' = ({}^{X}_{B}A)'$ by 2.1. In any case $A_P \to \left({}^{X}_{B}A\right)_P$ is t-closed. Thus $A \to {}^{X}_{B}A$ is t-closed because t-closedness localizes and globalizes [**16**]. Therefore, $A \to B$ is X-t-closed by 4.6. Now we show (2). Assume that the hypotheses of (2) hold. If P is an element of Spec $(A) \setminus X$, we know that A' is an integral domain (see Section 0). Then $A' \to B'$is X'-t-closed, $P' \notin X'$ and Spec $(A') = \{0, P'\}$. If $A' \to B'$ is not t-closed, there is an elementary infra-integral morphism $A' \to A'[x] = C \subset B'$ with $x \notin A'$. Let $r \in A'$ such that $x^2 - rx = a$, $x^3 - rx^2 = b \in A'$ and denote by I the conductor of $A' \to C$. Observe that $C = A' + A'x$ and $ax = b$ so that a lies in I. First assume that $a \neq 0$. In this case, we have Min $(V_{A'}(I)) = V_{A'}(I) = \{P'\}$. Moreover, $A' \to C$ is X'-t-closed and x does not belong to A'. We deduce from 1.16 that $X' \not\subset D_{A'}(I) = \{0\}$ since $A' \to C$ is finite and $A' \to C$ is not X'-infra-integral. But in this case P' belongs to X' which is absurd. Therefore, $A' \to B'$ is t-closed. Now assume that $a = 0$. We have $x(x - r) = 0$. Setting $x = \frac{\beta}{t}$ and $r = \frac{u}{t}$ where $\beta \in B$, $u \in A$ and $t \in A \setminus P$, it follows that there is some $v \in A \setminus P$ such that $(v\beta)^2 - (vu)(v\beta) = 0$. If e is an idempotent in A and B such that $0 : v\beta = Be$, we get $v\beta e = 0$ and $v\beta - vu = ye$ for some $y \in B$ which combine to yield $0 = v\beta e = vue + ye$. Thus we have $ye = -vue \in A$ so that $v\beta \in A$. Hence we are led to the contradiction $x = \frac{v\beta}{vt} \in A'$. Therefore, in any case, $A' \to B'$ is t-closed. $\square$

Corollary 4.22. *Let $A \to B$ be a finite injective birational morphism between one-dimensional integral domains with conductor $I \neq A$ and*

$X \subset \mathrm{Spec}\,(A)$ *an admissible subset. Then* $A \to B$ *is* X*-*t*-closed if and only if the only maximal ideal in* A_P *and* B_P *is* I_P*, respectively* PA_P*, for every* $P \in \mathrm{Spec}\,(A) \setminus X$.

Proof. According to [**16**, 4.7], $A_P \to B_P$ is t-closed if and only if PA_P is the only maximal ideal in A_P and B_P. To conclude use 4.21. $\square$

Corollary 4.23. *Let* $f : A \to B$ *and* $g : B \to C$ *be finite injective birational morphisms between one-dimensional integral domains and* $X \subset \mathrm{Spec}\,(A)$ *an admissible subset. Letting* $Y = {}^a f^{-1}(X)$*, then* $A \to C$ *is* X*-*t*-closed if and only if* $A \to B$ *is* X*-*t*-closed and* $B \to C$ *is* Y*-*t*-closed.*

Proof. Thanks to 4.15, we need only to show that $C \to B$ is Y-t-closed when $A \to C$ is X-t-closed. Let $Q \in \mathrm{Spec}\,(B) \setminus Y$ be a prime ideal lying over $P \in \mathrm{Spec}\,(A) \setminus X$. Then $A_P \to C_P$ is t-closed and so is $A_P \to B_P$; then PA_P is the only maximal ideal of B_P by 4.22. Moreover, $B_P \to C_P$ is t-closed [**16**, 4.8]. Therefore, $B_Q \to C_Q$ is t-closed. The conclusion follows from 4.21, (1). $\square$

We give now two results characterizing X-t-closed or X-seminormal morphisms $A \to B$. The first one can be obtained by using that $A \to B$ is X-t-closed, respectively X-seminormal, if and only if $A \to {}^X_B A$ is t-closed, respectively seminormal, and [**16, 25**]. Remember also that $\mathrm{Ass}_A(M) \subset \mathrm{Supp}_A(M) \subset \mathrm{S}(\mathrm{Ass}_A(M))$ for an A-module M (S denotes the specialization operator).

Proposition 4.24. *Let* $A \to B$ *be an injective morphism and* $X \subset \mathrm{Spec}\,(A)$ *an admissible subset. The following statements are equivalent*:

(1) $A \to B$ *is* X*-*t*-closed.*

(2) $A_P \to B_P$ *is* X_P*-*t*-closed for all* $P \in \mathrm{Spec}\,(A)$.

(3) $A_P \to B_P$ *is* X_P*-*t*-closed for all* $P \in \mathrm{Max}\,(A)$.

(4) $A_P \to B_P$ *is* X_P*-*t*-closed for all* $P \in \mathrm{Supp}_A(B/A)$.

(5) $A_P \to B_P$ *is* X_P-*t-closed for all* $P \in \operatorname{Ass}_A(B/A)$.

Similar statements hold for seminormality.

Theorem 4.25. *Let* $f : A \to B$ *be an injective morphism and* $X \subset \operatorname{Spec}(A)$ *an admissible subset. The following statements are equivalent*:

(1) $A \to B$ *is* X-*t-closed.*

(2) $A/J \to B/J$ *is* (X/J)-*t-closed for some common ideal* J *of* A *and* B.

(3) *For every finite morphism* $g : A \to C \subset B$ *and conductor* J *such that* $X \subset D(J)$, *there is an injection* $\theta : \operatorname{Min}(V_A(J)) \to \operatorname{Min}(V_C(J))$ *such that* ${}^a g \circ \theta = \operatorname{Id}$ *and* J *is a radical ideal of* C.

Proof. The equivalence of (1) and (2) is obvious (see 4.10). Assume that $A \to B$ is X-t-closed and let $A \to C$ be an A-algebra satisfying the hypotheses of (3). Then $A \to C$ is t-closed by 4.15 and 1.16. Indeed, we have $A = {}^X_C A \cap {}^t_C A$ and $C = {}^X_C A$. Hence $A \to C$ is seminormal so that J is a radical ideal in C [**25**]. Then the existence of θ is given by [**16**, 3.11] because $A = {}^+_B A = {}^t_B A$. Thus we have proved (1) $\Rightarrow$ (3). To show the converse, we can reduce to the case where $A \to B$ is integral. Assume that the hypotheses of (3) hold. It is enough to show that $A \to {}^X_B A$ is t-closed. Then the t-closedness criterion (2) of [**16**, 3.15] applies. Indeed, for any $b \in {}^X_B A \setminus A$, we have $C = A[b] \subset {}^X_B A$. Hence $A \to C$ is a finite X-isomorphism with conductor J so that $X \subset D(J)$ by 1.16. $\square$

The next result is given only for S-t-closedness, S a multiplicative subset, because things are complicated for an ordinary admissible subset.

Proposition 4.26. *Let* $A \to B$ *be an injective ring morphism and* S *a multiplicative subset of* A.

(1) $A \to B$ *is* S-*t-closed if* $A[[z]] \to B[[z]]$ *is* S-*t-closed.*

(2) $A[[z]] \to B[[z]]$ *is* S-*t-closed if* $A \to B$ *is integral and* S-*t-closed.*

Proof. (1) is obvious. Observe that $C = {}_{B[[z]]}^{S}A[[z]] \subset \left({}_{B}^{S}A\right)[[z]]$. Indeed, let $\sum b_i z^i \in C$; there is some $s \in S$ such that $sb_i \in A$ whence $b_i \in {}_{B}^{S}A$. Assume that $A \to B$ is S-t-closed and integral, then $A \to {}_{B}^{S}A$ is t-closed and integral. According to [**17**, 2.23], $A[[z]] \to \left({}_{B}^{S}A\right)[[z]]$ is t-closed. The above observation shows that $A[[z]] \to {}_{B[[z]]}^{S}A[[z]]$ is t-closed whence $A[[z]] \to B[[z]]$ is S-t-closed. $\square$

5. seminormal or t-closed rings along admissible subsets. We recall the following definitions (when A is a ring, $\mathrm{Tot}\,(A)$ is its total quotient ring and A' its integral closure).

Definition 5.1. Let A be a ring, then A is said to be:

(1) [**25**] seminormal if for each pair $(x, y) \in A^2$ such that $x^3 = y^2$ there is some $t \in A$ such that $x = t^2$, $y = t^3$.

(2) [**17**] t-closed if A is a weak Baer ring and for each triple $(x, y, r) \in A^3$ such that $x^3 + rxy - y^2 = 0$ there is some $t \in A$ such that $x = t^2 - rt$, $y = t^3 - rt^2$.

For instance, an absolutely flat ring is t-closed and seminormal.

We restrict our theory to the class of decent rings (rings A such that $\mathrm{Tot}\,(A)$ is absolutely flat whence seminormal and t-closed). See our paper [**21**] where decent schemes are defined. Indeed, reduced rings with a finite minimal spectrum are decent as well as weak Baer rings. This point of view allows us to give a unified treatment for integral domains and reduced Noetherian rings. Moreover, we have the following result.

Proposition 5.2 [**21**]. *Let A be a decent ring with integral closure A' and $A \to B$ a flat epimorphism, then we have $B' = A' \otimes_A B$, $\mathrm{Tot}\,(B) = \mathrm{Tot}\,(A) \bigotimes_A B$ and B is a decent ring.*

A decent ring A is seminormal, respectively t-closed, if and only if $A \to \mathrm{Tot}\,(A)$ (or $A \to A'$) is a seminormal morphism, respectively a t-closed morphism [**21**, 5.11]. Thus we are led to the following definition.

Definition 5.3. Let A be a decent ring and $X \subset \mathrm{Spec}\,(A)$ an admissible subset. If $A \to \mathrm{Tot}\,(A)$ is X-seminormal, respectively X-t-closed, then A is said to be X-seminormal, respectively X-t-closed. Thus an X-t-closed or X-seminormal ring is reduced.

A decent ring A is X-seminormal, respectively X-t-closed, if and only if $A \to A'$ is X-seminormal, respectively X-t-closed.

We are able to give useful results for X-t-closedness only when the ring is assumed to be a weak Baer ring. In fact, this has already been observed for t-closedness in our previous papers. Moreover, decentness does not insure that an X-t-closure is a weak Baer ring, unlike the t-closed case. Here are key results.

Definition 5.4. A ring morphism $f : A \to B$ is said to be minimalizing if ${}^a f(\mathrm{Min}\,(B)) \subset \mathrm{Min}\,(A)$.

Lemma 5.5. *Let $f : A \to B$ be an injective morphism between decent rings. Then there is a commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathrm{Tot}\,(A) & \longrightarrow & \mathrm{Tot}\,(B) \end{array}$$

if and only if $A \to B$ is minimalizing.

(1) *When $A \to B$ is minimalizing,* $\mathrm{Tot}\,(A) \to \mathrm{Tot}\,(B)$ *is injective.*

(2) *A faithfully flat morphism is minimalizing. In this case, the above commutative diagram is cartesian.*

Proof. Assume that the commutative diagram exists and let Q be a minimal prime ideal of B. There is a prime ideal S in $\mathrm{Tot}\,(B)$ lying over P and over a minimal prime ideal R in $\mathrm{Tot}\,(A)$ (this ring is absolutely flat). Now ${}^a f(Q)$ is a minimal prime ideal because $A \to \mathrm{Tot}\,(A)$ is flat. Conversely, assume that $A \to B$ is minimalizing and let r be a regular element in A. Then $f(r)$ cannot be a zero divisor for, if not, $f(r)$ lies in some minimal prime ideal and so does r. Therefore, the commutative diagram exists. In this case, $\mathrm{Tot}\,(A) \to \mathrm{Tot}\,(B)$ is injective because $A \to \mathrm{Tot}\,(A)$ is a flat epimorphism [**10**, IV, 3.4]. The going-down property holds for a faithfully flat morphism $A \to B$. Thus such a morphism is minimalizing. Read in [**21**, Proof of 4.30] that the diagram is cartesian. $\square$

Proposition 5.6. *Let $f : A \to B$ be a minimalizing injective ring morphism between decent rings.*

Let $X \subset \mathrm{Spec}\,(A)$ be an admissible subset and let $Y = {}^a f^{-1}(X)$.

(1) *If B is Y-seminormal and $A \to B$ is X-seminormal, then A is X-seminormal.*

(2) *If A is X-seminormal, then $A \to B$ is X-seminormal.*

Proof. Consider the diagram of 5.5. Since $A \to \mathrm{Tot}\,(A)$ is a flat epimorphism and $A \to \mathrm{Tot}\,(B)$ is injective, then $\mathrm{Tot}\,(A) \to \mathrm{Tot}\,(B)$ is an injective ring morphism between absolutely flat rings whence t-closed and seminormal, [**17**, 2.1]. Therefore, this morphism is seminormal [**25**, 3.4] whence Z-seminormal for any admissible subset Z of $\mathrm{Spec}\,(\mathrm{Tot}\,(A))$ by 4.2. Then the results are easy consequences of 4.15. $\square$

Lemma 5.7. *Let A be a ring.*

(1) *If $A \to B$ is an injective ring morphism such that B is a weak Baer ring and* $\mathrm{Bool}\,(A) = \mathrm{Bool}\,(B)$, *then A is a weak Baer ring and f is minimalizing.*

(2) *If A is a weak Baer ring, S a multiplicative subset of A and f the canonical morphism $A \to A_S$, then $f(\mathrm{Bool}\,(A)) = \mathrm{Bool}\,(A_S)$.*

It follows that a decent ring A is a weak Baer ring if and only if $\mathrm{Bool}\,(A) = \mathrm{Bool}\,(\mathrm{Tot}\,(A))$.

Proof. Under the hypotheses of (1), it is easy to prove that A is a weak Baer ring. We show that there is a commutative diagram as in 5.5. If $r \in A$ is a regular element, then r is regular in B since we have $0 :_B r = Be$ where e is an idempotent lying in A so that $e = 0$. Therefore, the commutative diagram exists. To get (2), we need only to show that for any idempotent $\frac{a}{s}$ of A_S there is $e \in \mathrm{Bool}\,(A)$ such that $f(e) = \frac{a}{s}$. There is some $\sigma \in S$ such that $(\sigma a)^2 = (s\sigma)(\sigma a)$ so that $\sigma a \in 0 : (\sigma a - \sigma s) = Ae$ where e is an idempotent. Then $e\sigma a = e\sigma s$ and $\sigma ae = \sigma a$ combine to yield $\frac{a}{s} = \frac{e}{1}$. □

Proposition 5.8. *Let $f : A \to B$ be an injective ring morphism where B is a weak Baer ring and* $\mathrm{Bool}(A) = \mathrm{Bool}(B)$. *Let $X \subset$* $\mathrm{Spec}(A)$ *be an admissible subset and let $Y = {}^a f^{-1}(X)$. Then A is a weak Baer ring. Moreover, we have*

(1) *If B is Y-t-closed and $A \to B$ is X-t-closed, then A is X-t-closed.*

(2) *If A is X-t-closed, then $A \to B$ is X-t-closed.*

Proof. Observe that A is a weak Baer ring, f is minimalizing by 5.7, (1) and $\mathrm{Bool}(\mathrm{Tot}(A)) = \mathrm{Bool}(\mathrm{Tot}(B))$ by 5.7, (2). It follows that $\mathrm{Tot}(A) \to \mathrm{Tot}(B)$ is t-closed since $\mathrm{Tot}(A)$ is t-closed [**17**, 1.6]. Now we can argue as in 5.6. □

We are going to give characterizations of X-seminormal and X-t-closed rings with respect to the flat epimorphism associated to X. We need the definitions of new classes of morphisms.

Definition 5.9. Let $\varphi : A \to E$ be a ring morphism. We say that

(1) φ is almost seminormal if for any pair $(x, y) \in A^2$ such that $x^3 = y^2$ and $\varphi(x) = u^2$, $\varphi(y) = u^3$ for some $u \in E$, there is some t in A such that $x = t^2$, $y = t^3$.

(2) φ is almost t-closed if for any triple $(x, y, r) \in A^3$ such that $x^3 + rxy - y^2 = 0$ and $\varphi(x) = u^2 - \varphi(r)u$, $\varphi(y) = u^3 - \varphi(r)u^2$ for some $u \in E$, there is some $t \in A$ such that $x = t^2 - rt$, $y = t^3 - rt^2$.

Clearly, $\psi \circ \varphi$ is almost seminormal, respectively almost t-closed, when φ and ψ are almost seminormal, respectively almost t-closed. Moreover, if $\psi \circ \varphi$ is almost seminormal, respectively almost t-closed, so is φ.

Descent principle 5.10. *Let $A \to E$ be an injective ring morphism and $u \in A$, $t \in E$.*

(1) *If $u^2 = t^2$ and $u^3 = t^3$, then $t = u$ so that t lies in A.*

(2) *If E is a weak Baer ring,* $\mathrm{Bool}\,(A) = \mathrm{Bool}\,(E)$ *and there is some $r \in A$ such that $u^2 - ru = t^2 - rt$ and $u^3 - ru^2 = t^3 - rt^2$, then t lies in A.*

Proof. Under the hypotheses of (1), we get $u = t$ by [**25**, 3.1]. Assume that the hypotheses of (2) hold. There are idempotents e, $e' = 1-e$, f in A such that $0 : x = Be$ where $x = t^2 - rt$ and $t = (r-ue')(1-f)+fue'$ so that $t \in A$ [**17**, 1.2]. $\square$

Proposition 5.11. *Let $\varphi : A \to E$ be an injective ring morphism.*

(1) *φ is almost seminormal if and only if φ is seminormal.*

(2) *If E is a weak Baer ring and* $\mathrm{Bool}\,(A) = \mathrm{Bool}\,(E)$*, then φ is almost t-closed if and only if φ is t-closed.*

Proof. Obviously, t-closedness implies almost t-closedness and similarly for seminormality. Conversely, assume that φ is almost seminormal. If x, $y \in A$ are such that $x = u^2$, $y = u^3$ where $u \in E$ so that $x^3 = y^2$, there is some $t \in A$ such that $x = u^2 = t^2$ and $y = u^3 = t^3$. By the descent principle, φ is seminormal. Now assume that φ is almost t-closed. Let x, $y \in A$ be such that $x = u^2 - ru$ and $y = u^3 - ru^2$ where $u \in B$ so that $x^3 + rxy - y^2 = 0$. There is some $t \in E$ such that $x = u^2 - ru = t^2 - rt$ and $y = u^3 - ru^2 = t^3 - rt^2$. Again, φ is t-closed by the descent principle. $\square$

Definition 5.12. Let A be a ring, $X \subset \mathrm{Spec}\,(A)$ an admissible subset and $G \subset A$ generating an ideal I of A such that $X \subset D(I)$.

(1) A pair $(x, y) \in A^2$ is said to be X-seminormal for G if for all $a \in G$ there is some $z_a \in A$ such that $a^2x = z_a^2$ and $a^3y = z_a^3$.

(2) A triple $(x, y, r) \in A^3$ is said to be X-t-closed for G if for all $a \in G$ there is some $z_a \in A$ such that $a^2x = z_a^2 - arz_a$ and $a^3y = z_a^3 - arz_a^2$.

Proposition 5.13. *Let A be a decent ring and $X \subset$* $\mathrm{Spec}\,(A)$ *an admissible subset.*

(1) *A is X-seminormal if and only if for any pair $(x, y) \in A^2$ such that $x^3 = y^2$ there is some $t \in A$ such that $x = t^2$, $y = t^3$ whenever the*

pair (x, y) is X-seminormal for some subset, respectively finite subset, G of A.

(2) *If A is a weak Baer ring, then A is X-t-closed if and only if for any triple $(x, y, r) \in A^3$ such that $x^3 + rxy - y^2 = 0$ there is some $t \in A$ such that $x = t^2 - rt$, $y = t^3 - rt^2$ whenever (x, y, r) is X-t-closed for some subset, respectively finite subset, G of A.*

Proof. We give a proof for t-closedness. Recall that $\mathrm{Tot}\,(A)$ is t-closed. To begin with, assume that the ring A is X-t-closed and let $(x, y, r) \in A^3$ be an X-t-closed triple for some subset G of A generating the ideal I and such that $x^3 + rxy - y^2 = 0$. There is some $t \in \mathrm{Tot}\,(A)$ such that $x = t^2 - rt$, $y = t^3 - rt^2$ because $\mathrm{Tot}\,(A)$ is t-closed. Then we have $X \subset D(I)$ and for all $a \in G$ there is some $z_a \in A$ such that $a^2x = z_a^2 - arz_a$ and $a^3y = z_a^3 - arz_a^2$. We get also for $a \in G$ the relations $a^2x = (at)^2 - ar(at)$, $a^3x = (at)^3 - ar(at)^2$. Then by the descent principle at belongs to A. Indeed, $\mathrm{Bool}\,(A) = \mathrm{Bool}\,(\mathrm{Tot}\,(A))$ by 5.7. Then $It \subset A$ and $x = t^2 - rt$, $y = t^3 - rt^2 \in A$ combine to yield that $t \in A$ because $A \to \mathrm{Tot}\,(A)$ is X-t-closed. Hence A satisfies the property. Conversely, assume that this property holds. Let $t \in \mathrm{Tot}\,(A)$ be such that $t^2 - rt = x$, $t^3 - rt^2 = y$ lie in A for some r in A and let I be an ideal such that $X \subset D(I)$ and $It \subset A$. If G is a generating set of I which can be assumed to be finite by quasi-compactness of X, we get $x^3 + rxy - y^2 = 0$ and $a^2x = (at)^2 - ra(at)$, $a^3y = (at)^3 - ar(at)^2$ for $a \in I$. Therefore, (x, y, r) is an X-t-closed triple for G so that $t^2 - rt = u^2 - ru$ and $t^3 - rt^2 = u^3 - ru^2$ for some $u \in A$. The descent principle shows again that $t \in A$. Thus $A \to \mathrm{Tot}\,(A)$ is X-t-closed. $\square$

Theorem 5.14. *Let A be a decent ring and $X \subset \mathrm{Spec}\,(A)$ an admissible subset.*

(1) *A is X-seminormal if and only if for any pair $(x, y) \in A^2$ such that $x^3 = y^2$ there is some $t \in A$ such that $x = t^2$, $y = t^3$ whenever there exist $a_1, \dots, a_n, z_1, \dots, z_n \in A$ such that $X \subset D(a_1, \dots, a_n)$ and $a_i^2x = z_i^2$, $a_i^3y = z_i^3$ for $i = 1, \dots, n$.*

(2) *If A is a weak Baer ring, then A is X-t-closed if and only if for any triple $(x, y, r) \in A^3$ such that $x^3 + rxy - y^2 = 0$ there is some $t \in A$ such that $x = t^2 - rt$, $y = t^3 - rt^2$ whenever there*

exist $a_1, \dots, a_n$, $z_1, \dots, z_n \in A$ *such that* $X \subset D(a_1, \dots, a_n)$ *and* $a_i^2 x = z_i^2 - a_i r z_i$, $a_i^3 y = z_i^3 - a_i r z_i^2$ *for* $i = 1, \dots, n$.

Proof. Translate 5.12 and 5.13. □

When $X = \mathcal{X}(A_S)$, S a multiplicative subset, a ring A is said to be S-seminormal, respectively S-t-closed, if A is X-seminormal, respectively X-t-closed. The characterization of S-seminormality or S-t-closedness is much simpler. We recover Yanagihara's definition of S-seminormality [**26**].

Corollary 5.15. *Let A be a decent ring and S a multiplicative subset.*

(1) *A is S-seminormal if and only if for any pair $(x,y) \in A^2$ such that $x^3 = y^2$ there is some $t \in A$ such that $x = t^2$, $y = t^3$ whenever there are $s \in S$ and $z \in A$ such that $z^2 = s^2 x$, $z^3 = s^3 y$.*

(2) *If A is a weak Baer ring, then A is S-t-closed if and only if for any triple $(x,y,r) \in A^3$ such that $x^3 + rxy - y^2 = 0$ there is some $t \in A$ such that $x = t^2 - rt$, $y = t^3 - rt^2$ whenever there are $s \in S$ and $z \in A$ such that $z^2 - rsz = s^2 x$, $z^3 - rsz^2 = s^3 y$.*

Proof. Instead of using 5.14, observe that an element $b \in \mathrm{Tot}\,(A) = T$ belongs to ${}^S_T A$ if and only if there is some $s \in S$ such that $sb \in A$; then A is S-seminormal if and only if $A \to {}^S_T A$ is seminormal. □

Lemma 5.16. *Let $f : C \to B$ be an injective ring morphism where B is an absolutely flat ring and I a common ideal of C and B.*

(1) *$p : C \to C/I$ is almost seminormal.*

(2) *$p : C \to C/I$ is almost t-closed if in addition* $\mathrm{Bool}\,(C) = \mathrm{Bool}\,(B)$.

Proof. (1) is an easy consequence of the descent principle applied to $C/I \to B/I$ since B is seminormal and C is a pullback. Here is a proof for t-closedness. Obviously, C is the pullback defined by $q : B \to B/I$ and $g : C/I \to B/I$. Let $(x,y,r) \in C^3$ be a triple such that $x^3 + rxy - y^2 = 0$ and $p(x) = u^2 - p(r)u$, $p(y) = u^3 - p(r)u^2$ for some $u \in C/I$. Observe that there is an idempotent $e = xx' \in C$

where x' is the quasi-inverse of x in B ($x^2x' = x$ and ${x'}^2x = x'$). Now there exists $v = yx' \in B$ such that $x = v^2 - rv$, $y = v^3 - rv^2$ since B is absolutely flat [**17**, 2.1]. Besides, we have $q(x) = g(u)^2 - q(r)g(u)$, $q(y) = g(u)^3 - q(r)g(u)^2$ whence $q(y) = g(u)q(x)$. It follows that $q(y)q(x') = g(u)q(e) = g(up(e))$ while $q(y)q(x') = q(v)$. From $g(up(e)) = q(v)$, we deduce that v belongs to C. Hence we have proved that p is almost t-closed. $\square$

Theorem 5.17. *Let A be a decent ring, respectively a weak Baer ring, and $X \subset \mathrm{Spec}\,(A)$ an admissible subset associated to the flat epimorphism $A \to E$. The following statements are equivalent*:

(1) *A is X-seminormal, respectively X-t-closed.*

(2) *$\pi_X : A \to \prod_{P\in X} A_P$ is almost seminormal, respectively almost t-closed.*

(3) *$A \to E$ is almost seminormal, respectively almost t-closed.*

Proof. We give a proof for t-closedness. Assume that A is a weak Baer ring. Set $B = \mathrm{Tot}\,(A)$ and $C = {}^{X}_{B}A$. In view of 2.23, there is a factorization $A \to C \to B$ where B is an absolutely flat ring and there is a common ideal I of C and B such that $A \to E = A \to C \to C/I$. Moreover, we know that $A \to B$ is X-t-closed if and only if $A \to C$ is t-closed (see 4.6). It follows that $A \to E$ is almost t-closed if and only if $A \to B$ is X-t-closed. Indeed, $C \to C/I$ is almost t-closed by 5.16, $A \to C$ is injective and $\mathrm{Bool}\,(A) = \mathrm{Bool}\,(C) = \mathrm{Bool}\,(B)$ by 5.7. Therefore, (1) is equivalent to (3). Assume that the condition (2) holds. Let $(x, y, r) \in A^3$ be such that $x^3 + rxy - y^2 = 0$ and let $G \subset A$ generate an ideal I such that $X \subset D(I)$ and for all $a \in G$ there is some $z_a \in A$ such that $a^2x = z_a^2 - arz_a$ and $a^3y = z_a^3 - arz_a^2$. Now, if P is a prime ideal in X, there is some $a \in G \setminus P$. Therefore, there is an element $u \in \prod_{P\in X} A_P$ such that $\pi_X(x) = u^2 - \pi_X(r)u$, $\pi_X(y) = u^3 - \pi_X(r)u^2$. Since π_X is almost t-closed, we get some $t \in A$ such that $x = t^2 - rt$, $y = t^3 - rt^2$. Thus (2) implies (1). Conversely, assume that A is X-t-closed. Define ϕ_P to be the canonical morphism $A \to A_P$ for $P \in X$ and let $(x, y, r) \in A^3$ be such that $x^3 + rxy - y^2 = 0$. Assume that there is some $u_P \in A_P$ such that $\phi_P(x) = u_P^2 - \phi_P(r)u_P$, $\phi_P(y) = u_P^3 - \phi_P(r)u_P^2$ for every $P \in X$.

We can write $s_P^2 x = z_P^2 - rs_P z_P$ and $s_P^3 y = z_P^3 - rs_P z_P^2$ for some $s_P \in A \setminus P$ and $z_P \in A$. Define I to be the ideal generated by the set G of all elements s_P for $P \in X$ so that$X \subset D(I)$. Since A is X-t-closed, there is some $t \in A$ such that $x = t^2 - rt$, $y = t^3 - rt^2$ by 5.13. Therefore, π_X is almost t-closed. □

Remark 5.18. A more direct proof can be obtained for seminormality since for any ring B the canonical morphism $B \to \prod_{M \in \mathrm{Max}\,(B)} B_M$ is seminormal [**25**]. This is no longer true for t-closedness.

The following result will be useful.

Proposition 5.19. *Let $A \to B \to C$ be injective morphisms and $X \subset \mathrm{Spec}\,(A)$ an admissible subset.*

(1) *If $A \to C$ is X-subintegral and B is X-seminormal, then $B = C$.*

(2) *Assume in addition that C is a weak Baer ring and that $\mathrm{Bool}\,(C) = \mathrm{Bool}\,(A)$. If $A \to C$ is X-infra-integral and B is X-t-closed, then $B = C$.*

Proof. Assume that the hypotheses of (2) hold and set $Y = {}^a f^{-1}(X)$ where f is $A \to B$. Observe that $B \to C$ is Y-infra-integral by 3.2, (4). Now $B \to C$ is Y-t-closed by 5.8, (2). Thus we get $B = C$ (see 4.9). □

Let S be a multiplicative subset of a ring A. We recall that the large quotient ring $A_{[S]}$ with respect to S is defined to be the subset of all elements $x \in \mathrm{Tot}\,(A) = T$ such that $sx \in A$ for some $s \in S$. Now let $X \subset \mathrm{Spec}\,(A)$ be an admissible subset. The preceding definition suggests setting $A_{[X]} = {}^X_T A$ so that an element $x \in T$ lies in $A_{[X]}$ if and only if there is some ideal I such that $Ix \subset A$ and $X \subset D(I)$. Let A' be the integral closure of A in T. We set $A^X = {}^X_{A'} A$ so that we can identify X with its inverse image in $\mathrm{Spec}\,(A^X)$.

Lemma 5.20. *Let A be a decent ring and $X \subset \mathrm{Spec}\,(A)$ an admissible subset. Then $A^X = (A^X)^X$. Moreover, we have*:

(1) *A^X is X-seminormal.*

(2) *If A is a weak Baer ring, then A^X is X-t-closed.*

Proof. Clearly, we have $A^X = (A^X)^X$ since A^X is the X-closure of A in A'. Assume that A is a weak Baer ring. Then $A^X \to A'$ is X-closed whence X-t-closed. Now A' is t-closed whence Y-t-closed where Y is the inverse image of X in $\mathrm{Spec}\,(A')$. Therefore, we can use 5.8, (1) to conclude. □

Proposition 5.21. *Let $f : A \to B$ be a minimalizing injective ring morphism. Let X be an admissible subset of* $\mathrm{Spec}\,(A)$ *and $Y = {}^a f^{-1}(X)$.*

(a) *$A \to B$ induces an injective ring morphism $A^X \to B^Y$.*

(b) *Assume in addition that* $\mathrm{Tot}\,(A) \to \mathrm{Tot}\,(B)$ *is an isomorphism; if $A \to B$ is seminormal, respectively t-closed, so is $A^X \to B^Y$.*

Proof. Obviously, $A \to B$ induces a ring morphism $A' \to B'$. Hence $A \to B$ induces a ring morphism $A^X = {}^X_{A'}A \to {}^Y_{B'}B = B^Y$ by 2.16. Now assume that $A \to B$ is t-closed, $\mathrm{Tot}\,(A) \simeq \mathrm{Tot}\,(B)$ and let $b \in B^Y$ such that $b^2 - rb = \alpha$, $b^3 - rb^2 = \beta \in A^X$ and $r \in A^X$. There is an ideal L of B such that $Lb \subset B$ and $Y \subset D(L)$. Setting $K = f^{-1}(L)$ so that $X \subset D(K)$ by 2.29, (1), there are ideals F, G, H of A such that $X \subset D(F)$, $D(G)$, $D(H)$ and $F\alpha$, $G\beta$, $Hr \subset A$. Letting $I = F \cap G \cap H \cap K$, we get $X \subset D(I)$ and $Ib \subset B$, $I\alpha, I\beta$, $Ir \subset A$ while $(ab)^2 - (ar)(ab)$, $(ab)^3 - (ar)(ab)^2 \in A$ for any $a \in I$. It follows that $Ib \subset A$. Now b is integral over $A^X \subset A'$ whence $b \in A'$. Therefore, b lies in A^X. Hence $A^X \to B^Y$ is t-closed. □

We recall that for a decent ring A with total quotient ring T, the seminormalization of A is given by ${}^+A = {}^+_T A = {}^+_{A'} A$ while its t-closure is given by ${}^t A = {}^t_T A = {}^t_{A'} A$.

Proposition 5.22. *Let A be a decent ring, respectively a weak Baer ring, and X an admissible subset of* $\mathrm{Spec}\,(A)$*. The following statements are equivalent*:

(1) *A is X-seminormal, respectively X-t-closed.*

(2) *$A \to A_{[X]}$ is seminormal, respectively t-closed.*

(3) *$A \to A^X$ is seminormal, respectively t-closed.*

(4) *$A \to {}^+A$, respectively $A \to {}^tA$ is X-closed.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are straightforward since $A_{[X]} = {}^X_T A$ and $A^X = {}^X_{A'} A$ (see Definitions 5.3 and 4.6). Then (1) $\Leftrightarrow$ (4) follows from 4.7. □

Corollary 5.23. *Let A be a weak Baer ring and $X \subset \operatorname{Spec}(A)$ an admissible subset. The following implications hold*

$$A \text{ is } t\text{-closed} \implies \begin{cases} A & \text{is } X\text{-}t\text{-closed} \\ A & \text{is seminormal} \end{cases} \implies A \text{ is } X\text{-seminormal.}$$

Proposition 5.24. *Let A be a decent ring, respectively a weak Baer ring, and X an admissible subset of* $\operatorname{Spec}(A)$. *Then A is seminormal, respectively t-closed, if and only if $A \to A^X$ is seminormal, respectively t-closed, and A^X is seminormal, respectively t-closed.*

Proof. Assume that A is t-closed so that $A \to \operatorname{Tot}(A)$ is t-closed and so is $A \to A^X$. Now, let $b \in A'$ and $r \in A^X$ such that $b^2 - rb = \alpha$, $b^3 - rb^2 = \beta \in A^X$. There is some ideal I of A such that $X \subset D(I)$ and $I\alpha$, $I\beta$, $Ir \subset A$. Then we have $(xb)^2 - (xr)(xb)$, $(xb)^3 - (xr)(xb)^2 \in A$ for any $x \in I$. We get $Ib \subset A$ by t-closedness of A so that b lies in A^X. Therefore, A^X is t-closed. The converse is [**17**, 1.6]. □

Proposition 5.25. *Let $A \to B$ be a subintegral, respectively infra-integral, morphism such that* $\operatorname{Tot}(A) = \operatorname{Tot}(B)$ *and* $X \subset \operatorname{Spec}(A)$ *an admissible subset. Then $A = {}^X_B A$ if A is X-seminormal, respectively a weak Baer X-t-closed ring.*

Proof. Assume that A is a weak Baer X-t-closed ring and $A \to B$ is infra-integral. Then $A \to {}^X_B A$ is X-infra-integral. Moreover, $A \to B$ is integral so that $B \subset A'$ and the X-t-closedness of $A \to A'$ implies the X-t-closedness of $A \to {}^{X}_{B}A$. The conclusion follows from 4.9. $\square$

Definition 5.26. Let A be a decent ring, respectively a weak Baer ring, with total quotient ring T, and let $X \subset \operatorname{Spec}(A)$ be an admissible subset. We define the X-seminormalization and X-t-closure of A to be respectively

$$\overset{+}{\underset{X}{}}A = \overset{(X,+)}{\underset{T}{}}A = \overset{(X,+)}{\underset{A'}{}}A \quad \text{and} \quad \overset{t}{\underset{X}{}}A = \overset{(X,t)}{\underset{T}{}}A = \overset{(X,t)}{\underset{A'}{}}A.$$

A is X-seminormal, respectively X-t-closed, if and only if $A = {}^{+}_{X}A$, respectively $A = {}^{t}_{X}A$.

When $X = \mathcal{X}(A_S)$, we set ${}^{+}_{X}A = {}^{+}_{S}A$ and ${}^{t}_{X}A = {}^{t}_{S}A$.

Proposition 5.27. *Let A be a decent ring, respectively a weak Baer ring, with total quotient ring T and $X \subset \operatorname{Spec}(A)$ an admissible subset.*

(1) *${}^{+}_{X}A$, respectively ${}^{t}_{X}A$ is the smallest A-subalgebra C of T (or A') with structural morphism $g : A \to C$ such that C is ${}^{a}g^{-1}(X)$-seminormal, respectively ${}^{a}g^{-1}(X)$-t-closed.*

(2) *${}^{+}_{X}A$, respectively ${}^{t}_{X}A$ is the largest A-subalgebra C of T (or A') such that $A \to C$ is X-infra-integral, respectively X-subintegral.*

In particular, A is X-seminormal, respectively X-t-closed, if and only if $A = {}^{+}_{X}A$, respectively $A = {}^{t}_{X}A$.

Proof. Use 4.3 and 4.8. $\square$

Remark 5.28. When $X = \mathcal{X}(A_S)$, S a multiplicative subset, the statement (1) in 5.27 can be given in a much simpler form (indeed, we have ${}^{a}g^{-1}X = \mathcal{X}(C_S)$).

(1) *${}^{+}_{S}A$, respectively ${}^{t}_{S}A$, is the smallest A-subalgebra C of T, or A', such that C is S-seminormal, respectively S-t-closed.*

Corollary 5.29. *Let A be a decent ring, respectively a weak Baer ring, $X \subset \operatorname{Spec}(A)$ an admissible subset and Y^{+}, respectively Y^{t}, the inverse image of X in $\operatorname{Spec}({}^{+}A)$, respectively in $\operatorname{Spec}({}^{t}A)$. The* *following statements hold*:

$$
\begin{gathered}
{}^{+}_{X}A = {}^{+}A \cap A^{X} = {}^{+}A \cap A_{[X]} = {}^{+}_{A^X}A = {}^{+}_{A_{[X]}}A = {}^{X}_{{}^{+}A}A \\
{}^{+}_{X}A \subset {}^{+}(A^{X}) \subset ({}^{+}A)^{Y+} \\
{}^{t}_{X}A = {}^{t}A \cap A^{X} = {}^{t}A \cap A_{[X]} = {}^{t}_{A^X}A = {}^{t}_{A_{[X]}}A = {}^{X}_{{}^{t}A}A \\
{}^{t}_{X}A \subset {}^{t}(A^{X}) \subset ({}^{t}A)^{Y^t}.
\end{gathered}
$$

Proof. We provide a proof for seminormality. The first statement follows from 4.1, 4.2 and 4.4. Clearly, we have ${}^{+}_{X}A \subset A^{X} \subset {}^{+}(A^{X})$. Now ${}^{+}A$ is seminormal and so is $({}^{+}A)^{Y+}$ by 5.24. Then observe that $A^{X} \subset ({}^{+}A)^{Y+}$ by 5.21. The relation ${}^{+}(A^{X}) \subset ({}^{+}A)^{Y+}$ is a consequence of [**16**, 3.5]. $\square$

Lemma 5.30. *Let $f : A \to B$ be a minimalizing injective morphism between decent rings, $X \subset \operatorname{Spec}(A)$ an admissible subset and $Y = {}^{a}f^{-1}(X)$. There is a commutative diagram with injective morphisms*

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
{}^{+}_{X}A & \longrightarrow & {}^{+}_{Y}B.
\end{array}
$$

A similar statement holds for X-t-closure when A and B are weak Baer rings.

Proof. Assume that the hypotheses hold. In view of 5.21 and 5.5, $A \to B$ induces a ring morphism $C = A^{X} \to B^{Y} = D$. Now $A \to B$ induces a ring morphism ${}^{t}_{C}A \to {}^{t}_{D}B$ (see [**16**, 2.2]). Then it is enough to observe that ${}^{t}_{C}A = {}^{t}_{X}A$ by 5.29. $\square$

Theorem 5.31. *Let A be a decent ring, respectively a weak Baer ring, and $X \subset \operatorname{Spec}(A)$ an admissible subset. Then ${}^{+}_{X}A$ is a decent* *ring, respectively* $\overset{t}{\underset{X}{}}A$ *is a weak Baer ring, and* $s_X : A \to \overset{+}{\underset{X}{}}A$, $t_X : A \to \overset{t}{\underset{X}{}}A$ *are injective minimalizing morphisms.*

Moreover, any injective minimalizing ring morphism $f : A \to B$ *where* B *is a decent* ${}^a f^{-1}(X)$*-seminormal ring, respectively a* ${}^a f^{-1}X$*-*t*-closed weak Baer ring, can be factored* $A \to \overset{+}{\underset{X}{}}A \to B$*, respectively* $A \to \overset{t}{\underset{X}{}}A \to B$.

Proof. First observe that $A \to \overset{+}{\underset{X}{}}A$ is minimalizing. Indeed, if P is a minimal prime ideal of $\overset{+}{\underset{X}{}}A$, there is a minimal prime ideal Q of $\mathrm{Tot}\,(A)$ lying over P. Then $Q \cap A$ is a minimal prime ideal by flatness of $A \to \mathrm{Tot}\,(A)$. Now use 5.30 and $B = \overset{+}{\underset{{}^a f^{-1}(X)}{}}B$. $\square$

Remark 5.32. In the seminormal case, the factorization $A \to \overset{+}{\underset{X}{}}A \to B$ is unique. Indeed, $A \to \overset{+}{\underset{X}{}}A$ is X-subintegral whence subintegral. A subintegral morphism is clearly a radicial morphism. It is well known that a radicial morphism is an epimorphism of the category of reduced ring. In the t-closed case, the factorization $A \to \overset{t}{\underset{X}{}}A \to B$ is unique if $f : A \to B$ is a tight morphism of A-modules, that is to say for any nonzero element $b \in B$ there is some $a \in A$ such that $f(a)b \in f(A)$ and $f(a)b \neq 0$ (this property holds for $A \to \overset{t}{\underset{X}{}}A \subset \mathrm{Tot}\,(A)$). In fact, assume that there are two different morphisms $h, g : \overset{t}{\underset{X}{}}A \to B$ such that $f = g \circ t_X = h \circ t_X$. There is some $x \in \overset{t}{\underset{X}{}}A$ such that $b = h(x) - g(x) \neq 0$. If f is tight, there is some $a \in A$ such that $f(a)b = f(\alpha)$ and $f(\alpha) \neq 0$. Now, there is a regular element r such that $rx \in A$. It follows that $f(r\alpha) = 0$ so that $r\alpha = 0$, a contradiction.

Proposition 5.33. *Let* A *be a decent ring, respectively a weak Baer ring,* $X \subset \mathrm{Spec}\,(A)$ *an admissible subset,* $f : A \to E$ *a flat epimorphism and set* $Y = {}^a f^{-1}(X)$*. Then* E *is a decent ring, respectively a weak Baer ring, and we have*

$$\overset{+}{\underset{Y}{}}E = \left(\overset{+}{\underset{X}{}}A\right) \bigotimes_A E, \quad \textit{resp.} \quad \overset{t}{\underset{Y}{}}E = \left(\overset{t}{\underset{X}{}}A\right) \bigotimes_A E.$$

In particular, when $E = A_S$*,* S *a multiplicative subset, we have*

$$\overset{+}{\underset{X_S}{}}A_S = \left(\overset{+}{\underset{X}{}}A\right)_S, \quad \textit{resp.} \quad \overset{t}{\underset{X_S}{}}A_S = \left(\overset{t}{\underset{X}{}}A\right)_S.$$

It follows that if A is X-seminormal, respectively X-t-closed, so is E.

Proof. Read in [**21**] that if A is decent, respectively a weak Baer ring, so is E. Now we have $\mathrm{Tot}\,(E) = \mathrm{Tot}\,(A) \otimes_A E$ by 5.2. Hence, we are in a position to apply 4.10, (1). □

Corollary 5.34. *Let A be a decent ring, respectively a weak Baer ring, and $X \subset \mathrm{Spec}\,(A)$ an admissible subset. The following statements are equivalent*:

(1) *A is X-seminormal, respectively X-t-closed.*

(2) *A_P is X_P-seminormal, respectively X_P-t-closed, for all $P \in \mathrm{Spec}\,(A)$.*

(3) *A_P is X_P-seminormal, respectively X_P-t-closed, for all $P \in \mathrm{Max}\,(A)$.*

(4) *A_P is X_P-seminormal, respectively X_P-t-closed, for all $P \in \mathrm{Supp}_A(A'/A)$.*

(5) *A_P is X_P-seminormal, respectively X_P-t-closed, for all $P \in \mathrm{Ass}_A(A'/A)$.*

If in addition A is Noetherian, (1) *is equivalent to the following statement*:

(6) *A_P is X_P-seminormal, respectively X_P-t-closed, for all $P \in \mathrm{Spec}\,(A)$ such that $\mathit{Prof}_A(A_P) = 1$.*

Proof. Use 5.33 and $\mathrm{Tot}\,(A_P) = \mathrm{Tot}\,(A)_P$ (see 5.2). For the last assertion, read the proof of [**17**, 2.8]. □

We recall that if A is decent, respectively a weak Baer ring, so is the polynomial ring $A[z]$ (see [**21**]). We denote by j the canonical morphism $A \to A[z]$.

Proposition 5.35. *Let A be a decent ring, respectively a weak Baer ring, $X \subset \mathrm{Spec}\,(A)$ an admissible subset and $Y = {}^a j^{-1}(X)$. Then we* *have*

$$\left(\overset{+}{X}A\right)[z] = \overset{+}{Y}A[z] = \overset{+}{\Sigma_X}A[z], \; \textit{resp.} \; \left(\overset{t}{X}A\right)[z] = \overset{t}{Y}A[z] = \overset{t}{\Sigma_X}A[z].$$

Moreover, when $X = \mathcal{X}(A_S)$ *we have* $\left(\overset{+}{S}A\right)[z] = \overset{+}{S}A[z]$, *respectively* $\left(\overset{t}{S}A\right)[z] = \overset{t}{S}A[z]$. *Therefore, the following statements are equivalent*:

(1) A *is* X*-seminormal, respectively* X*-*t*-closed.*

(2) $A[z]$ *is* Y*-seminormal, respectively* Y*-*t*-closed.*

(3) $A[z]$ *is* Σ_X*-seminormal, respectively* Σ_X*-*t*-closed.*

Proof. Apply 4.10, (4) with $B = A'$ since $A'[z]$ is the integral closure of $A[z]$ [**21**, 4.35]. □

6. Examples and properties of seminormal or t-closed rings along admissible subsets.

Proposition 6.1. *Let* A *be a decent ring, respectively a weak Baer ring, and* X *an admissible subset of* $\operatorname{Spec}(A)$. *Then* A *is* X*-seminormal, respectively* X*-*t*-closed, if and only if an element* $x \in \operatorname{Tot}(A)$ *belongs to* A *whenever there is some ideal* I *of* A *such that* $X \subset D(I)$, $Ix \subset A$ *and* $x^n \in A$, *respectively there is some* $r \in A$ *such that* $x^{n+1} - rx^n \in A$, *for large* n.

Proof. We know that A is X-t-closed if and only if $A \to A_{[X]}$ is t-closed (see 5.22), while x lies in $A_{[X]}$ if there is some ideal I of A such that $Ix \subset A$ and $X \subset D(I)$. Now apply [**17**, 2.15] to $A \to A_{[X]}$. In the seminormal case, use the similar result [**7**, 1.1]. □

Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of rings and A the product ring. Then A is a weak Baer, respectively decent, ring if every A_λ is a weak Baer, respectively decent, ring. Furthermore, we have $\operatorname{Tot}(A) = \prod_{\lambda\in\Lambda} \operatorname{Tot}(A_\lambda)$. We denote by p_λ the canonical surjective morphism $A \to A_\lambda$.

Proposition 6.2. *Let* $\{A_\lambda\}_{\lambda\in\Lambda}$ *be a family of decent, respectively weak Baer, rings and* A *the decent, respectively weak Baer, product ring.*

Let $X \subset \operatorname{Spec}(A)$ be an admissible subset and consider the admissible subsets $X_\lambda = {}^a p_\lambda{}^{-1}(X)$. Then A is X-seminormal, respectively X-t-closed, if and only if every A_λ is X_λ-seminormal, respectively X_λ-t-closed.

Proof. Assume that A is X-t-closed. Let $b_\lambda \in \operatorname{Tot}(A_\lambda)$ be such that there are some ideal I_λ of A_λ and some element $r_\lambda \in A_\lambda$ satisfying $X_\lambda \subset D(I_\lambda)$ and $b_\lambda^2 - r_\lambda b_\lambda$, $b_\lambda^3 - r_\lambda b_\lambda^2 \in A_\lambda$. Setting $J = I_\lambda \times \prod_{\mu \neq \lambda} A_\mu$, then J is an ideal of A such that $X \subset D(J)$ by 2.29, (1). Denote by b and r the elements of $\operatorname{Tot}(A)$ and A with components b_λ, r_λ and zero elsewhere. We get $bJ \subset A$ and $b^2 - rb$, $b^3 - rb^2 \in A$. Therefore, b belongs to A and b_λ to A_λ. It follows that A_λ is X_λ-t-closed. Conversely, assume that every A_λ is X_λ-t-closed and let $b = (b_\lambda) \in \operatorname{Tot}(A)$, $r = (r_\lambda) \in A$ and J an ideal of A such that $X \subset D(J)$, $b^2 - rb$, $b^3 - rb^2 \in A$. It follows that $X_\lambda = {}^a p_\lambda{}^{-1}(X) \subset D(JA_\lambda)$ and $b_\lambda J A_\lambda \subset A_\lambda$. Then we get $a_\lambda \in A_\lambda$ and $a \in A$. Hence, A is X-t-closed. □

Next we give some results using cartesian squares. We recall that decentness is not descended by faithfully flat morphisms [**21**, 1.6] while to be a weak Baer ring is descended by such morphisms [**21**, 1.12].

Proposition 6.3. *Let $A \to A'$ be a faithfully flat morphism, X an admissible subset of* $\operatorname{Spec}(A)$ *and $Y = {}^a f^{-1}(X)$.*

(1) *If A and A' are decent rings and A' is Y-seminormal, then A is X-seminormal.*

(2) *If A' is a Y-t-closed weak Baer ring, then A is an X-t-closed weak Baer ring.*

Proof. In each case we have a cartesian square as in 5.5. Then use 4.11. □

Proposition 6.4. *Let A be a decent, respectively weak Baer, ring, $f_1, \dots, f_n \in A$ such that* $\operatorname{Spec}(A) = D(f_1) \cup \cdots \cup D(f_n)$ *and $X \subset \operatorname{Spec}(A)$ an admissible subset. Then every localization A_{f_i} is X_{f_i}-seminormal, respectively X_{f_i}-t-closed, if and only if A is X-seminormal, respectively X-t-closed.*

Proof. We give a proof for seminormality. According to 5.33, A_{f_i} is X_{f_i}-seminormal when A is X-seminormal. We show the converse. The morphism $f : A \to \prod A_{f_i} = B$ is faithfully flat. In view of 6.2, B is ${}^a f^{-1}(X)$-seminormal since $A \to B \to A_{f_i}$ is the canonical morphism. Then use 6.3 to end. □

Proposition 6.5. *Let $A \to B$ be an injective ring morphism where A and B are decent and $A \to B$ minimalizing, respectively B is a weak Baer ring and* $\mathrm{Bool}\,(A) = \mathrm{Bool}\,(B)$. *Assume that there is a commutative cartesian square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

where $C \to D$ is injective. Let $X \subset \mathrm{Spec}\,(A)$ be an admissible subset and set $Y = {}^a f^{-1}(X)$ and $Z = {}^a g^{-1}(X)$. If $C \to D$ is Z-seminormal, respectively Z-t-closed, and B is Y-seminormal, respectively Y-t-closed, then A is X-seminormal, respectively X-t-closed.

Proof. We deduce from 4.11 that $A \to B$ is X-seminormal, respectively X-t-closed. To end, use 5.6 and 5.8. □

Remark 6.6. In the above result, assume that $X = \mathcal{X}(A_S)$ where S is a multiplicative subset of A. We can replace Y and Z by $f(S)$ and $g(S)$.

Proposition 6.7. *Let B be a weak Baer ring, $\{A_\lambda\}_{\lambda \in \Lambda}$ a family of subrings of B such that* $\mathrm{Bool}\,(A_\lambda) = \mathrm{Bool}\,(B)$ *for each index λ and A the intersection of the family. Then A is a weak Baer ring such that* $\mathrm{Bool}\,(A) = \mathrm{Bool}\,(B)$. *Moreover, let X_λ be an admissible subset of* $\mathrm{Spec}\,(A_\lambda)$ *for each index λ and $f_\lambda : A \to A_\lambda$ the canonical injection. Set $Y = \cup_{\lambda \in \Lambda}\, {}^a f_\lambda(X_\lambda)$. If every A_λ is X_λ-seminormal, respectively X_λ-t-closed, and X is an admissible subset such that $Y \subset X$, for instance Y_a, then A is X-seminormal, respectively X-t-closed.*

Proof. A is a weak Baer ring by 5.7 and f_λ induces an injective ring morphism $\mathrm{Tot}\,(A) \to \mathrm{Tot}\,(A_\lambda)$ by 5.7 and 5.5. Assume that every A_λ is X_λ-seminormal. Let $b \in \mathrm{Tot}\,(A)$ and I an ideal of A be such that $X \subset D(I)$, $Ib \subset A$ and $b^2, b^3 \in A$. It follows that ${}^a g_\lambda(X_\lambda) \subset D(I)$ so that $X_\lambda \subset {}^a f_\lambda^{-1}(D(I)) = D(IA_\lambda)$. Then we have $(IA_\lambda)b \subset A_\lambda$ and $b^2, b^3 \in A_\lambda$ whence b lies in A_λ. Hence, b belongs to A and A is X-seminormal. □

Remark 6.8. Under the hypotheses of 6.7, assume that $X_\lambda = \mathcal{X}\left(A_{\lambda S_\lambda}\right)$ where S_λ is a multiplicative subset of A_λ and consider the multiplicative subset $S = A \cap \cap_\lambda S_\lambda$ of A. It is obvious that ${}^a f_\lambda(X_\lambda) \subset \mathcal{X}(A_S)$. Therefore, A is S-seminormal, respectively S-t-closed, whenever every A_λ is S_λ-seminormal, respectively S_λ-t-closed.

Proposition 6.9. *Let A be a decent ring, respectively weak Baer ring, and $X \subset \mathrm{Spec}\,(A)$ an admissible subset.*

(1) *Let $X \subset Y$ be an admissible subset, for instance, X_m. Then we have*
$$\overset{+}{\underset{Y}{}}A \subset \overset{+}{\underset{X}{}}A, \quad \textit{resp.}\ \overset{t}{\underset{Y}{}}A \subset \overset{t}{\underset{X}{}}A.$$
It follows that if A is X-seminormal, respectively X-t-closed, then A is Y-seminormal, respectively Y-t-closed.

(2) *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of admissible subsets such that $X = \cup_{\lambda\in\Lambda} X_\lambda$. Then we have*
$$\overset{+}{\underset{X}{}}A = \bigcap_{\lambda\in\Lambda} \overset{+}{\underset{X_\lambda}{}}A, \quad \textit{resp.}\ \overset{t}{\underset{X}{}}A = \bigcap_{\lambda\in\Lambda} \overset{t}{\underset{X_\lambda}{}}A.$$
Hence, A is X-seminormal, respectively X-t-closed, whenever A is X_λ-seminormal, respectively X_λ-t-closed, for every λ.

(3) *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of admissible subsets stable under finite intersections such that $X = \cap_{\lambda\in\Lambda} X_\lambda$. Then we have*
$$\overset{+}{\underset{X}{}}A = \bigcup_{\lambda\in\Lambda} \overset{+}{\underset{X_\lambda}{}}A, \quad \textit{resp.}\ \overset{t}{\underset{X}{}}A = \bigcup_{\lambda\in\Lambda} \overset{t}{\underset{X_\lambda}{}}A.$$
Hence, A is X-seminormal, respectively X-t-closed, if and only if A is X_λ-seminormal, respectively X_λ-t-closed, for each λ.

Proof. All these results are easy consequences of 2.17. For instance, set $T = \operatorname{Tot}(A)$. To show (2), observe that $\overset{+}{_X}A = \overset{+}{_T}A \cap \overset{X}{_T}A = \cap \overset{+}{_{X_\lambda}}A$ by 2.17 and 5.29. $\square$

Proposition 6.10. *Let A be a decent, respectively weak Baer, ring and $X \subset \operatorname{Spec}(A)$ an admissible subset. Then A is X-seminormal, respectively X-t-closed, if and only if A_P is X_P-seminormal, respectively X_P-t-closed, for every prime ideal $P \notin X$.*

Proof. One implication follows from 5.34. Assume that A_P is X_P-seminormal for all $P \in \operatorname{Spec}(A) \setminus X$. Then recall that for every prime ideal P we have $\overset{+}{_{X_P}}A_P = \big(\overset{+}{_X}A\big)_P$ so that $A_P = \big(\overset{+}{_X}A\big)_P$ when $P \notin X$ while $\big(A_{[X]}\big)_P = A_P$ for P in X so that $A_P = ({}^+A)_P \cap \big(A_{[X]}\big)_P = \big(\overset{+}{_X}A\big)_P$ by 5.29. Therefore, A is X-seminormal. $\square$

Proposition 6.11. *Let A be a decent, respectively weak Baer, ring and $X \subset \operatorname{Spec}(A)$ an admissible subset.*

(1) *A is X-seminormal, respectively X-t-closed, if A_P is seminormal, respectively t-closed, for every $P \notin X$.*

(2) *Assume in addition that A is an X-seminormal, respectively X-t-closed, one-dimensional weak Baer ring. Then A_P is seminormal, respectively t-closed, for every $P \notin X$.*

In particular, if A is a one-dimensional local integral domain with maximal ideal $M \notin X$, then A is X-seminormal, respectively X-t-closed, if and only if A is seminormal, respectively t-closed.

Proof. In view of 6.10 and since seminormality implies X-seminormality, we get (1) and similarly for t-closedness. We show (2) for t-closedness. Then A' is one-dimensional if A is one-dimensional. Take any prime ideal $P \notin X$. Then $A_P \to A'_P$ is X_P-t-closed by 5.8, (2) since A_P is X_P-t-closed. If A_P is zero-dimensional, then A_P is a field isomorphic to $\operatorname{Tot}(A_P)$. In this case, A_P is t-closed. If A_P is one-dimensional, so is A'_P. Moreover, we have $\operatorname{Bool}(A_P) = \operatorname{Bool}(A'_P)$ by 5.7 since $\operatorname{Bool}(A') = \operatorname{Bool}(A)$ and A' is a weak Baer ring by 5.7 since an idempotent of $\operatorname{Tot}(A)$ belongs to A'. Now we can use 4.21 to conclude. $\square$

Remark 6.12. Let A be a one-dimensional local integral domain with maximal ideal $M \in X$; then A is X-t-closed. Indeed, $A_M \to \left(A_{[X]}\right)_M$ is an isomorphism.

Proposition 6.13. *Let A be a one-dimensional integral domain and $X \subset \operatorname{Spec}(A)$ an admissible subset. Let $f : A \to B \subset A'$ defining an A-subalgebra of A'. If A is X-t-closed, respectively X-seminormal, then B is ${}^af^{-1}(X)$-t-closed, respectively ${}^af^{-1}(X)$-seminormal.*

Proof. Assume that the hypotheses hold. Let $Q \notin {}^af^{-1}(X)$ be a prime ideal of B lying over $P \notin X$ in A. Then A_P is seminormal, respectively t-closed, by 6.11, (2). Now $A_P \to B_P$ is integral. We deduce from [**1**, 2.5], respectively [**19**, 2.5], that B_P is seminormal, respectively t-closed, and so is B_Q since this ring is a localization of B_P. According to 6.11, (1), B is ${}^af^{-1}(X)$-seminormal, respectively ${}^af^{-1}(X)$-t-closed. □

Proposition 6.14. *Let $f : A \to B$ be a finite injective minimalizing morphism such that A and B are one-dimensional weak Baer rings and $\operatorname{Tot}(A) \to \operatorname{Tot}(B)$ is an isomorphism. Let $X \subset \operatorname{Spec}(A)$ be an admissible subset. Then A is X-t-closed and B is ${}^af^{-1}(X)$-t-closed if and only if every A-subalgebra C of B, with structural morphism $g : A \to C$, is ${}^ag^{-1}(X)$-t-closed.*

Proof. One implication is obvious. Conversely, assume that A is X-t-closed and B is ${}^af^{-1}(X)$-t-closed and let $g : A \to C$ defining an A-subalgebra of B. Consider $Q \notin {}^ag^{-1}(X)$ lying over $P \notin X$ in A. Then A_P is t-closed by 6.11, (2) and an integral domain since A is a weak Baer ring. Now observe that $\operatorname{Tot}(B_P) = \operatorname{Tot}(B) \otimes_A A_P$ because $B_P = B \otimes_A A_P$ and $\operatorname{Tot}(B_P) = \operatorname{Tot}(B) \otimes_B B_P$. Indeed, B is decent and $B \to B_P$ is a flat epimorphism (see 5.2). Furthermore, $A_P \to B_P$ is minimalizing so that there is a morphism $\operatorname{Tot}(A_P) \to \operatorname{Tot}(B_P)$. Now look at the following diagram

$$\begin{array}{ccccc} A & \longrightarrow & \mathrm{Tot}\,(A) & \longrightarrow & \mathrm{Tot}\,(B) \\ \downarrow & & \downarrow & & \downarrow \\ A_P & \longrightarrow & \mathrm{Tot}\,(A_P) & \longrightarrow & \mathrm{Tot}\,(B_P). \end{array}$$

The rectangular diagram is commutative and co-cartesian by the above observation. It follows that the right diagram is commutative because $A \to \mathrm{Tot}\,(A)$ is an epimorphism. Now, the left diagram is co-cartesian. We get that the right diagram is co-cartesian. Therefore, $\mathrm{Tot}\,(A_P) \to \mathrm{Tot}\,(B_P)$ is an isomorphism so that B_P is an integral domain because $\mathrm{Tot}\,(A_P)$ is a field. From the finiteness of $A \to B$ and since $\mathrm{Tot}\,(A)$ is isomorphic to $\mathrm{Tot}\,(B)$, we get that the conductor I of $A \to B$ contains a regular element of A whence $I_P \neq 0$. Then B_P is a local ring. Indeed, the conductor I_P of $A_P \to B_P$ is nonzero. If $I_P = A_P$ we are done. If not, P_P belongs to $\mathrm{Min}\,(V(I_P))$ and $A_P \to B_P$ is t-closed whence seminormal by [**17**, 1.6]. Thus we can use [**16**, 3.11]. There is a unique prime ideal RB_P of B_P lying over P_P. Hence, $B_P = B_R$ is a one-dimensional integral domain and $R \notin {}^a f^{-1}(X)$. From 6.11, (2), we deduce that B_R is t-closed. Now, observe that Q is the only prime ideal of C lying over P so that $C_P = C_Q$. Therefore, we get $A_P \subset C_Q \subset B_R$. According to [**19**, 1.8], B_Q is t-closed. Thus B is ${}^a g^{-1}(X)$-t-closed by 6.11, (1). $\square$

Proposition 6.15. *Let A be a weak Baer ring and $X \subset \mathrm{Spec}\,(A)$ an admissible subset. Then A is X-t-closed if and only if for every finite injective morphism $g : A \to C \subset A'$ with conductor J and such that $X \subset D(J)$, there is an injective map $\theta : \mathrm{Min}\,(V_A(J)) \to \mathrm{Min}\,(V_C(J))$ such that ${}^a g \circ \theta = Id$ and J is a radical ideal of C.*

Proof. A is X-t-closed if and only if $A \to A'$ is X-t-closed. Then use 4.25 with $B = A'$. $\square$

Proposition 6.16. *Let A be a weak Baer ring and $X \subset \mathrm{Spec}\,(A)$ an admissible subset. Then A is X-seminormal, respectively X-t-closed, if and only if A/M is (X/M)-seminormal, respectively (X/M)-t-closed, for every minimal prime ideal M of A.*

Proof. Using 5.14, we can mimic the argument of [**17**, Proof of 2.5]. $\square$

Proposition 6.17. *Let A be a one-dimensional weak Baer ring and $X \subset \mathrm{Spec}\,(A)$ an admissible subset. If A is X-seminormal, respectively X-t-closed, then A/M is seminormal, respectively t-closed, for every minimal prime ideal M such that $M \notin X$.*

Proof. Assume that A is X-t-closed and let $M \in \mathrm{Min}\,(A) \setminus X$. Then a prime ideal P of A such that $M \subset P$ does not belong to X because X is stable under generalizations. It follows that A_P is t-closed by 6.11. Now observe that A_P is isomorphic to $(A/M)_{(P/M)}$ since A_P is an integral domain so that $M_P = 0$. Therefore, A/M is t-closed by [**17**, 2.8]. $\square$

Remark 6.18. In some of the next results, we will use the following observations:

(1) Let A be a ring, $X \subset \mathrm{Spec}\,(A)$ an admissible subset and S a multiplicative subset of A. Denote by $\varphi_S : A \to A_S$ the canonical morphism. Then we have $X_S = {}^a{\varphi_S}^{-1}(X)$.

(2) Let $f : A \to B$ be an injective morphism, $X \subset \mathrm{Spec}\,(A)$ an admissible subset and S a multiplicative subset of A. Then we have ${}^a{f_S}^{-1}(X_S) = \left({}^a f^{-1}(X)\right)_S$.

Recall that a ring A is said to be a Mori ring if $A \to A'$ is finite.

Proposition 6.19. *Let A be a weak Baer ring, $X \subset \mathrm{Spec}\,(A)$ an admissible subset and $f : A \to A'$ the canonical morphism. Assume that A is X-seminormal.*

(1) *If the map $\mathrm{Spec}\,(A') \setminus {}^a f^{-1}(X) \to \mathrm{Spec}\,(A) \setminus X$ is bijective, then A is X-t-closed.*

(2) *If A is a one-dimensional X-t-closed Mori ring, then the map $\mathrm{Spec}\,(A') \setminus {}^a f^{-1}(X) \to \mathrm{Spec}\,(A) \setminus X$ is bijective.*

Proof. To show (1), use 4.16, (2) with $B = A'$. We get $A = {}^{+}_{X}A = {}^{(X,+)}_{A'}A = {}^{(X,t)}_{A'}A = {}^{t}_{X}A$. Conversely, if A is one-dimensional and X-t-closed, the map $\mathrm{Spec}\,(A'_P) \setminus {}^a f_P^{-1}(X_P) \to \mathrm{Spec}\,(A_P) \setminus X_P$ is bijective for every $P \in \mathrm{Spec}\,(A)$ by 4.17 since $\mathrm{Tot}\,(A_P) \to \mathrm{Tot}\,(A'_P)$ is an isomorphism (see the beginning of the proof of 6.14), and A_P and A'_P are one-dimensional integral domains. To see that the map $\mathrm{Spec}\,(A') \setminus {}^a f^{-1}(X) \to \mathrm{Spec}\,(A) \setminus X$ is bijective, we need only to show that this map is injective. Let Q, $R \in \mathrm{Spec}\,(A') \setminus {}^a f^{-1}(X)$ be such that $Q \cap A = R \cap A = P \notin X$. By using 6.18, we get that $Q_P = R_P$ so that $Q = R$. $\square$

Corollary 6.20. *Let A be a one-dimensional integral Mori domain. Assume that the conductor I of $A \to A'$ is a radical ideal in A'. Let $X \subset \mathrm{Spec}\,(A)$ be an admissible subset. Then A is X-t-closed if and only if every prime ideal $P \in \mathrm{Min}\,(V_A(I)) \setminus X$ has a unique upper in A'.*

Proof. A is seminormal by [**16**, 4.9]. Then 6.19 gives the conclusion since $D_{A'}(I) \to D_A(I)$ is bijective. $\square$

Let A be a ring. We denote by $\mathrm{Spec}_1(A)$ the set of all height-one prime ideals of A.

Theorem 6.21. *Let A be a weak Baer Noetherian Mori ring such that condition (S_2) holds for A. Let I be the conductor of $f : A \to A'$ and $I = \cap_{i=1}^n Q_i$ an irredundant primary representation where Q_i is P_i-primary. Let $X \subset \mathrm{Spec}\,(A)$ an admissible subset. The following statements are equivalent.*

(1) *A is X-t-closed.*

(2) *A_P is X_P-t-closed for every $P \in \mathrm{Spec}_1(A) \setminus X$.*

(3) *A_P is t-closed for every $P \in \mathrm{Spec}_1(A) \setminus X$.*

(4) *The two following conditions hold:*

(a) *$Q_i = P_i$ whenever P_i belongs to $V_{A'}(I) \setminus {}^a f^{-1}(X)$ and $P_i \cap A$ belongs to $\mathrm{Min}\,(V_A(I))$.*

(b) *${}^a f$ induces a bijection $\mathrm{Spec}_1(A') \setminus {}^a f^{-1}(X) \to \mathrm{Spec}_1(A) \setminus X$.*

Proof. Observe that $\mathrm{Dim}\,(A_P) = \mathrm{Prof}\,(A_P) = 1$ when $P \in \mathrm{Spec}_1(A)$ is a consequence of condition (S_2). Then (2) $\Rightarrow$ (1) follows from 5.34, (6) while (1) $\Rightarrow$ (2) is a consequence of 5.34. We get (2) $\Leftrightarrow$ (3) by 6.11 applied to A_P where $\mathrm{Dim}\,(A_P) = 1$ and $P_P \notin X_P$ (indeed, A_P is an integral domain). Next we show that (1) $\Rightarrow$ (4). To begin with, we prove (4) (a). Let Q be an R-primary ideal of the representation of I such that $R \in V_{A'}(I) \setminus {}^af^{-1}(X)$ and $R \cap A = P \in \mathrm{Min}\,(V_A(I)) \setminus X$. Localizing with respect to P, we get $I_P = \cap\,(Q_k)_P$ where $P_k \cap A \subset P$. This last condition is equivalent to $P_k \cap A = P$ since $I \subset P_k \cap A$ and $P \in \mathrm{Min}\,(V_A(I))$. Set $B = A_P$, $B' = (A')_P$, $J = J_P$ and $Y = X_P$. Then B is an integral domain with integral closure B' and J is the conductor of $g : B \to B'$. Setting R_k and S_k for the localizations of P_k and Q_k at P, we get an irredundant primary representation $J = \cap S_k$ where S_k is a P_k-primary ideal and $R_k \cap B = P_P$. Then we have ${}^ag^{-1}(Y) \subset D(R_k)$. Indeed, let $Q \in {}^ag^{-1}(Y)$ be such that $R_k \subset Q$; since ${}^ag^{-1}(Y)$ is admissible whence stable under generalizations, R_k belongs to ${}^ag^{-1}(Y) = \left({}^af^{-1}(X)\right)_P$ by 6.18. It follows that $P \in X$, a contradiction. Now observe that a power of R_k is contained in S_k from which we deduce that ${}^ag^{-1}(Y) \subset D(J)$. In view of 2.29, we get $Y \subset D(J)$. We deduce from 4.25 that there is an injective map $\theta : \mathrm{Min}\,(V_B(J)) \to \mathrm{Min}\,(V_{B'}(J))$ such that ${}^ag \circ \theta = Id$ and J is a radical ideal. Therefore, there is only one prime ideal R_k lying over P_P because such an ideal lies in $\mathrm{Min}\,(V_{B'}(J))$. Moreover, we get $J = R_k = S_k$ since J is a radical ideal. It follows that $Q = R$. Thus (4)(a) is proved. Now we show (4)(b). Let P be a prime ideal in $\mathrm{Spec}_1(A) \setminus X$. If P belongs to $D_A(I)$, then $A_P \to A'_P$ is an isomorphism whence A_P is t-closed. If P belongs to $V_A(I)$, then P lies in $\mathrm{Min}\,(V_A(I))$. Indeed, the height of P is 1 and I contains a regular element since A is a Mori ring. Using the above notations, when proving (4)(a), we have $Y \subset D(J)$. It follows that ${}^{X_P}_{A'_P}A_P = A'_P$ by 1.16. In this case, A_P is t-closed. According to [**17**, 3.15], ${}^af_P$ induces a bijection $\mathrm{Spec}_1(A'_P) \to \mathrm{Spec}_1(A_P)$ for every P in $\mathrm{Spec}_1(A) \setminus X$. Then (4)(b) follows. Now we show that (4) $\Rightarrow$ (3). Let $P \in \mathrm{Spec}_1(A) \setminus X$. When $P \in D_A(I)$, we have an isomorphism $A_P \to A'_P$ so that A_P is t-closed. Assume that $P \in V_A(I)$. Arguing as above, we get $P \in \mathrm{Min}\,(V_A(I))$. According to (b), there is a unique prime ideal P_i lying over P. Furthermore, we have $P_i = Q_i$ by (a) so that $I_P = (Q_i)_P = (P_i)_P$ is a maximal ideal of A'_P. In this case, A_P is t-closed by [**17**, 3.5] and (3) follows. $\square$

We say that a prime ideal P of a ring A is unibranched if there is only one prime ideal in A' lying over P.

Theorem 6.22. *Let A be a weak Baer ring and $X \subset \operatorname{Spec}(A)$ an admissible subset. If A is X-seminormal and if the elements of* $\operatorname{Ass}_A(A/Aa) \setminus X$ *are unibranched for each regular element $a \in A$, then A is X-t-closed.*

Proof. Under the hypotheses, we need only to settle that the seminormal morphism $A \to A^X$ is t-closed (see 5.22). Assume the contrary, then there are an element $b \in A' \setminus A$, an $r \in A$ and an ideal I of A such that $b^2 - rb$, $b^3 - rb^2 \in A$, $X \subset D(I)$ and $Ib \subset A$. Set $B = A[b]$, then $A \to B$ is seminormal and we can replace I by the conductor of $A \to B$. Since $A \to B$ is seminormal, I is a radical ideal in B. Let P be an element of $\operatorname{Min}(V_A(I))$, then P does not belong to X. There is a regular element $a \in A$ such that $ab \in A$ since $A \to B$ is finite. It follows that $P \in \operatorname{Ass}(A/Aa) \setminus X$ whence P is unibranched. Therefore, there is only one prime ideal Q in B lying over P. Now localize at P. The conductor of $A_P \to B_P$ is I_P and I_P is a radical ideal so that $I_P = P_P$. Since $A \to B$ is integral, we have $B_P = B_Q$, $I_P = Q_P$ and the residual extension $k(P) \to k(Q)$ is an isomorphism because $A \to B$ is elementary infra-integral. It follows that $A_P \to B_P$ is an isomorphism, contradicting $P \in \operatorname{Supp}_A(B/A) = V_A(I)$. Therefore, b belongs to A and A is X-t-closed. $\square$

Corollary 6.23. *Let A be a weak Baer Noetherian ring and $X \subset$* $\operatorname{Spec}(A)$ *an admissible subset. If A is X-seminormal, if a prime ideal P in A is unibranched whenever $P \notin X$ and $\mathit{Prof}\,(A_P) = 1$, then A is X-t-closed.*

Proof. The proof is the same as the proof of [**17**, 4.7]. $\square$

Corollary 6.24. *Let A be a weak Baer Noetherian ring and $X \subset$* $\operatorname{Spec}(A)$ *an admissible subset. If A is X-seminormal and if the elements of* $\operatorname{Ass}_A(A'/A) \setminus X$ *are unibranched, then A is X-t-closed.*

Proof. Similar to the proof of [**17**, 4.9]. $\square$

Remark 6.25. Let A be a decent ring, respectively weak Baer ring, and $X \subset \mathrm{Spec}(A)$ an admissible subset. An extreme case of X-seminormality, respectively X-t-closedness, is obtained when $A = A_{[X]}$. We saw that this condition holds if and only if $X = V(I)$ where I is a pure radical ideal of A (see 2.24 and take $B = \mathrm{Tot}(A)$). Therefore, if A is an integral domain, we have either $I = 0$ or $I = A$ since any $x \in I$ can be written $x = xy$ where y lies in I. It follows that either $X = \mathrm{Spec}(A)$ or $A = \mathrm{Tot}(A)$.

Proposition 6.26. *Let A be an integral domain and S a multiplicative subset of A.*

(1) *If $A[[z]]$ is S-t-closed so is A.*

(2) *If in addition A and A' are Noetherian, then $A[[z]]$ is S-t-closed if A is S-t-closed.*

Proof. By [**17**, 2.22], $A \to A[[z]]$ is t-closed whence S-t-closed. Then (1) follows by 5.8. When A and A' are Noetherian, $A'[[z]]$ is completely integrally closed whence S-t-closed. To get (2), use 4.26 which asserts that $A[[z]] \to A'[[z]]$ is S-t-closed and 5.8. ◻

Proposition 6.27. *Let A be a local ring, $X \subset \mathrm{Spec}(A)$ an admissible subset and $f : A \to \hat{A}$, $g : A \to A^h$ its completion and Henselization.*

(1) *If A is Noetherian, analytically irreducible and $\hat{A}$ is ${}^af^{-1}(X)$-t-closed, then A is X-t-closed. A similar result holds for seminormality.*

(2) *If A is reduced, unibranched and if A^h is ${}^ag^{-1}(X)$-t-closed, respectively ${}^ag^{-1}(X)$-seminormal, then A is X-t-closed, respectively X-seminormal.*

Proof. f and g are faithfully flat and $\hat{A}$, respectively A^h, is an integral domain under the hypotheses of (1), respectively (2). The result follows from 6.3. ◻

Proposition 6.28. *Let A be an integral domain and G a group of automorphisms of A. Let A^G be the associated ring of invariants and $S \subset A^G$ a multiplicative subset. If A is S-seminormal, respectively*

S-t-closed, so is A^G.

Proof. Use 5.15 and argue as in the proof of [**17**, 3.14]. $\square$

We end this section by giving some examples of S-t-closed or S-seminormal quadratic orders.

Definition 6.29. Let $A \to B$ be an injective ring morphism and p a prime integer. Then $A \to B$ is said to be p-seminormal, respectively p-t-closed, if $b \in B$ lies in A whenever b^2, b^3, $pb \in A$, respectively there is some $r \in A$ such that $b^2 - rb$, $b^3 - rb^2$, $pb \in A$.

Swan gave the definition of p-seminormality [**25**] and Yanagihara characterized this notion by means of S-seminormality [**26**].

Lemma 6.30. *Let $A \to B$ be an injective ring morphism, p a prime integer and $S = \{p^n\}_{n \in \mathbf{N}}$. Then $A \to B$ is p-seminormal, respectively p-t-closed, if and only if $A \to B$ is S-seminormal, respectively S-t-closed. Similarly, a decent ring, respectively weak Baer ring, A is p-seminormal, respectively p-t-closed, if and only if A is S-seminormal, respectively S-t-closed.*

Proof. The seminormal case is proved in [**26**]. Obviously, S-t-closedness implies p-t-closedness. Assume that $A \to B$ is p-t-closed and let $b \in B$, $r \in A$ such that $b^2 - rb$, $b^3 - rb^2$, $p^n b \in A$ for some integer n. Then we get $p^{n-1}b \in A$ because $(p^{n-1}b)^2 - (p^{n-1}r)(p^{n-1}b)$, $(p^{n-1}b)^3 - (p^{n-1}r)(p^{n-1}b)^2$, $p(p^{n-1}b) \in A$. An easy induction shows that $b \in A$. $\square$

In the following, we consider a square-free integer d and the field extension $\mathbf{Q} \to \mathbf{Q}\left(\sqrt{d}\right)$. The integral closure of $\mathbf{Z}$ in $\mathbf{Q}\left(\sqrt{d}\right)$ is well known to be a free $\mathbf{Z}$-module A' with basis $\{1, \omega_d\}$ and $A' = \mathbf{Z}[\omega_d]$ while its algebraic orders are the subrings $A = \mathbf{Z}[n\omega_d]$ where n is a positive integer (see for instance [**20**]). Now let p be a prime integer and S the associated multiplicative subset. Then A^S is the set of all elements $b \in A'$ such that $p^n b \in A$ for some integer n. If p is a prime integer, we denote by v_p the associated valuation.

Lemma 6.31. *With the above notations,* $A^S = \mathbf{Z}[m\omega_d]$ *where* $n = mp^{v_p(n)}$.

Proof. Let $x = a + b\omega_d \in A'$ where $a, b \in \mathbf{Z}$. Then $p^i x \in A$ is equivalent to n divides $p^i b$. Letting $n = mp^{v_p(n)}$, we get easily that the last condition holds if and only if m divides b. The result follows. □

Remark 6.32. If $\gcd(p, n) = 1$, then A is p-t-closed by 5.29 since $A^S = A$.

Theorem 6.33. *Let* $n = p_1^{e_1} \cdots p_s^{e_s}$ *where* $p_1, \ldots, p_s$ *are prime integers and consider the multiplicative subset* $S = \{p_i^l\}_{l \in \mathbf{N}}$ *of* $A = \mathbf{Z}[n\omega_d]$. *Then we have*:

(1) ${}^{+}_{S}A = {}^{t}_{S}A = A^S$ *if* p_i *is ramified in* A'.

(2) ${}^{t}_{S}A = A^S$ *and* ${}^{+}_{S}A = \mathbf{Z}\left[p_i \prod_{j \neq i} p_j^{e_j} \omega_d\right]$ *if* p_i *is decomposed in* A'.

(3) ${}^{t}_{S}A = {}^{+}_{S}A = \mathbf{Z}\left[p_i \prod_{j \neq i} p_j^{e_j} \omega_d\right]$ *if* p_i *is inert in* A'.

Proof. Define E_1, respectively E_2, to be the set of all decomposed, respectively inert, prime integers p_j and set $m_1 = \prod_{p_j \in E_1} p_j$, $m_2 = \prod_{p_j \in E_2} p_j$, $m = \prod_{j \neq i} p_j^{e_j}$. Then we have $A^S = \mathbf{Z}[m\omega_d]$, $A^+ = \mathbf{Z}[m_1 m_2 \omega_d]$ and $A^t = \mathbf{Z}[m_2 \omega_d]$ by [**20**, 5.7]. Moreover, we know that ${}^{+}_{S}A = A^S \cap A^+$ and ${}^{t}_{S}A = A^S \cap A^t$. If p_i is ramified, then $m_1 m_2$ divides m. In this case, we get $A^S \subset A^+ \subset A^t$ so that ${}^{+}_{S}A = {}^{t}_{S}A = A^S$. Assume that p_i is decomposed and let $x \in A^S \cap A^+$. There are integers a, b, c, f such that $x = a + m_1 m_2 b\omega_d = c + mf\omega_d$. It follows that p_i divides f. Therefore, we get ${}^{+}_{S}A = \mathbf{Z}\left[p_i \prod_{j \neq i} p_j^{e_j} \omega_d\right]$ while $A^S \subset A^t$ gives ${}^{t}_{S}A = A^S$. If p_i is inert, the same argument proves (3). □

Corollary 6.34. *With the above hypotheses where* p_i *divides* n, *we have*

(1) *If* p_i *is ramified,* A *is neither* p_i*-seminormal nor* p_i*-*t*-closed.*

(2) *If p_i is decomposed, A is p_i-seminormal if and only if $v_{p_i}(n) = 1$ but A is never t-closed.*

(3) *If p_i is inert, A is p_i-seminormal, respectively p_i-t-closed, if and only if $v_{p_i}(n) = 1$.*

Proof. A is p-seminormal, respectively p-t-closed, if and only if $A = {}^{+}_{S}A$, respectively $A = {}^{t}_{S}A$. $\square$

Example 6.35. With the above notations, let p, q be prime integers such that p is inert and q is decomposed in A'.

(1) $\mathbf{Z}[pq^2\omega_d]$ is p-t-closed and is not seminormal.

(2) $\mathbf{Z}[q\omega_d]$ is seminormal and is not q-t-closed.

(3) $\mathbf{Z}[pq\omega_d]$ is p-t-closed and seminormal and is not t-closed.

7. Appendix on flat epimorphisms. In the previous sections, we have been dealing with admissible subsets. Now we intend to give some information about affine open subsets of a spectrum. Let O be an affine subset of a spectrum $\operatorname{Spec}(A)$. We know that there is a flat epimorphism of finite presentation $f : A \to B$ such that ${}^a f(\operatorname{Spec}(B)) = O$. A result of Ferrand quoted by Lazard asserts that a ring morphism $f : A \to B$ is an epimorphism if and only if ${}^a f$ is injective, the residual extensions of f are isomorphic, $\Omega_A(B) = 0$, and the kernel of $B \otimes_A B \to B$ is finitely generated [**10**, IV, 1.1.5]. The last condition holds when $A \to B$ is of finite type, as an A-algebra. Now if f is a flat epimorphism of finite presentation, then f is an etale morphism since unramified (in French: *net*) and flat [**23**]. We recall that a ring morphism $A \to B$ is said to be etale standard if there exist a monic polynomial $P(z) \in A[z]$ and $Q(z) \in A[z]$ such that $B = (A[z]/(P(z)))_{Q(z)}$ and $P'(z)$ is a unit in B; this last condition is equivalent to

$$Q(z) \in \sqrt{(P(z)) + (P'(z))}.$$

Proposition 7.1. *Let $f : A \to B$ be a ring morphism of finite presentation. The following statements are equivalent*:

(1) *f is a flat epimorphism.*

(2) *f is an etale morphism with isomorphic residual extensions and ${}^a f$ is injective.*

(3) *f is an etale morphism and $K \bigotimes_A B = 0$ or $K \to K \bigotimes_A B$ is an isomorphism for every ring morphism $A \to K$ where K is a field, respectively an algebraically closed field.*

(4) *${}^a f$ is injective and for every prime ideal Q of B lying over P in A there exist $g \in B \setminus Q$, $h \in A \setminus P$ and an etale standard epimorphism $A_h \to B_g$ such that the following diagram commutes*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_h & \longrightarrow & B_g. \end{array}$$

Proof. In view of the above considerations, (1) is equivalent to (2). Now let P be a prime ideal of A and consider its associated fiber morphism $\mathbf{k}(P) \to \mathbf{k}(P) \otimes_A B = F_P$; the spectrum of F_P is well known to be homeomorphic with ${}^a f^{-1}(P)$. Moreover, the residual extensions of f are the residual extensions of all fiber morphisms. If $A \to K$ is a ring morphism where K is a field, let P be the kernel of $A \to K$. Then P is a prime ideal of A and $A \to K$ can be factored $A \to \mathbf{k}(P) \to K$. Since $\mathbf{k}(P) \to K$ is faithfully flat, it descends isomorphisms and nullity. Thus (3) implies (2). Assume that the hypotheses of (2) hold and let $A \to K$ be a ring morphism, K a field. Then $K \to K \otimes_A B$ is an epimorphism by (1) whence an isomorphism if $K \otimes_A B \neq 0$ [**10**, IV, 1.1.3]. Therefore, (2) $\Rightarrow$ (3). If (1) holds, then $A \to B$ is etale and an epimorphism. We deduce from the local structure of etale morphisms theorem that there exists a commutative diagram as in (3) where $A_h \to B_g$ is etale standard [**23**, V, 1.1]. Moreover, $A_h \to B_g$ is an epimorphism since $A \to B \to B_g$ is an epimorphism. Thus we get (1) $\Rightarrow$ (4). The hypothesis of (4) being granted, we get that $A \to B$ is etale by [**23**]. Moreover, $A_P \to B_Q$ is an isomorphism for $Q \in \mathrm{Spec}\,(B)$ lying over P since $A_h \to B_g$ is a flat epimorphism (see 1.1). Hence, condition (2) follows. $\square$

Remark 7.2. It follows that a flat epimorphism of finite presentation is locally on the spectrum an etale standard epimorphism. We say that an affine open subscheme U of $\mathrm{Spec}\,(A)$ is standard if there is a ring morphism $\varphi : A \to A_h \to B$ where $h \in A$ and $A_h \to B$ is an etale standard epimorphism such that ${}^a\varphi(\mathrm{Spec}\,(B)) = U$. Let A be a ring and O an affine open subset of $\mathrm{Spec}\,(A)$. It follows from the above result that $O = U_1 \cup \cdots \cup U_n$ where U_i is a standard affine open subset. Conversely, let O be a subset of $\mathrm{Spec}\,(A)$ such that $O = U_1 \cup \cdots \cup U_n$ where every U_i is standard affine and assume that the U_i do not meet. Then every U_i is associated to a flat epimorphism of finite presentation $A \to B_i$. Now consider the canonical morphism $A \to \prod B_i = B$. A prime ideal P of A belongs to only one U_i so that $B_P = (B_i)_P$ for some i. By 1.1, $A \to B$ is a flat epimorphism with spectral image O. Moreover, $A \to B$ is of finite presentation because so is every $A \to B_i$. Thus O is an affine open subset.

Therefore, the knowledge of affine open subsets relies on the study of etale standard epimorphisms. This is done below and provides examples of nonclassical flat epimorphisms.

Let A be a ring and $P(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n \in A[z]$ a monic polynomial with degree $n > 0$. Let $\sigma_1, \dots, \sigma_n \in A[z_1, \dots, z_n]$ be the elementary symmetric polynomials. We consider the splitting ring of $P(z)$

$$A' = A[z_1, \dots, z_n]/(\sigma_1 + a_{n-1}, \dots, \sigma_n - (-1)^n a_0).$$

The canonical ring morphism $A \to A'$ turns A' into a free module of rank $n!$. Define $x_1, \dots, x_n$ to be the classes of $z_1, \dots, z_n$ in A'. Then we have $P(z) = (z - x_1)\cdots(z - x_n)$. The elements x_i are called the virtual zeros of $P(z)$. Clearly, if $A \to B$ is any ring morphism such that $P(z)$ splits in B, there is a ring morphism $A' \to B$ factoring $A \to B$ through $A \to A'$. If $Q(z) \in A[z]$, we can consider the related polynomial of Tschirnhauss

$$T_{Q(z)} = (z - Q(x_1))\cdots(z - Q(x_n)) = z^n - r_{n-1}z^{n-1} + \cdots + (-1)^n r_0 \in A'[z].$$

Then the elements $r_0, \cdots, r_{n-1}$ belong to A since they are symmetric with respect to the elements x_i. It can be proved that $T_{Q(z)}$ is the characteristic polynomial of $Q(x)$ where x is the class of z in $A[z]/(P(z))$. Moreover, let $f : A \to B$ be a ring morphism such that $P^f(z) = (z - b_1)\cdots(z - b_n)$. Then we have $T_{Q(z)} = (z - Q^f(b_1))\cdots(z - Q^f(b_n))$.

Definition 7.3. Let A be a ring, $P(z) \in A[z]$ a monic polynomial with degree $n > 0$ and $Q(z) \in A[z]$. The elements $r_0, \dots, r_{n-1}$ are called the hyper-resultants of $P(z)$ and $Q(z)$ and are denoted, respectively, by $\mathrm{Res}_i((P(z), Q(z))$.

Observe that $\pm\mathrm{Res}_0(P(z), Q(z))$ is nothing but the classical resultant of $P(z)$ and $Q(z)$. In a similar way, the elements $\mathrm{H}_i(P(z)) = \mathrm{Res}_i(P(z), P'(z))$ of A are called the hyper-discriminants of $P(z)$.

Proposition 7.4. *Let A be an integral domain, $P(z) = (z - a_1) \cdots (z - a_n)$ and $Q(z)$ in $A[z]$. In order that $Q(a_i) = 0$ for $p \leq n$ elements a_i, it is necessary and sufficient that $Res_i(P(z), Q(z)) = 0$ for $i = 0, \dots, p-1$.*

Proof. Straightforward. □

The following result may have an interest for its own sake. To abbreviate, we say that a ring B is over a ring A if there is a ring morphism $A \to B$.

Lemma 7.5. *Let A be a ring and $P(z) \in A[z]$ a monic polynomial with degree $n > 0$. Then $P(z)$ has at least a simple zero in every algebraically closed field K over A if and only if*

$$\big(H_0(P(z)), \dots, H_{n-1}(P(z))\big) = A.$$

Proof. The condition on hyper-discriminants means that for any prime ideal P of A there is some H_i which does not lie in P. Hence, for any algebraically closed field K over A, there is some $i \in \{0, \dots, n-1\}$ such that $H_i \neq 0$ in K. It follows from the above result that some zero of $P(z)$ is not a zero of $P'(z)$, that is to say $P(z)$ has a simple zero in K. To see the converse, observe that there is a factorization $A \to \mathbf{k}(P) \to K$ where $P = \mathrm{Ker}\,(A \to K)$. □

Theorem 7.6. *Let A be a ring, $P(z) \in A[z]$ a monic polynomial with degree $n > 0$ and $Q(z) \in A[z]$. Then $f : A \to \big(A[z]/(P(z))\big)_{Q(z)} = B$ is a flat epimorphism if and only if the following conditions hold*:

(a) $Q(z) \in \sqrt{(P(z)) + (P'(z))}$.

(b) $\mathrm{Res}_0(P(z), Q(z)), \ldots, \mathrm{Res}_{n-2}(P(z), Q(z)) \in \mathrm{Nil}\ (A)$.

Proof. Condition (a) means that $A \to B$ is an etale morphism. Assume that (a) holds. According to 7.1, (3), we can reduce to the case where A is an algebraically closed field K. We first examine when the ring $C = \big(K[z]/(z-k)^n\big)_{Q(z)}$ is zero or is isomorphic to K. The ring $K[z]/(z-k)^n$ has only one prime ideal $(x-k)$, x the class of z. Therefore, C equals zero if and only if $Q(x) \in (x-k)$, that is to say $Q(k) = 0$. Now, if $C \neq 0$, then C is a field if and only if $(x-k)C = 0$, that is to say there is some integer p such that $Q(z)^p(z-k) \in (z-k)^n$. Then $n > 1$ leads to the contradiction $Q(k) = 0$ whence $n = 1$. Conversely, C is a field if $n = 1$ and $C \neq 0$ and in this case $K \to C$ is an isomorphism. We go back to $K \to B$. The ring B is isomorphic to a product of finitely many rings $C_j = (K[z]/(z-x_j)^{n_j})_{Q(z)}$, $j = 1, \ldots, p$, where $Z = \{x_j\}$ is the set of all zeros of $P(z)$ in K with multiplicity n_j. According to the previous remarks, $B = 0$ if and only if $Q(x_1), \ldots, Q(x_p) = 0$ that is to say, $r_i = \mathrm{Res}_i(P(z), Q(z)) = 0$ for $i = 0, \ldots, n-1$. Furthermore, B is a field if and only if the following condition is fulfilled ($\star$): $p-1$ elements in Z are zeros of $Q(z)$ and the other is a simple zero of $P(z)$ and is not a zero of $Q(z)$. Now set $P(z) = (z-a_1)\cdots(z-a_n) \in K[z]$. Then condition ($\star$) implies that $Q(a_1) = \cdots = Q(a_{n-1}) = 0$ and $Q(a_n) \neq 0$ for a suitable choice of the indexes. Conversely, assume that this last condition holds and that (a) is satisfied. There is some integer s such that $Q(z)^s \in (P(z), P'(z))$ so that a_n is a simple zero of $P(z)$. Thus assuming that (a) holds, the condition ($\star$) is equivalent to $r_i = 0$ for $i = 0, \ldots, n-2$ and $r_{n-1} \neq 0$. Therefore, when A is any ring, the condition (a) being assumed, $A \to B$ is a flat epimorphism if and only if $\mathrm{Spec}\,(A) = V(r_0, \ldots, r_{n-1}) \cup ((V(r_0, \ldots, r_{n-2}) \cap D(r_{n-1})) = V(r_0, \ldots, r_{n-2})$. $\square$

Remark 7.7. In addition to the hypotheses of 7.6, assume that A is reduced. Then the condition (b) can be translated as follows. The characteristic polynomial of $Q(x)$ (x is the class of z in $A[z]/(P(z))$) is $z^{n-1}(z-a)$ where $a \in A$. Indeed, this is a consequence of the characterization of the Tschirnhauss polynomial as a characteristic polynomial.

By the Cayley-Hamilton Theorem, we get that $Q(z)^{n-1}(Q(z)-a) \in (P(z))$.

Remark 7.8. When $A \to (A[z]/(P(z)))_{Q(z)} = B$ is a flat epimorphism, then $P(z)$ has a unique simple zero in every algebraically closed field K over A such that $\mathrm{Ker}\,(A \to K) \in \mathcal{X}(B)$. Hence $P(z)$ has at least a simple zero in K and the other zeros are multiple. We get that $H_i \neq 0$ in K for some $i \in \{0, \dots, n-1\}$ for the first condition, while the second condition gives $H_0, \dots, H_{n-2} = 0$ in K. These conditions combine to yield $\mathcal{X}(B) \subset V(H_0, \dots, H_{n-2}) \cap D(H_{n-1})$.

Next we give some information about rings A in which every admissible subset $X = \mathcal{X}(A_S)$ for some multiplicative subset S of A. Let A be a ring and $X \subset \mathrm{Spec}\,(A)$. Then X is said to be expanded if a prime ideal P belongs to $\mathrm{G}\,(X)$ whenever $P \subset \cup\,[\,Q\,;\,Q \in X]$ while X is said to be fathomable if a finitely generated ideal I is contained in some element $P \in X$ whenever $I \subset \cup\,[\,Q\,;\,Q \in X]$ [**18**]. An expanded subset is fathomable [**18**, 2.1]. Moreover, when X is quasi-compact, X is expanded if and only if X is fathomable [**18**, 2.2].

Lemma 7.9. *Let A be a ring and X a subset of* $\mathrm{Spec}\,(A)$.

(1) *If X is stable under generalizations and expanded, then $X = X_m = \mathcal{X}(A_{S_X})$.*

(2) *If X is admissible, then X is fathomable if and only if $X = X_m = \mathcal{X}(A_{S_X})$.*

In particular, a flat epimorphism with a fathomable spectral image X is a localization $A \to A_{S_X}$.

Proof. To see (1), we need only to show that $X_m \subset X$ by 1.8. Let P be an element of X_m. We have $P \subset A \backslash S_X = \cup\,[\,Q\,;\,Q \in X]$ by [**18**, 1.2]. Since X is expanded, we get $P \in \mathrm{G}\,(X) = X$. Now an admissible subset X is quasi-compact and stable under generalizations. If in addition X is fathomable, then X is expanded and the conclusion of (2) follows from (1). Conversely, $\mathcal{X}(A_S)$ is fathomable for any multiplicative subset of A. $\square$

Remark 7.10. There exist rings A in which every subset of the spectrum is fathomable. In this case, A is called absolutely fathomable [**18**]. Such rings A are characterized by the following property. For every finitely generated ideal I of A, there is some $a \in A$ such that $\sqrt{Aa} = \sqrt{I}$, [**18**, 4.2]. Clearly, a quasi-compact open subset of $\mathrm{Spec}\,(A)$ is affine when A is absolutely fathomable. The previous result shows that every flat epimorphism $A \to B$ is a localization for such rings. Absolutely flat rings and Bézout rings are absolutely fathomable. A less trivial example is given by rings $\mathcal{C}(E)$ of real continuous functions defined on a topological space E [**18**, 3.20].

Remark 7.11. Any ring $\mathcal{C}(E)$ is seminormal. If f, $g : E \to \mathbf{R}$ are continuous functions such that $f^3 = g^2$, set $t = \sqrt[3]{g} \in \mathcal{C}(E)$ so that $g = t^3$; then $f \geq 0$ gives $f = t^2$.

Things are more complicated for t-closedness. Let E be a completely regular space. Unlike the literature, we denote by $V(f) \subset E$ the (closed) zero-set of $f \in \mathcal{C}(E)$ and its cozero-set by $D(f)$. Then an element f in $\mathcal{C}(E)$ is a regular element if and only if $D(f)$ is dense in E. The total quotient ring of $\mathcal{C}(E)$ can be calculated as follows. We have $\mathrm{Tot}\,(\mathcal{C}(E)) = \varinjlim \mathcal{C}(V)$, V ranging over all dense cozero-sets in E. Moreover, for two dense cozero-sets $U \supset V$, the canonical morphism $\mathcal{C}(U) \to \mathcal{C}(V)$ is injective so that two elements $f : U \to \mathbf{R}$, $g : V \to \mathbf{R}$ of the injective limit equal if and only if there is some dense cozero-set $W \subset U, V$ such that $f_{|W} = g_{|W}$ (see the representation theorem [**5**, 2.6]).

Proposition 7.12. *Let (E, d) be a connected metric space.*

(1) *$\mathcal{C}(E)$ is a decent ring.*

(2) *$\mathcal{C}(E)$ is an integral domain when $\mathcal{C}(E)$ is t-closed.*

Therefore, $\mathcal{C}(\mathbf{R})$ is not t-closed.

Proof. We first show that the total quotient ring of $\mathcal{C}(E)$ is absolutely flat so that this ring is decent. Let $f \in \mathcal{C}(U)$ where U is a dense cozero-set. Consider the open subset $V = D_U(f) \cup \mathrm{Int}\,(V_U(f))$. Then V is the union of two open subsets of U which do not meet. Furthermore, V is dense in U. Now V is also a dense open subset of E and is a cozero-set since E is a metric space ($V = D(h)$ where $h = d(\ ,E \setminus V)$). Next we define a continuous function $g : V \to \mathbf{R}$ by setting $g(x) = f(x)^{-1}$ if $x \in D_U(f)$ and $g(x) = 0$ if $x \in \mathrm{Int}\,(V_U(f))$. Obviously, we have $(f^2g)_{|V} = f_{|V}$. It follows that $\mathrm{Tot}\,(\mathcal{C}(E))$ is absolutely flat. Next, assume that $\mathcal{C}(E)$ is t-closed. Then each idempotent of $\mathrm{Tot}\,(\mathcal{C}(E))$ belongs to $\mathcal{C}(E)$. But an idempotent function $e : E \to \mathbf{R}$ is 0 or 1 since E is connected. Indeed, $V(e)$ is open and closed. It follows that $\mathrm{Tot}\,(\mathcal{C}(E))$ is a field since any element of an absolutely flat ring can be written eu where e is an idempotent and u a unit. In this case, $\mathcal{C}(E)$ is an integral domain. □

Recall that a topological space is said to be basically disconnected, respectively extremally disconnected, if every cozero-set, respectively open set, has an open closure.

Proposition 7.13. *Let E be a topological space. Then the following statements are equivalent*:

(1) $\mathcal{C}(E)$ *is a t-closed ring* (*whence a weak Baer ring*).

(2) E *is basically disconnected.*

(3) $\mathcal{C}(E)$ *is a weak Baer ring.*

(4) $\mathrm{Min}\,(\mathcal{C}(E))$ *is compact and* $\mathcal{C}(E)$ *is normal.*

Therefore, $\mathcal{C}(E)$ is t-closed when E is extremally disconnected.

Proof. According to [**2**, Remark 3], $\mathcal{C}(E)$ is a weak Baer ring if and only if E is basically disconnected. Now, $\mathcal{C}(E)/P$ is an integrally closed domain for any prime ideal P [**24**, p. 301]. Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follows from [**17**, 2.9]. Recall that a reduced ring is a weak Baer ring if and only if $\mathrm{Min}\,(A)$ is compact and the irreducible components do not meet. Now, in view of [**8**, 0.6.5.1], a ring A is normal if and only if A_P is an integrally closed domain for every prime ideal P or equivalently, A is reduced, A/M is an integrally closed domain for every minimal prime ideal M and the irreducible components do not meet. This completes the proof. □

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