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ON THE FORM OF CORRELATION FUNCTION FOR A CLASS OF NONSTATIONARY FIELD WITH A ZERO SPECTRUM

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ABSTRACT. The present paper is devoted to the derivation of an explicit form of linearly representable random fields in the form $h(x_1, x_2) = \exp\{i(x_1A_1 + x_2A_2)\}h$, where $h \in H$, H is a Hilbert space, operators A_1, A_2 are such that $A_1A_2 = A_2A_1$ and $C^3 = 0$ where $C = A_1^*A_2 - A_2A_1^*$.

The results obtained are the generalization of theorem proved by Livshits and Yantsevitch [**4**] and Yantsevich and Abbaui [**6**].

It is shown that a rank of nonstationary of field $h(x_1, x_2)$ depends not only on a degree of nonself conjugation of A_1, A_2 but on a degree of nilpotency of commutator $C(C^3 = 0)$.

In the present paper an explicit form of correlation function when the spectrum of A_1 and A_2 lies in zero is derived.

1. Preliminary information.

1.1. Let us consider a vector field $h(x)$ depending of two variables $x = (x_1, x_2) \in \mathbf{R}^2$ with values in the Hilbert space H.

In this paper we will suppose that $h(x)$ depends on x as $h(x) = Z_xh$ where $Z_x = \exp[i(x_1A_1 + x_2A_2)]$. In this case A_1 and A_2 are such operators in the Hilbert space H for which $A_1A_2 = A_2A_1$. We shall call an operator function Z_x to be an two-parameter commutative semigroup. The main tool of correlation theory for vector fields in a Hilbert space H is a correlation function [**4**]:

$$K(x, y) = \langle h(x), h(y)\rangle, \tag{1}$$

where $x, y \in \mathbf{R}^2$. For twice permutational classes of linear operators $\{A_1, A_2\}$, $(A_1A_2 = A_2A_1, A_1^*A_2 = A_2A_1^*)$. Generalizing the results

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given by Livshits and Jantsevich [**4**], Yantsevich and Abbaui [**6**] have introduced partial infinitesimal correlation functions (ICF) by relations (under the assumption that $K(x_1, x_2, y_1, y_2)$ is a twice differentiable function):

(2)

$$W_1(x_1, x_2, y_1, y_2) = -\frac{\partial K(x_1+\tau_1, x_2, y_1+\tau_1, y_2)}{\partial \tau_1}\,|_{\tau_1=0}$$
$$W_2(x_1, x_2, y_1, y_2) = -\frac{\partial K(x_1, x_2+\tau_2, y_1, y_2+\tau_2)}{\partial \tau_2}\,|_{\tau_2=0}$$
$$W(x_1, x_2, y_1, y_2) = -\frac{\partial^2 K(x_1+\tau_1, x_2+\tau_2, y_1+\tau_1, y_2+\tau_2)}{\partial \tau_1 \partial \tau_2}\,|_{\tau_1\tau_2=0}$$

W_1, W_2 and W are not independent.

Indeed:

$$\int_o^{-y_1} W(x_1+\tau_1, x_2+\tau_2, y_1+\tau_1, y_2+\tau_2)\, d\tau_1$$
$$= \left[\frac{\partial}{\partial \tau_2}K(x_1-y_1, x_2+\tau_2, 0, y_2+\tau_2) - \frac{\partial}{\partial \tau_2}K(x_1, x_2+\tau_2, y_1, y_2+\tau_2)\right]$$
$$= -W_2(x_1-y_1, x_2+\tau_2, 0, y_2+\tau_2) + W_2(x_1, x_2+\tau_2, y_1, y_2+\tau_2).$$

Similarly it is easy to get:

$$\int_o^{-y_2} W(x_+\tau_1, x_2+\tau_2, y_1+\tau_1, y_2+\tau_2)\, d\tau_2$$
$$= -W_1(x_1+\tau_1, x_2-y_2, y_+\tau_1, 0) + W_1(x_1+\tau_1, x_2, y_1+\tau_1, y_2).$$

(3) $$\int_0^{-y_1}\int_0^{-y_2} W(x_1+\tau_1, x_2+\tau_2, y_1+\tau_1, y_2+\tau_2)\, d\tau_1\, d\tau_2$$
$$= K(x_1-y_1, x_2, y_2, 0, 0) - K(x_1-y_1, x_2, 0, y_2)$$
$$- K(x_1, x_2-y_2, y_1, 0) - K(x_1, x_2, y_1, y_2).$$

Let us remember that the field $h(x)$ in H is called dissipative if $(A_1)_I \geq 0$. As in the one-dimension case it is easy to establish [**4, 6**] that:

$$\lim_{\tau_1\to\infty} K(x_1+\tau_1, x_2, y_1+\tau_1, y_2) = K^1_\infty(x_1-y_1, x_2, y_2);$$

(4) $$\lim_{\tau_2\to\infty} K(x_1, x_2+\tau_1, y_1, y_2+\tau_1) = K^2_\infty(x_2-y_2, x_1, y_1);$$

$$\lim_{\tau_1,\tau_2\to\infty} K(x+\tau, y+\tau) = K_\infty(x-y).$$

If the correlation function depends only on a difference in arguments then a field is called a stationary field [**4**] (in just this way a stationary was defined by Kolmogorov).

Then the formula (3) may be presented in the form:

$$\begin{aligned} K(x,y) = &\int_0^\infty \int_0^\infty W(x+\tau, y+\tau)\, d\tau_1\, d\tau_2 \\ &+ K^1_\infty(x_1-y_1, x_2, y_2) + K^2_\infty(x_2-y_2, x_1, y_1) \\ &+ K_\infty(x-y). \end{aligned} \tag{5}$$

$K_\infty(x-y)$ is a Hermitian-positive function which may be considered as a stationary field correlation function, $K^1_\infty(x_1-y_1, x_2, y_2)$ (as well as $K^2_\infty(x_2-y_2, x_1, y_1)$) in variable x_1-y_1 is a Hermitian-positive function for each x_2, y_2, and , as a function of x_2, y_2, is a dissipative curve of one variable in H. Thus, essentially everything is determined by the infinitesimal correlation function $W(x,y)$.

1.2. Let us introduce as in [**4, 6**] a rank of nonstationarity.

We recall that the rank of nonstationary of function $h(x)$ of twice permutational system of linear operators A_1, A_2 is the greatest rank of quadratic form

$$\sum_{\alpha,\beta=1}^{n} W(x_\alpha, x_\beta)\zeta_\alpha\bar{\zeta}_\beta, \quad x_\alpha \in \mathbf{R}^2,\ \zeta_\alpha \in \mathbf{C}, \quad n < \infty.$$

It is not difficult to show that the rank of nonstationarity for the present case coincides with the dimension of space H_0 where $H_0 = \overline{(A_1)_I H} \bigcap \overline{(A_2)_I H}$ (here as usual $(A_k)_I = (A_k - A_k^*)/(2i)$ [**4**]) and in addition

$$W(x,y) = 4\langle (A_1)_I (A_2)_I\, h(x), h(y)\rangle. \tag{6}$$

The derivation of formula (6): From formula (2) it follows that

$$\begin{aligned} W_1(x_1,x_2,y_1,y_2) &= -\frac{\partial K(x+\tau_1,x_2,y_1+\tau_1,y_2)}{\partial \tau_1}\,|_{\tau_1=0} \\ &= -\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial y_1}\right)K(x_1,x_2,y_1,y_2) \\ &= -\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial y_1}\right)\langle Z_x h, Z_y h\rangle \\ &= -\langle iA_1 Z_x h, Z_y h\rangle - \langle Z_x h, iA_1 Z_y h\rangle \\ &= \left\langle \frac{A_1-A_1^*}{i}\, Z_x h_1 Z_y h\right\rangle = 2\langle (A_1)_I h(x), h(y)\rangle. \end{aligned}$$

Similarly,

$$W_2(x_1,x_2,y_1,y_2) = 2\langle (A_2)_I h(x), h(y)\rangle.$$

Therefore,

$$W(x_1,x_2,y_1,y_2) = -\left(\frac{\partial}{\partial x_2}+\frac{\partial}{\partial y_2}\right)W_1(x_1,x_2,y_1,y_2).$$

Then we get that

$$\begin{aligned} W(x_1,x_2,y_1,y_2) &= -\left(\frac{\partial}{\partial x_2}+\frac{\partial}{\partial y_2}\right)\langle 2(A_1)_I h(x), h(y)\rangle \\ &= -\langle 2(A_1)_I iA_2 h(x), h(y)\rangle - \langle 2(A_1)_I h(x), iA_2 h(y)\rangle \\ &= 2\left\langle \frac{(A_1)_I A_2 - A_2^*(A_1)_I}{i}\, h(x), h(y)\right\rangle. \end{aligned}$$

As A_1 and A_2 are twice permutable then,

$$W(x,y) = 2\left\langle (A_1)_I \frac{A_2-A_2^*}{i}\, h(x), h(y)\right\rangle = 4\langle (A_1)_I (A_2)_I h(x), h(y)\rangle$$

For the case $\dim H_0 = 1$, i.e. when the rank of nonstationarity of vector field $h(x)$ is equal to one, we get

$$W(x,y) = \Phi(x)\overline{\Phi(y)}, \tag{7}$$

where $\Phi(x) = \langle h(x), h_0\rangle$

2. Correlation functions and spectral representation for the twice premutational fields of rank 1.

2.1. Let us consider a vector field $h(x_1, x_2) = \exp(ix_1A_1 + ix_2A_2)h$, where $h \in H, H_0 = \overline{(A_1)_I H} \cap \overline{(A_2)_I H}$, $\dim H_0 = 1$ and operators A_1 and A_2 are twice permutable. As $H_0 = \overline{(A_1)_I H} \cap \overline{(A_2)_I H}$ is univariable and the operator $4(A_1)_I(A_2)_I$ is self-adjoint, then general theory gives

$$4(A_1)_I(A_2)_I h = \langle h, h_0\rangle h_0$$

for any $h \in H$. Therefore from formula (6) it follows that

$$\begin{aligned} W(x_1, x_2, y_1, y_2) &= \langle 4(A_1)_I(A_2)_I h(x), h(y)\rangle \\ &= \big\langle \langle h(x_1, x_2), h_0\rangle h_0, h(y_1, y_2)\big\rangle \\ &= \langle h(x_1, x_2), h_0\rangle \langle h_0, h(y_1, y_2)\rangle \\ &= \Phi(x_1, x_2) \cdot \overline{\Phi(y_1, y_2)} \end{aligned}$$

where

$$\Phi(x_1, x_2) = \langle h(x_1, x_2), h_0\rangle = \langle \exp(ix_1A_1 + ix_2A_2)h, h_0\rangle.$$

As it was shown in [**6**], then the ICF of vector field $h(x_1, x_2)$ has the form

$$W(x_1, x_2, y_1, y_2) = \Phi(x_1, x_2)\overline{\Phi(y_1, y_2)},$$

where $\Phi(x_1, x_2) = \langle \exp(ix_1A_1 + ix_2A_2)h, h_0\rangle$, $h_0 \in H_0$, $\|h_0\| = 1$, $2\,\mathrm{Im}\,A_1 2\,\mathrm{Im}\,A_2 h_0 = \lambda_0 h_0$ and λ_0 is a real number.

Applying the well-known Risse-Danford representation for functions of operators [**4,5**],

Using relation [**4**]

$$\exp(tA) = -\frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)(A - \lambda I)^{-1}\,d\lambda$$

where Γ is a closed path that contains all the spectrum of operator A, one can represent the function $\Phi(x_1, x_2)$ in the form

$$\Phi(x_1, x_2) = \left(\frac{1}{2\pi i}\right)^2 \oint_{\Gamma_1}\oint_{\Gamma_2} \exp(i\lambda_1 x_1 + i\lambda_2 x_2) \langle (A_1 - \lambda_1 I)^{-1}(A_2 - \lambda_2 I)^{-1}h, h_0\rangle\,d\lambda_1,\,d\lambda_2. \tag{8}$$

Closed path Γ_k includes the spectrum of operator A_k, $k = 1, 2$. When calculating integrals in (8) one can pass to any system of operators $\overset{\bullet}{A}_1, \overset{\bullet}{A}_2$, acting in Hilbert space $\overset{\bullet}{H}$, which are unitary equivalent to the original operators A_1, A_2:

$$((A_1-\lambda_1 I)^{-1}(A_2-\lambda_2 I)^{-1}h, h_0)_H = ((\overset{\bullet}{A}_1-\lambda_1 I)^{-1}(\overset{\bullet}{A}_2-\lambda_2 I)^{-1}g, g_0)_{\overset{\bullet}{H}},$$

where $\overset{\bullet}{A}_k U = U A_k$, $k = 1, 2$, and U is a unitary operator acting from H in $L^2(D)$,

$$D = [0 \times l_1] \times [0, l_2], \quad U h_0 = g_0.$$

Then the function $\Phi(x_1, x_2)$ is presented in the form

$$\Phi(x_1, x_2) = \left(-\frac{1}{2\pi i}\right)^2 \oint_{\Gamma_1}\oint_{\Gamma_2} \exp\left(i\lambda_1 x_1 + i\lambda_2 x_2\right) \\ \times \left\langle (\overset{\bullet}{A}_1 - \lambda_1 I)^{-1}(\overset{\bullet}{A}_2 - \lambda_2 I)^{-1})g, g_0\right\rangle d\lambda_1\, d\lambda_2.$$

2.2. Let us consider a case when the function $h(x_1, x_2)$ belongs to class $K_{11}^{(1)}$, i.e., the spectrum of each operator A_k, $k = 1, 2$, is contracted in zero. Then [**7**] the model space $\overset{\bullet}{H}$ coincides with $L^2(D), D = [0, l_1] \times [0, l_2], l_1, l_2 < \infty$.

The operators $\overset{\bullet}{A}_1$ and $\overset{\bullet}{A}_2$ are defined in $L^2(D)$ as follows:

$$\overset{\bullet}{A}_1 f(x, y) = -i\int_x^{l_1} f(t, y)\, dt; \qquad \overset{\bullet}{A}_2 f(x, y) = -i\int_y^{l_2} f(x, \tau)\, d\tau,$$

where x and y are one dimensional. Due to the unitary equivalence H_o is mapped by operator U on $\overset{\bullet}{H}_0 = 2\mathrm{Im}\,\overline{\overset{\bullet}{A}_1\overset{\bullet}{H}} \cap 2\mathrm{Im}\,\overline{\overset{\bullet}{A}_2\overset{\bullet}{H}}$ which is a subspace of constant functions from $L^2(D)$, therefore $h_0(x, y) \equiv 1$, and $\overset{\bullet}{h}_0 = f(x, y)$.

It is not difficult to show that

$$(\overset{\bullet}{A}{}_1^{*} - \lambda_1 I)^{-1}(\overset{\bullet}{A}{}_2^{*} - \lambda_2 I)^{-1} h_0(x, y) = \frac{1}{\lambda_1\lambda_2}\exp\left(\frac{ix}{\lambda_1} + \frac{iy}{\lambda_2}\right).$$

Then

$$\Phi(x_1,x_2)=\left(-\frac{1}{2\pi i}\right)^2\oint_{\Gamma_1}\oint_{\Gamma_2}\exp\left(i\lambda_1x_1+i\lambda_2x_2\right)$$
$$\times\left[\int_D\frac{1}{\lambda_1\lambda_2}\exp\left(\frac{i\zeta_1}{\lambda_1}+\frac{i\zeta_2}{\lambda_2}\right)f(\zeta_1,\zeta_2)\,d\zeta_1\,d\zeta_2\right]d\lambda_1\,d\lambda_2$$
$$=\left(-\frac{1}{2\pi i}\right)^2\int_D\left[\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{1}{\lambda_1}\exp\left(i\lambda_1x_1+\frac{i\zeta_1}{\lambda_1}\right)\right.$$
$$\left.\times\frac{1}{\lambda_2}\exp\left(i\lambda_2x_2+\frac{i\zeta_2}{\lambda_2}\right)d\lambda_1\,d\lambda_2\right]f(\zeta_1,\zeta_2)\,d\zeta_1\,d\zeta_2$$

and finally

$$\Phi(x_1,x_2)=\int_0^{l_1}\int_0^{l_2}J_0(2\sqrt{x_1\zeta_1}\,)J_0(2\sqrt{x_2\zeta_2}\,)f(\zeta_1,\zeta_2)\,d\zeta_1\,d\zeta_2,$$

where

$$J_0(2\sqrt{x_1\zeta_1}\,)=\sum_{n=1}^{\infty}\frac{(-1)^n(x_1\zeta_1)^n}{(n!)^2}.$$

3. Correlation functions for commutative systems of operators in case of nilpotentness of the commutator $C = [A_1^*, A_2](C^3 = 0)$.

3.1. Similar to the class of twice permutable system of linear operator for the vector field

$$h(x)=Z_xh,\quad x=(x_1,x_2)\in\mathbf{R}^2,\quad Z_x=\exp\left[i(x_1A_1+x_2A_2)\right],\quad h\in H,$$

where the system of operators $\{A_1,A_2\}$ is such that

$$[A_1,A_2]=0,\quad C=[A_1^*,A_2],\quad C^3=0,\quad C^2\neq 0 \tag{9}$$

we introduce the correlation functions

(10)

$$W_1(x_1,x_2,y_1,y_2)=-\frac{\partial}{\partial\tau_1}K(x_1+\tau_1,x_2,y_1+\tau_1,y_2)\,|_{\tau_1=0}$$
$$W_2(x_1,x_2,y_1,y_2)=-\frac{\partial}{\partial\tau_2}K(x_1,x_2+\tau_2,y_1,y_2+\tau_2)\,|_{\tau_2=0}$$
$$W(x_1,x_2,y_1,y_2)=-\frac{\partial^2}{\partial\tau_1\partial\tau_2}K(x_1+\tau_1,x_2+\tau_2,y_1+\tau_1,y_2+\tau_2)\,|_{\tau_1=\tau_2=0}$$

It is not difficult to see that for the case of vector field $h(x)$ one can obtain [**3**]

$$\begin{aligned} W_1(x_1,x_2,y_1,y_2) &= 2\langle (A_1)_I h(x), h(y)\rangle \\ W_2(x_1,x_2,y_1,y_2) &= 2\langle (A_2)_I h(x), h(y)\rangle \\ W(x_1,x_2,y_1,y_2) &= \langle Dh(x), h(y)\rangle. \end{aligned} \tag{11}$$

Here operator D is self-adjoint and is of the form

$$D = 2i\big(A_2^*(A_1)_I - (A_1)_I A_2\big) = 2i\big(A_1^*(A_2)_I - (A_2)_I A_1\big). \tag{12}$$

Let us show that D may be represented as (12). From formula (10) using differentiation rules one can easily get

$$\begin{aligned} W_1(x_1,x_2,y_1,y_2) &= -\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right) K(x,y) \\ &= -\Big\langle \frac{\partial}{\partial x_1} h(x), h(y)\Big\rangle - \Big\langle h(x), \frac{\partial}{\partial y_1} h(y)\Big\rangle \\ &= \big\langle -iA_1 h(x), h(y)\big\rangle + \big\langle h(x), iA_1 h(y)\big\rangle \\ &= 2\Big\langle \left(\frac{A_1 - A_1^*}{i}\right) h(x), h(y)\Big\rangle. \end{aligned}$$

Then we can find $W(x,y)$

$$\begin{aligned} W(x,y) &= -\frac{\partial}{\partial x_2}\, 2\langle (A_1)_I h(x), h(y)\rangle - \frac{\partial}{\partial y_2}\, \langle 2(A_1)_I h(x), h(y)\rangle \\ &= -\,\langle 2i(A_1)_I A_2 h(x), h(y)\rangle - \langle 2(A_1)_I h(x), iA_2 h(y)\rangle \end{aligned}$$

that is the proof (12). Elementary evaluations show that the operator D in (12) can be reduced to

$$D = C + 4(A_2)_I (A_1)_I = C^* + 4(A_1)_I (A_2)_I. \tag{13}$$

In what follows,in order to render a concrete form of operator D we confine ourselves to the systems of linear operators that satisfy the next theorem proved in [**7**]. A system of operators A_1, A_2 is called a simple system [4] if there is no subspace in H which, reducing the operators A_1 and A_2, a contraction on which is self-adjoint at least for one of operator A_k.

Theorem 1. *Let us assume that a simple commuting system of linear operators A_1, A_2 is such that*:

1. $C^3 = 0$, $\dim CH = 2$
2. $\dim H_0 = 1$, $H_0 = 1$, $H_0 = \overline{(A_1)_I H} \cap \overline{(A_2)_I H}$
3. $\overline{(A_1)_I C^k H} \subset C^k H$, $(A_2)_I C^{*p} H \subset C^{*p} H$, $k, p = 1, 2$.

Then the space H is decomposed into the orthogonal sum $H = H_1 \oplus H_2 \oplus H_3$, where H_k reduces A_1 and subspaces H_3 and $H_2 \oplus H_3$ are invariant relative to A_2 and the contractions of system $\{A_1, A_2\}$ on H_k are twice permutable.

This theorem has been proved in [**7**]. In what follows we assume that a system of linear operators $\{A_1, A_2\}$ satisfies the assumption of Theorem 1. Let $C^2 H = \{\lambda h_3\}$, $CH \ominus C^2 H = \{\mu h_2\}$ and $C^{*2} H = \{\lambda g_3\}$, $C^* H \ominus C^{*2} H = \{\mu g_2\}$. It is obvious that $h_3 \perp g_3, g_2$ and $g_3 \perp h_3, h_2$. This readily follows from the condition $C^3 = C^{*3} = 0$. One can easily see that $H_3 \cap H_0 = \{\lambda h_3\}$, $H_2 \cap H_0 = \{2 h_2\}$. Let us denote by h_1 a vector such that $\{\lambda \tilde{h}_1\} = H_1 \cap H_0$ and introduce the following vectors:

$$
\begin{aligned}
h_1 &= \tilde{h}_1 = \langle \tilde{h}_1, g_3 \rangle g_3, \\
g_1 &= \tilde{g}_1 = \langle \tilde{g}_1, h_3 \rangle h_3,
\end{aligned}
$$

where the vector g_1 is such that $g_1 + h_3 + g_2 + g_3 = h_0$, where h_0 is a basis vector of space H_0.

Then it is easy to see that

$$
DH = H_D = \operatorname{span}\{h_3, h_2, h_1, g_1, g_2, g_3, \}.
$$

Thus, the operator D, corresponding to the defect of being non-stationary, maps H into a six-dimensional space.

Let us find an explicit form of self-adjoint operator D defined in H_D. Really, it is easy to see that

$$
Dh_3 = Ch_3 + 4(A_2)_I (A_1)_I h_3 = 4(A_2)_I \alpha_3 h_3,
$$

where $(A_1)_I h_3 = \alpha_3 h_3$. Therefore

$$
\langle Dh_3, g_2 \rangle = 0 \quad \text{and} \quad \langle Dh_3, g_3 \rangle = 0.
$$

Similarly one can obtain

$$Dh_2 = Ch_2 + 4(A_2)_I(A_1)_I h_2 = \mu h_3 + 4(A_2)_I \alpha_2 h_2,$$

where $(A_1)_I h_2 = \alpha_2 h_2$. Thus $\langle Dh_2, g_3\rangle = 0$. By repeating the same arguments one can obtain

$$\langle Dh_3, h_1\rangle = 0, \quad \langle Dg_3, h_2\rangle = 0, \quad \langle Dg_2, h_3\rangle = 0.$$

Hence, we have proved the following lemma.

Lemma 1. *The matrix of the operator D in the basis $\{h_1, h_2, h_3, g_1, g_2, g_3\}$ of the space H_D can be written in the form*

$$\begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} & 0 & 0 \\ d_{12} & d_{22} & d_{23} & d_{24} & d_{25} & 0 \\ d_{13} & d_{23} & d_{33} & d_{34} & d_{35} & d_{36} \\ d_{14} & d_{24} & d_{34} & d_{44} & d_{45} & d_{64} \\ 0 & d_{25} & d_{35} & d_{45} & d_{55} & d_{56} \\ 0 & 0 & d_{36} & d_{46} & d_{56} & d_{66} \end{pmatrix} \tag{14}$$

where $d_{k,s} \in \mathbf{R}$ are real numbers.

Thus D is a generalization of Jacobian matrix, namely D is a semi-diagonal matrix.

Consequently

$$Dh = \sum_{\alpha,\beta=1}^{6} \langle h, l_\alpha\rangle d_{\alpha,\beta} l_\beta, \tag{15}$$

where $l_1 = h_3,\ l_2 = h_2,\ l_3 = h_1,\ l_4 = g_1,\ l_5 = g_2,\ l_6 = g_3$.
Here as above we denote $\|l_k\| = 1$, $k = 1, \dots, 6$, and $d_{\alpha,\beta} = \langle Dl_\alpha, l_\beta\rangle$.

3.2. Now let us consider the infinitesimal correlation function $W(x, y)$ (11):

$$W(x, y) = \langle Dh(x), h(y)\rangle.$$

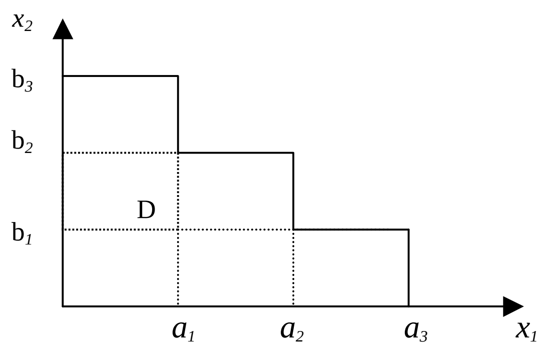
Then by virtue of (15) one can obtain

$$W(x,y) = \sum_{\alpha,\beta=1}^{6} \langle h(x), l_\alpha\rangle d_{\alpha,\beta} \overline{\langle h(y), l_\beta\rangle}.$$

Denote $\Phi_\alpha(x) = \langle h, \exp[i(x_1A_1^* + x_2A_2^*)]l_\alpha\rangle$, $\alpha = 1, 2, \dots, 6$, then

$$W(x,y) = \sum_{\alpha,\beta=1} \Phi_\alpha(x)\cdot d_{\alpha,\beta}\overline{\Phi_\beta(y)}. \tag{16}$$

Let us find the form of functions $\Phi_\alpha(x)$. Note, first of all, that the functions $\Phi_\alpha(x)$ are invariant relatively under unitary equivalence and hence we can use the model presentation which was derived in [**7**]. As is obvious from these models the vector-functions $\exp[-i(x_1A_1^* + x_2A_2^*)]l_\alpha$ generate subspaces L_α which are invariant relative to the operators A_1^* and A_2^* where the contractions of the operators A_1^* and A_2^* on L_α are twice permutable. Let us denote images of vectors $\{l_\alpha\}$ under unitary equivalence (which is realized by the model construction) by $\{h_\alpha\}$, and denote the image of h by $f(x_1, x_2)$ which is a function in the space $L_2(D)$, where domain D has the form



Then l_1 is a function equal to zero outside domain $[0, a_1]\times[b_2, b_3]$ and is a constant in this domain. Similarly, l_2 is a constant in $[0, a_2]\times[b_1, b_2]$, l_3 is that in $[0, a_3]\times[0, b_1]$, l_4 is that in $[0, a_1]\times[0, b_2]$, l_5 is that in $[a_1, a_2]\times[0, b_2]$, and last l_6 is a constant in $[a_2, a_3]\times[0, b_1]$.

Since the spectrum of each operator A_1, A_2 lies in zero and we are in the frames of assumptions of Theorem 1, one obtains, in virtue of the formulas given in Section 2

$$
(17)\quad
\begin{aligned}
\Phi_1(x_1,x_2) &= \int_0^{a_1}\int_{b_2}^{b_3} f(\zeta_1,\zeta_2)J_0(2\sqrt{x_1\zeta_1}\,)J_0(2\sqrt{x_2\zeta_2}\,)\,d\zeta_1\,d\zeta_2\\
\Phi_2(x_1,x_2) &= \int_0^{a_2}\int_{b_1}^{b_2} f(\zeta_1,\zeta_2)J_0(2\sqrt{x_1\zeta_1}\,)J_0(2\sqrt{x_2\zeta_2}\,)\,d\zeta_1\,d\zeta_2\\
\Phi_3(x_1,x_2) &= \int_0^{a_2}\int_0^{b_1} f(\zeta_1,\zeta_2)J_0(2\sqrt{x_1\zeta_1}\,)J_0(2\sqrt{x_2\zeta_2}\,)\,d\zeta_1\,d\zeta_2\\
\Phi_4(x_1,x_2) &= \int_0^{a_1}\int_0^{b_2} f(\zeta_1,\zeta_2)J_0(2\sqrt{x_1\zeta_1}\,)J_0(2\sqrt{x_2\zeta_2}\,)\,d\zeta_1\,d\zeta_2\\
\Phi_5(x_1,x_2) &= \int_{a_1}^{a_2}\int_0^{b_2} f(\zeta_1,\zeta_2)J_0(2\sqrt{x_1\zeta_1}\,)J_0(2\sqrt{x_2\zeta_2}\,)\,d\zeta_1\,d\zeta_2\\
\Phi_6(x_1,x_2) &= \int_{a_2}^{a_3}\int_0^{b_1} f(\zeta_1,\zeta_2)J_0(2\sqrt{x_1\zeta_1}\,)J_0(2\sqrt{x_2\zeta_2}\,)\,d\zeta_1\,d\zeta_2
\end{aligned}
$$

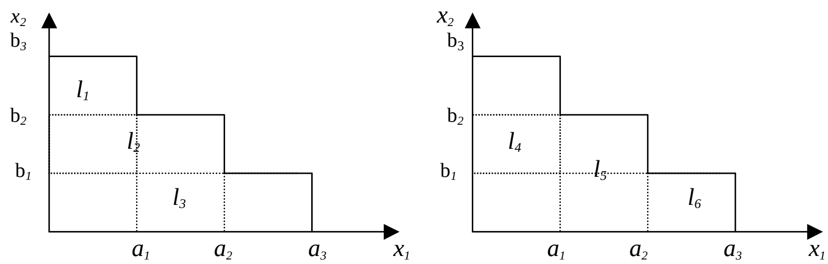
where $J_0(z)$ is the Bessel function

$$J_0(z) = \sum_0^{\infty} \frac{(-1)^n (z/2)^{2k}}{(K!)^2}$$

Thus, one can formulate the following theorem.

Theorem 2. *Assume that a system of linear operators* $\{A_1, A_2\}$ *satisfies the propositions of Theorem* 1 *where the spectrum of each operator* A_k *lies in zero. Then the infinitesimal correlation function* $W(x,y)$ (11) *is represented in the form* (16) *where* $d_{\alpha,\beta} \in \mathbf{R}$, *and the functions* $\Phi_\alpha(x)$ *are defined in* (17).

To evaluate $d_{k,s}$ we represent l_k graphically in the pictures



where l_k are normalized constants in indicated areas ($\|l_k\|_{L^2(D)} = 1$). So that

$$l_k = \frac{S_{D_k}}{\sqrt{G_{D_k}}},$$

where S_{D_k} is the characteristic function of the domain D_k, which is shown in the pictures for l_k and G_{D_k} is the area.

For example,

$$l_1 = \frac{S_{[0,a_1]\times[b_2,b_3]}}{\sqrt{a_1(b_3 - b_2\,)}}, \qquad l_2 = \frac{S_{[0,a_2]\times[b_1,b_2]}}{\sqrt{a_2(b_2 - b_1\,)}},$$

etc.

Let us evaluate Dl_1:

$$Dl_1 = \big(C + 4(A_2)_I(A_1)_I\big)l_1 = 4(A_2)_I(A_1)_I l_1 = 2(A_2)_I a_1 b_1$$

(as $2(A_1)_I$ realizes integration in variable x_1). After the integration x_2 which is carried out by operator $(A_2)_I$ one can get

$$Dl_1 = \frac{a_1(b_3 - b_3)}{\sqrt{a_1(b_3 - b_2)}} \cdot S_{[0,a_1]\times[0,b_3]} = \sqrt{a_1(b_3 - b_2)}\,\sqrt{a_1 b_3}\, l_4$$

to evaluate $d_{1,1}$ it is necessary to find

$$\begin{aligned} d_{1,1} &= \langle Dl_1, l_1\rangle \\ &= \left\langle \sqrt{a_1(b_3 - b_2)}\, S_{[0,a_1]\times[0,b_3]},\ \frac{S_{[0,a_1]\times[b_2,b_3]}}{\sqrt{a_1(b_3 - b_2)}} \right\rangle \\ &= a_1(b_3 - b_2). \end{aligned}$$

Then $d_{1,2} = \langle Dl_1, l_2\rangle$ we can derive that

$$\begin{aligned} d_{1,2} &= \Big\langle \sqrt{a_1(b_3 - b_2\,)} S_{[0,a_1]\times[0,b_3]}, \frac{S_{[0,a_2]\times[b_1,b_2]}}{\sqrt{a_2(b_2 - b_1\,)}} \Big\rangle \\ &= \sqrt{\frac{a_1(b_3 - b_2\,)}{a_2(b_2 - b_1)}} \cdot \sqrt{a_1(b_2 - b_3\,)} \\ &= a_1 \sqrt{\frac{b_3 - b_2}{a_2}}\ . \end{aligned}$$

Let us evaluate

$$\begin{aligned} d_{1,3} &= \langle Dl_1, l_3\rangle \\ &= \Big\langle \sqrt{a_1(b_3 - b_2)}\ S_{[0,a_1]\times[0,b_3]}, \frac{S_{[0,a_3]\times[0,b_1]}}{\sqrt{a_3 b_1}} \Big\rangle \\ &= \frac{\sqrt{a_1(b_3 - b_2)}}{\sqrt{a_3 b_1}} \sqrt{a_1 b_1} \ = \ a_1 \sqrt{\frac{b_3 - b_2}{a_3}}\ . \end{aligned}$$

Also

$$\begin{aligned} d_{1,4} &= \langle Dl_1, l_4\rangle = \big\langle \sqrt{a_1(b_3 - b_2\,)}\ \sqrt{a_1 b_3} l_4,\ l_4 \big\rangle \\ &= a_1 \sqrt{b_3(b_3 - b_2\,)}\ . \end{aligned}$$

Since $Dl_1 = a_1\sqrt{b_3(b_3 - b_2)}\, l_4$ and $l_4 \perp l_5$ and $l_4 \perp l_6$ we derive that $d_{1,5} = d_{1,6} = 0$. Finally

$$\begin{aligned} d_{1,1} &= a_1(b_3 - b_2) \\ d_{1,2} &= a_1 \sqrt{\frac{b_3 - b_2}{a_2}} \\ d_{1,3} &= a_1 \sqrt{\frac{b_3 - b_2}{a_3}} \\ d_{1,4} &= a_1 \sqrt{(b_3 - b_2) b_2} \\ d_{1,5} &= 0 \\ d_{1,6} &= 0 \end{aligned}$$

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