ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 1, Spring 2003

NESTED SEQUENCES OF BALLS, UNIQUENESS OF HAHN-BANACH EXTENSIONS AND THE VLASOV PROPERTY

PRADIPTA BANDYOPADHYAY AND ASHOKE K. ROY

ABSTRACT. In this work we characterize when a single linear functional dominated by a sublinear functional p on a subspace of a real vector space has a unique extension to the whole space dominated by p in terms of nested sequences of "p-balls" in a quotient space. This is then specialized to obtain characterizations of the phenomenon when a single linear functional on a subspace of a Banach space has unique norm-preserving extension to the whole space, thus localizing and generalizing some recent work of Oja and Põldvere. These results are used to characterize w^*-asymptotic norming properties in terms of nested sequences of balls in X extending the notion of Property (V) introduced by Sullivan. A variety of examples and applications of the main results are also presented.

1. Introduction. We work with *real* scalars. For a Banach space X, we denote by $B(X), S(X)$ and $B(x,r)$, or $B[x,r]$, respectively, the closed unit ball, the unit sphere and the open, or closed, ball of radius $r > 0$ around $x \in X$. When X is just a vector space, we will denote linear functionals on X by f, g, etc., while for a Banach space X, elements of the dual X^* will be denoted by x^*, y^*, etc.

Definition 1.1. A closed subspace Y of a Banach space X is said to be a U-subspace of X if for any $y^* \in Y^*$ there exists a unique Hahn-Banach (i.e., norm-preserving) extension of y^* in X^*.

X is said to be Hahn-Banach smooth if X is a U-subspace of X^{**} under the canonical embedding of X in X^{**}.

1991 AMS *Mathematics Subject Classification.* 46A22, 46B04, 46B20.
Key words and phrases. Sublinear functionals, nested sequences of (p-)balls, (p-)U-subspaces, ideals, (w^*-) asymptotic norming properties, Hahn-Banach smoothness, Vlasov Property.
Received by the editors on September 13, 2000, and in revised form on June 18, 2001.

U-subspaces were first systematically studied by Phelps in [**13**], who referred to them as "subspaces with Property U." Our terminology is borrowed from [**2**].

Oja and Põldvere [**11**] have obtained characterizations of U-subspaces, Hahn-Banach smoothness and some other geometric notions in terms of nested sequences of balls.

Definition 1.2. A nested sequence of balls in a Banach space X is a sequence $\{B_n = B(x_n, r_n)\}$ of open balls in X such that for all $n \geq 1$, $B_n \subseteq B_{n+1}$ and $r_n \uparrow \infty$.

In Sections 2 and 3 we localize results of [**11**] to characterize when a single linear functional on a subspace of a Banach space has a unique Hahn-Banach extension to the whole space. Indeed we do this in a much more general set-up and, in Section 2, actually characterize when a single linear functional f dominated by a sublinear functional p on a subspace Y of a real vector space X has a unique extension $\hat{f}$ dominated by p on the whole of X. As in [**11**], our characterization also is primarily in terms of nested sequences of "p-balls," see Definition 2.8. This "purely linear space" result is clearly of interest inasmuch as the general form of the Hahn-Banach extension theorem used in applications is most frequently in this form, see [**3**] or [**5**].

Specializing these results to Banach spaces, in Section 3, we get characterizations of U-subspaces and of ideals that are U-subspaces. In these cases our results and proofs, though inspired by [**11**], are somewhat different. In particular, for localizing results of [**11**], we use as a running thread of our arguments, a rather elementary and well-known criterion for the uniqueness of the dominated extension, Lemma 2.6. And, in addition, we have one quantitative criterion.

Since this paper was written, another paper on a similar theme by Oja and Põldvere [**12**] has appeared. This paper provides some additional characterizations to those of our Theorems 3.1 and 3.6.

In Section 4 we explore some applications of our results in different areas—both in the "purely linear space" context as well as in the context of Banach spaces—and also discuss examples that illustrate and clarify our results. For instance, in this section, we characterize the uniqueness of positive extension of a positive functional, use it and other results to show that a positive linear functional of norm 1 on the subspace c, of all convergent sequences, has a unique Hahn-Banach extension as well as a unique extension as a positive functional to ℓ^∞, the space of all bounded sequences, if and only if its restriction to c_0, the space of all sequences converging to 0, is already of norm 1. We also discuss the uniqueness question in an important integral representation theorem due to Strassen [**17**] with diverse applications in probability theory and related areas, where the Hahn-Banach extension procedure with respect to a sublinear functional is crucially applied.

In Section 5 we explore another aspect of nested sequences of (norm) balls in Banach spaces. Our starting point is the following result of Vlasov [**18**] (see also [**11**, Theorem 2]).

Theorem 1.3. *X^* is strictly convex if and only if the union of any nested sequence of balls in X is either the whole of X or an open half-space.*

Later Sullivan [**16**] introduced a stronger notion which he called the Property (V), see Definition 5.1—we will call it the Vlasov Property to avoid confusion with Pełczynski's Property (V)—and showed that X has the Vlasov Property if and only if X is Hahn-Banach smooth and X^* is strictly convex. In [**1**] this was used to show that the Vlasov Property is equivalent to w^*-ANP-Π'.

For the definitions of asymptotic norming properties (ANP for short), and their w^*-versions, see Section 5. For various geometric notions related to w^*-ANPs, refer to [**1, 8, 9**].

In Section 5, we obtain a simpler reformulation of the Vlasov Property and use it to directly prove that it is equivalent to w^*-ANP-Π'. This approach leads naturally to other "Vlasov-like" properties, see Definition 5.9, and the main object of this section is to establish their equivalence with other w^*-ANPs, Theorem 5.10.

It is known that Hahn-Banach smoothness is equivalent to w^*-ANP-III [**8**]. Apart from the fact that everything we do involves nested sequences of balls, it is this last result that ties the results of this last section with the rest of the paper.

2. Main results. Let X be a real vector space. We denote by $X^{\#}$ the space of all linear functionals on X. Let $p : X \to \mathbf{R}$ be a sublinear functional.

Considering the Banach space case, it is clear that the uniqueness of the extension does not make much sense unless the norm is preserved. In general, too, one needs to impose such restrictions.

Definition 2.1. Let X be a vector space, p a sublinear functional and Y a subspace. Let

$$Y_p^* = \{f \in Y^{\#} : \text{there exists } K > 0 \text{ such that } f \leq Kp \text{ on } Y\}.$$

Notice that Y_p^* is a cone, i.e., closed under addition and multiplication by nonnegative scalars. For $f \in Y_p^*$, let

$$N_Y(f) = \sup\{f(y) : y \in Y, p(y) \leq 1\}.$$

We will simply write $N(f)$ if there is no scope of confusion.

Lemma 2.2. *For $f \in Y_p^*$, $f(Y) \leq N(f)p(y)$ for all $y \in Y$.*

Proof. Let $K > 0$ be such that $f \leq Kp$ on Y. Then, clearly, $0 \leq N(f) \leq K$. If $p(y) \leq 0$, then $f(y) \leq Kp(y) \leq N(f)p(y)$. And if $p(y) > 0$, then $f(y/p(y)) \leq N(f)$, and hence $f(y) \leq N(f)p(y)$. ◻

Remark 2.3. If $N(f) = 0$, then $f \equiv 0$.

Definition 2.4. We say that Y is a p-U-subspace of X, if every $f \in Y_p^*$ has a *unique* extension $\hat{f} \in X^{\#}$ with $\hat{f} \leq N(f)p$ on X.

We may and will assume that $N(f) = 1$ in the sequel.

In the discussion of uniqueness of extension, the quotient space X/Z, where $Z = \ker(f)$ in Y, comes naturally into the picture.

For any subspace Z of X, by analogy with the quotient norm, it is natural to define $\widetilde{p}$ on the quotient space X/Z by

$$\widetilde{p}(x + Z) = \inf\{p(x + z) : z \in Z\}.$$

But this $\widetilde{p}$ may assume $-\infty$ as a value. For example, on $X = C[0,1]$, for $p(f) = \sup\{f(x) : x \in [0,1]\}$, $Z = \{\text{all constant functions}\}$ and $i(x) = x$, the identity function on $[0,1]$, $\widetilde{p}(i + Z) = -\infty$.

However, in our context, i.e., when $f \in Y_p^*$ and $Z = \ker(f)$ in Y, $\widetilde{p}$ is a proper sublinear functional on X/Z, as $\widetilde{p}(x + Z) \geq \hat{f}(x) > -\infty$, where $\hat{f}$ is some extension of f dominated by p.

Observe that, in this case, f is also a well-defined functional on Y/Z.

Lemma 2.5. *Let $f \in Y_p^*$ with $N_Y(f) = 1$, $Z = \ker(f)$ in Y and let $y_0 \in Y$ be such that $f(y_0) = 1$. Then $\widetilde{p}(y_0 + Z) = 1$ and $N_{Y/Z}(f) = 1$.*

Proof. Observe that $f(y) = f(y + z) \leq p(y + z)$ for all $y \in Y$, $z \in Z$, and hence, $f(y) \leq \widetilde{p}(y + z)$ for all $y \in Y$. In particular, $\widetilde{p}(y_0 + Z) = r$, say, and $r \geq 1$.

Claim. *$rf(y) \leq p(y)$ for all $y \in Y$.*

Any $y \in Y$ is of the form $y = \alpha y_0 + z$ for some $\alpha \in \mathbf{R}$ and $z \in Z$. Then $f(y) = \alpha$. If $\alpha = 0$, $y \in Z$ and the claim is clearly true. If $\alpha > 0$, $r \leq p(y_0 + z/\alpha) = p(y/\alpha) = p(y)/\alpha$ and the claim follows. And if $\alpha < 0$, then $rf(y) = r\alpha \leq \alpha = f(y) \leq p(y)$. Hence, the claim.

It now follows that $N(f) = 1 \leq 1/r$. That is, $r \leq 1$ and therefore $r = 1$.

It also follows that $N_{Y/Z}(f) = 1$. □

From the proof of the analytic form of the Hahn-Banach theorem (see, e.g., [**4**, Theorem 21.1]) we get the following elementary and well-known criterion for uniqueness of extensions, which will be used repeatedly in the sequel.

Lemma 2.6. *Let X be a vector space, p a sublinear functional and Y a subspace. Let $f \in Y^\#$ such that $f \leq p$ on Y. Let $x_0 \notin Y$. Then*

$$\sup\{f(y) - p(y - x_0) : y \in Y\} \leq \mathit{inf}\{f(y) + p(x_0 - y) : y \in Y\}$$

and α lies between these two numbers if and only if there exists an *extension* $\hat{f} \in X^{\#}$ *of* f *with* $\hat{f} \leq p$ *on* X *and* $\hat{f}(x_0) = \alpha$.

In particular if $f \in Y_p^*$ *with* $N_Y(f) = 1$, $Z = \ker(f)$ *in* Y *and* $y_0 \in Y$ *such that* $f(y_0) = 1$, *then the following are equivalent*:

(a) f *has a unique extension from* Y *to* X *dominated by* p.

(b) $\sup\{f(y) - p(y - x_0) : y \in Y\} = \inf\{f(y) + p(x_0 - y) : y \in Y\}$ *for all* $x_0 \in X \setminus Y$.

(c) f *has a unique extension from* Y/Z *to* X/Z *dominated by* $\widetilde{p}$.

(d) $\sup\{\alpha - \widetilde{p}(\alpha y_0 - x_0 + Z) : \alpha \in \mathbf{R}\} = \inf\{\alpha + \widetilde{p}(x_0 - \alpha y_0 + Z) : \alpha \in \mathbf{R}\}$ *for all* $x_0 \in X \setminus Y$.

Proof. Equivalence of (a) and (b) as well as of (c) and (d) follows from the first part. And one can easily check that both the right-hand and the left-hand expressions of (b) are equal to the corresponding expressions of (d). $\square$

Lemma 2.7. *Let* $Y \subseteq X$ *be a subspace and let* p *be a sublinear functional on* X *such that* $p \geq 0$ *on* Y. *Let* $x_0 \in X \setminus Y$ *be such that* $\delta = \widetilde{p}(x_0 + Y) > 0$. *Define* f *on* $V = Y \oplus \mathbf{R}x_0$ *by*

$$f(y + \alpha x_0) = \alpha, \quad \alpha \in \mathbf{R}, \quad y \in Y.$$

Then

$$\delta f(v) \leq p(v) \quad \textit{for all } v \in V \quad \textit{and } N_V(f) = \frac{1}{\delta}.$$

Consequently, there exists $\hat{f} \in X^{\#}$ *such that* $\hat{f} \equiv 0$ *on* Y, $\hat{f}(x_0) = 1$, $\hat{f} \leq p/\delta$ *on* X *and* $N_X(\hat{f}) = 1/\delta$.

Proof. To show $\delta f(v) \leq p(v)$ for all $v \in V$, we show that $\delta\alpha \leq p(y + \alpha x_0)$ for all $\alpha \in \mathbf{R}$, $y \in Y$. This is clearly true for $\alpha = 0$, by the assumption on p; and for $\alpha > 0$, by definition of δ. Now if $\alpha < 0$, let $\beta = -\alpha > 0$. We need to check $-\delta\beta \leq p(y - \beta x_0)$ or $\delta \geq -p(y/\beta - x_0)$. Now for any $y_1 \in Y$, $0 \leq p(y/\beta - y_1) \leq p(x_0 - y_1) + p(y/\beta - x_0)$. Thus $-p(y/\beta - x_0) \leq p(x_0 - y_1)$. Taking infimum over $y_1 \in Y$, we get $-p(y/\beta - x_0) \leq \delta$, as was to be shown.

It follows that $N_V(f) \leq 1/\delta$. To prove the equality, fix $0 < \eta < 1/\delta$ and choose $0 < \varepsilon < \eta\delta$. There exists $y_0 \in Y$ such that $p(x_0 - y_0) <$

$\delta(1+\varepsilon)$. Then

$$p\left(\frac{x_0 - y_0}{\delta(1+\varepsilon)}\right) < 1$$

and

$$f\left(\frac{x_0 - y_0}{\delta(1+\varepsilon)}\right) = \frac{1}{\delta(1+\varepsilon)} > \frac{1-\varepsilon}{\delta} > \frac{1}{\delta} - \eta.$$

Since η was arbitrary, this completes the proof. □

Now we come to nested sequences of p-balls.

Definition 2.8. For $x_0 \in X$ and $r > 0$, define the open p-ball of radius r around x_0 by $B_p(x_0, r) = \{x \in X : p(x_0 - x) < r\}$.

A nested sequence of p-balls is a sequence $\{B_n = B_p(x_n, r_n)\}$ of open p-balls in X such that for all $n \geq 1$, $B_n \subseteq B_{n+1}$ and $r_n \uparrow \infty$.

We adapt the proof of [**18**, Proposition 0.2] to obtain a necessary and sufficient condition for p-balls to be nested.

Lemma 2.9. *If $0 < r_1 < r_2$, then $B_p(x_1, r_1) \subseteq B_p(x_2, r_2)$ if and only if $p(x_2 - x_1) \leq r_2 - r_1$.*

Proof. Sufficiency is immediate from the triangle inequality.

Conversely, if $p(x_2 - x_1) \leq 0$, then there is nothing to prove. If $p(x_2 - x_1) > 0$, let $r_1 > \varepsilon > 0$ and put

$$x = x_1 - \frac{(r_1 - \varepsilon)(x_2 - x_1)}{p(x_2 - x_1)}.$$

Then $p(x_1 - x) = r_1 - \varepsilon < r_1$. Therefore, $p(x_2 - x) < r_2$. That is,

$$p(x_2 - x) = p\left(x_2 - x_1 + \frac{(r_1 - \varepsilon)(x_2 - x_1)}{p(x_2 - x_1)}\right) < r_2.$$

It follows that $p(x_2 - x_1) < r_2 - r_1 + \varepsilon$. And hence, $p(x_2 - x_1) \leq r_2 - r_1$. □

Lemma 2.10. *Let X be a vector space and p a sublinear functional on X. Let*

$$p_\infty(x_1, x_2) = \max\{p(x_1), p(x_2)\} \quad \text{and} \quad p_1(x_1, x_2) = p(x_1) + p(x_2).$$

Then both p_∞ and p_1 are sublinear functionals on $X \times X$.

Let $Y \subseteq X$ be a subspace. Let

$$\Delta_1 = \{(y, -y) : y \in Y\} \subseteq X \times X.$$

If $\{B_p(y_n, r_n)\}$ is a nested sequence of p-balls in X with centers in Y, $0 \in B_p(y_1, r_1)$ and $x \in X$ such that $p(x) \leq 1$, then

$$\inf_n \frac{\widetilde{p}_\infty((y_n - x, y_n + x) + \Delta_1)}{r_n} \geq 1 \implies \inf_n \frac{\widetilde{p}_1((y_n - x, y_n + x) + \Delta_1)}{r_n} \geq 2.$$

Recall that by $\widetilde{p}_\infty((y_n - x, y_n + x) + \Delta_1)$, we mean $\inf\{p_\infty(y_n - x + y, y_n + x - y) : y \in Y\}$. $\widetilde{p}_1((y_n - x, y_n + x) + \Delta_1)$ is defined similarly.

Proof. Let $\{B_p(y_n, r_n)\}$ be a nested sequence of p-balls in X with centers in Y such that $0 \in B_p(y_1, r_1)$ and $\inf_n \widetilde{p}_\infty((y_n - x, y_n + x) + \Delta_1)/r_n \geq 1$ for some $x \in X$ with $p(x) \leq 1$. Then, for all $n \geq 1$,

$$d_n = \widetilde{p}_\infty\left(\left(\frac{y_n - x}{r_n}, \frac{y_n + x}{r_n}\right) + \Delta_1\right) \geq 1.$$

Now Δ_1 is a linear subspace of $X \times X$ and $p_\infty \geq 0$ on Δ_1. By Lemma 2.7, therefore, there exists $(f_n, g_n) \in X^\# \times X^\#$ such that

$$\begin{aligned} &f_n\left(\frac{y_n - x}{r_n}\right) + g_n\left(\frac{y_n + x}{r_n}\right) = 1, \\ &f_n(y) - g_n(y) = 0 \quad \text{for all } y \in Y, \\ &f_n(x_1) + g_n(x_2) \leq \frac{1}{d_n} p_\infty(x_1, x_2), \end{aligned}$$

and

$$\begin{aligned} N_X(f_n) + N_X(g_n) &= \sup\{f_n(x_1) + g_n(x_2) : p_\infty(x_1, x_2) \leq 1\} \\ &= \frac{1}{d_n} \leq 1. \end{aligned} \tag{2.1}$$

Notice that, for all $n \geq 1$,

$$\begin{aligned} 1 \leq d_n &\leq \frac{1}{r_n} \max\{p(y_n - x), p(y_n + x)\} \\ &\leq \frac{1}{r_n}[p(y_n) + \max\{p(-x), p(x)\}]. \end{aligned}$$

Therefore,

$$1 \leq \liminf d_n \leq \limsup d_n \leq \limsup p\left(\frac{y_n}{r_n}\right) \leq 1,$$

as $p(y_n) < r_n$. And, hence, $\lim d_n = 1$.

By Lemma 2.9, the p_∞-balls $B_n = B_{p_\infty}((y_n - x, y_n + x), r_n)$ are nested. And

$$\begin{aligned} \inf (f_n, g_n)(B_n) &= f_n(y_n - x) + g_n(y_n + x) \\ &\qquad - r_n \sup\{(f_n, g_n)(u) : p_\infty(u) < 1\} \\ &= r_n - \frac{r_n}{d_n} = r_n\left(1 - \frac{1}{d_n}\right) \geq 0. \end{aligned}$$

It follows that for any $m \leq n$, $\inf (f_n, g_n)(B_m) \geq 0$, i.e.,

$$f_n(y_m - x) + g_n(y_m + x) - \frac{r_m}{d_n} \geq 0. \tag{2.2}$$

Now, following the proof of the locally convex version of the Banach-Alaoglu theorem (see e.g., [**15**, Theorem 3.15]), the set

$$\begin{aligned} V = \{&h \in (X \times X)^\# : \max\{p_\infty(x_1, x_2), p_\infty(-x_1, -x_2)\} \\ &\leq 1 \Longrightarrow h(x_1, x_2) \leq 1\} \end{aligned}$$

is w^*-compact and, by (2.1), $(f_n, g_n) \in V$. Thus, there exists $(f, g) \in X^\# \times X^\#$ which is a w^*-cluster point of $\{(f_n, g_n)\}$. Then

$$f(y) = g(y) \leq \frac{1}{2}p(y) \quad \text{for all } y \in Y \quad \text{and} \quad N_X(f) + N_X(g) \leq 1.$$

Moreover, from (2.2) we have

$$f(y_m - x) + g(y_m + x) \geq r_m \quad \text{for all } m \geq 1.$$

It follows that

$$\liminf_{m\to\infty} f\left(\frac{y_m}{r_m}\right) = \liminf_{m\to\infty} g\left(\frac{y_m}{r_m}\right) \geq \frac{1}{2}.$$

And, since

$$f\left(\frac{y_m}{r_m}\right) + g\left(\frac{y_m}{r_m}\right) \leq N_X(f) + N_X(g) \leq 1,$$

we have

$$\lim_{m\to\infty} f\left(\frac{y_m}{r_m}\right) = \lim_{m\to\infty} g\left(\frac{y_m}{r_m}\right) = \frac{1}{2}.$$

Consequently,

$$N_X(f) = N_X(g) = \frac{1}{2}.$$

Now, for any $y \in Y$,

$$\begin{aligned} 1 &\leq f\left(\frac{y_n - x}{r_n}\right) + g\left(\frac{y_n + x}{r_n}\right) \\ &= f\left(\frac{y_n - x - y}{r_n}\right) + g\left(\frac{y_n + x + y}{r_n}\right) \\ &\leq \frac{1}{2}\left[p\left(\frac{y_n - x - y}{r_n}\right) + p\left(\frac{y_n + x + y}{r_n}\right)\right]. \end{aligned}$$

It follows that

$$\frac{1}{2}\widetilde{p}_1\left(\left(\frac{y_n - x}{r_n}, \frac{y_n + x}{r_n}\right) + \Delta_1\right) \geq 1. \qquad \square$$

Remark 2.11. Observe that $f = g \leq p/2$ on Y and $N_X(f) = N_X(g) = 1/2$. But $r_m \leq f(y_m - x) + g(y_m + x) = 2f(y_m) + g(x) - f(x) \leq p(y_m) + g(x) - f(x)$. Therefore, $g(x) - f(x) \geq r_m - p(y_m) > 0$ since $0 \in B_p(y_m, r_m)$.

We are now ready for our main theorem.

Theorem 2.12. *Let X be a vector space, p a sublinear functional and Y a subspace. Let $f \in Y_p^*$ with $N(f) = 1$, $Z = \ker(f)$ in Y and $y_0 \in Y$ be such that $f(y_0) = 1$. Let*

$$\Delta_1(Y/Z) = \{(y+Z, -y+Z) : y \in Y\} \subseteq X/Z \times X/Z.$$

Then the following are equivalent:

(a) *f has a unique extension $\hat{f}$ to $X^{\#}$ with $\hat{f} \leq p$.*

(b) *If $\{B_{\widetilde{p}}(y_n + Z, r_n)\}$ is a nested sequence of $\widetilde{p}$-balls in X/Z such that the centers $\{y_n\} \subseteq Y$, $0 \in B_{\widetilde{p}}(y_1 + Z, r_1)$ and $\widetilde{p}(x+Z) \leq 1$, then*

$$\inf_n \frac{d_1((y_n - x + Z, y_n + x + Z), \Delta_1(Y/Z))}{r_n} < 2,$$

where $d_1((y_n - x + Z, y_n + x + Z), \Delta_1(Y/Z))$
$= \inf\{\widetilde{p}(y_n - x - y + Z) + \widetilde{p}(y_n + x + y + Z) : y + Z \in Y/Z\}$.

(c) *If $\{B_{\widetilde{p}}(y_n + Z, r_n)\}$ is a nested sequence of $\widetilde{p}$-balls in X/Z such that the centers $\{y_n\} \subseteq Y$, $0 \in B_{\widetilde{p}}(y_1 + Z, r_1)$ and $\widetilde{p}(x+Z) \leq 1$, then there exist $y \in Y$ and $n_0 \geq 1$ such that*

$$\widetilde{p}(y_{n_0} \pm (x - y) + Z) < r_{n_0}.$$

(d) *If $\{B_p(y_n, r_n)\}$ is a nested sequence of p-balls in X such that the centers $\{y_n\} \subseteq Y$, $0 \in B_p(y_1, r_1)$ and $p(x) \leq 1$, then there exist $y \in Y$ and $n_0 \geq 1$ such that*

$$\widetilde{p}(y_{n_0} \pm (x - y) + Z) < r_{n_0}.$$

Proof. (a) $\Rightarrow$ (b). Suppose (b) doesn't hold. Then there is a nested sequence $\{B_{\widetilde{p}}(y_n + Z, r_n)\}$ of $\widetilde{p}$-balls in X/Z with $\{y_n\} \subseteq Y$, $0 \in B_{\widetilde{p}}(y_1 + Z, r_1)$ (consequently, $\widetilde{p}(y_n + Z) < r_n$ for all $n \geq 1$) and $\widetilde{p}(x + Z) \leq 1$, such that

$$\inf_n \frac{d_1((y_n - x + Z, y_n + x + Z), \Delta_1(Y/Z))}{r_n} \geq 2.$$

Then for all $n \geq 1$,

$$\begin{aligned} r_n &\leq \frac{1}{2} d_1((y_n - x + Z, y_n + x + Z), \Delta_1(Y/Z)) \\ &\leq d_\infty((y_n - x + Z, y_n + x + Z), \Delta_1(Y/Z)), \end{aligned}$$

where

$$\begin{aligned} &d_\infty((y_n - x + Z, y_n + x + Z), \Delta_1(Y/Z)) \\ &= \inf\{\max\{\widetilde{p}(y_n - x - y + Z), \widetilde{p}(y_n + x + y + Z)\} : y + Z \in Y/Z\}. \end{aligned}$$

Observe that Lemma 2.10, applied to Y/Z as a subspace of X/Z, produces g, $h \in (X/Z)^\#$ such that $g = h \leq \widetilde{p}/2$ on Y/Z and

$$N_{X/Z}(g) = N_{X/Z}(h) = \frac{1}{2}.$$

But $g(x) \neq h(x)$, see Remark 2.11. Since $\dim(Y/Z) = 1$, we have that $f = 2g = 2h$ on Y/Z. Thus, uniqueness fails.

(b) $\Rightarrow$ (c). Suppose (c) doesn't hold. Then there is a nested sequence $\{B_{\bar{p}}(y_n + Z, r_n)\}$ of $\widetilde{p}$-balls in X/Z such that the centers $\{y_n\} \subseteq Y$, $0 \in B_{\widetilde{p}}(y_1 + Z, r_1)$ and $\widetilde{p}(x + Z) \leq 1$, such that for all $n \geq 1$,

$$d_\infty((y_n - x + Z, y_n + x + Z), \Delta_1(Y/Z)) \geq r_n.$$

By Lemma 2.10 applied to X/Z,

$$\inf_n \frac{d_1((y_n - x + Z, y_n + x + Z), \Delta_1(Y/Z))}{r_n} \geq 2,$$

contradicting (b).

(c) $\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (a). Let $p(x) \leq 1$. For $\alpha \in \mathbf{R}$, let $u(\alpha) = \alpha - \widetilde{p}(\alpha y_0 - x + Z)$ and $v(\alpha) = \alpha + \widetilde{p}(x - \alpha y_0 + Z)$. By Lemma 2.6, given $0 < \varepsilon < 1$, it suffices to find α and $\alpha' \in \mathbf{R}$ such that $v(\alpha) - u(\alpha') < \varepsilon$.

Following [**11**, Theorem 1 (e) $\Rightarrow$ (a)], let $\alpha_n = n + \varepsilon/(n+2) - \varepsilon/2$ for all $n \geq 1$. Then $0 < \widetilde{p}(\alpha_1 y_0 + Z) = \alpha_1 < 1$ and $0 < \widetilde{p}(\alpha_{n+1} y_0 - \alpha_n y_0 + Z) = \alpha_{n+1} - \alpha_n < 1$ for all $n \geq 1$. Inductively construct a sequence $\{y_n\}$

such that $y_n \in \alpha_n y_0 + Z$ and $0 < p(y_1) < 1$ and $0 < p(y_{n+1} - y_n) < 1$ for all $n \geq 1$. Then $\{B_p(y_n, n)\}$ is a nested sequence of p-balls such that the centers $\{y_n\} \subseteq Y$. Hence, by (d), there exist $y \in Y$ and $n_0 \geq 1$ such that

$$\widetilde{p}(y_{n_0} \pm (x - y) + Z) < n_0.$$

Let α_0 be such that $y \in \alpha_0 y_0 + Z$. It follows that

$$\widetilde{p}(\alpha_{n_0} y_0 \pm (x - \alpha_0 y_0) + Z) < n_0.$$

Therefore,

$$\begin{aligned} &v(\alpha_0 - \alpha_{n_0}) - u(\alpha_0 + \alpha_{n_0}) \\ &\quad = \widetilde{p}(x - (\alpha_0 - \alpha_{n_0}) y_0 + Z) + \widetilde{p}((\alpha_0 + \alpha_{n_0}) y_0 - x + Z) - 2\alpha_{n_0} \\ &\quad = \widetilde{p}(\alpha_{n_0} y_0 + (x - \alpha_0 y_0) + Z) + \widetilde{p}(\alpha_{n_0} y_0 - (x - \alpha_0 y_0) + Z) - 2\alpha_{n_0} \\ &\quad < 2n_0 - 2\alpha_{n_0} < \varepsilon. \end{aligned}$$

This completes the proof. $\square$

Remark 2.13. From the proof of (d) $\Rightarrow$ (a) above, it follows that it suffices to consider nested sequences of p-balls of the type $\{B_p(y_n, n)\}$ in all the statements of Theorem 2.12.

Theorem 2.14. *Let X be a vector space, p a sublinear functional and Y a subspace. Let*

$$\Delta_1 = \{(y, -y) : y \in Y\} \subseteq X \times X.$$

Then the following are equivalent:

(a) *Y is a p-U-subspace of X.*

(b) *If $\{B_p(y_n, r_n)\}$ is a nested sequence of p-balls in X with centers in Y, $0 \in B_p(y_1, r_1)$ and $p(x) \leq 1$, then*

$$\inf_n \frac{d_1((y_n - x, y_n + x), \Delta_1)}{r_n} < 2,$$

where $d_1((y_n - x, y_n + x), \Delta_1) = \inf\{p_1((y_n - x - y, y_n + x + y) : y \in Y\}$.

(c) *If* $\{B_p(y_n, r_n)\}$ *is a nested sequence of* p*-balls in* X *with centers in* Y, $0 \in B_p(y_1, r_1)$ *and* $p(x) \leq 1$, *then there exist* $y \in Y$ *and* $n_0 \geq 1$ *such that*

$$p(y_{n_0} \pm (x - y)) < r_{n_0}.$$

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) follows similarly as in Theorem 2.12.

(c) $\Rightarrow$ (a). Let $f \in Y_p^*$ with $N(f) = 1$ and $Z = \ker(f)$ in Y. We will show that f has a unique extension to X dominated by p. Let $\{B_p(y_n, r_n)\}$ be a nested sequence of p-balls in X such that the centers $\{y_n\} \subseteq Y$, $0 \in B_p(y_1, r_1)$ and $p(x) \leq 1$. By (c), there exist $y \in Y$ and $n_0 \geq 1$ such that

$$p(y_{n_0} \pm (x - y)) < r_{n_0}.$$

It follows that

$$\widetilde{p}(y_{n_0} \pm (x - y) + Z) < r_{n_0}.$$

By Theorem 2.12 (d) $\Rightarrow$ (a), f has a unique extension to X dominated by p. $\square$

Remark 2.15. Because of the nature of sublinear functionals, where $p(x) > 0$ doesn't necessarily imply $p(-x) > 0$, as we see by taking $p((x_n)) = \limsup x_n$ on l_∞, all the above results are one-sided in nature—e.g., we had to consider one-sided p-balls, etc. When p is a norm or a semi-norm, these difficulties do not arise and we can dispense with such restrictions.

3. The Banach space case. Specializing to Banach spaces, we get characterizations of U-subspaces, of which condition (c) below was first established in [**11**, Theorem 1].

Theorem 3.1. *Let* Y *be a subspace of a Banach space* X. *Then the following are equivalent*:

(a) Y *is a* U*-subspace of* X.

(b) *If* $\{B(y_n, r_n)\}$ *is a nested sequence of balls in* X *with centers in* Y, $0 \in B(y_1, r_1)$ *and* $\|x\| \leq 1$, *then*

$$\inf_n \frac{d_1((x - y_n, x + y_n), \Delta)}{r_n} < 2,$$

where d_1 refers to the distance in $X \oplus_{l_1} X$, and

$$\Delta = \{(y, y) : y \in Y\} \subseteq X \times X.$$

(c) *If $\{B(y_n, r_n)\}$ is a nested sequence of balls in X with centers in Y, $0 \in B(y_1, r_1)$ and $\|x\| \leq 1$, then there exist $y \in Y$ and $n_0 \geq 1$ such that*

$$\|x - y \pm y_{n_0}\| < r_{n_0}.$$

Reversing our earlier approach, we now deduce the local result as a consequence of the global one, the connection being given by (b) below.

Theorem 3.2. *Let Y be a subspace of a Banach space X, $y_0^* \in S(Y^*)$ and $Z = \ker(y_0^*)$ in Y. Then the following are equivalent*:

(a) *y_0^* has a unique Hahn-Banach extension to X^*.*

(b) *Y/Z is a U-subspace of X/Z.*

(c) *If $\{B(y_n + Z, r_n)\}$ is a nested sequence of balls in X/Z with centers in Y/Z, $0 \in B(y_1 + Z, r_1)$ and $\|x + Z\| \leq 1$, then*

$$\inf_n \frac{d_1((x - y_n + Z, x + y_n + Z), \Delta_{Y/Z})}{r_n} < 2,$$

where d_1 refers to the distance in $X/Z \oplus_{l_1} X/Z$, and

$$\Delta_{Y/Z} = \{(y + Z, y + Z) : y \in Y\} \subseteq X/Z \times X/Z.$$

(d) *If $\{B(y_n + Z, r_n)\}$ is a nested sequence of balls in X/Z with centers in Y/Z, $0 \in B(y_1 + Z, r_1)$ and $\|x + Z\| \leq 1$, then there exist $y \in Y$ and $n_0 \geq 1$ such that*

$$\|x - y \pm (y_{n_0} + Z)\| < r_{n_0}.$$

(e) *If $\{B(y_n, r_n)\}$ is a nested sequence of balls in X with centers in Y, $0 \in B(y_1, r_1)$ and $\|x\| \leq 1$, then there exist $y \in Y$ and $n_0 \geq 1$ such that*

$$\|x - y \pm (y_{n_0} + Z)\| < r_{n_0}.$$

Remark 3.3. Regarding the condition (c) above, if $y_0 \in Y$ is such that $y_0^*(y_0) = 1$ and we write $y_n + Z = \alpha_n y_0 + Z$, then it is easy to see that

$$\frac{1}{2}\, d_1((x - \alpha_n y_0 + Z, x + \alpha_n y_0 + Z), \Delta_{Y/Z}) \\ \leq d_\infty((x - y_n + Z, x + y_n + Z), \Delta_{Y/Z}) \leq |\alpha_n| + d(x, Y/Z),$$

where d_∞ refers to the distance in $X/Z \oplus_{l_\infty} X/Z$.

It is therefore tempting to conjecture that the condition in (c) could be replaced by the simpler condition

$$\sup_n [r_n - |\alpha_n|] > d(x, Y/Z).$$

But this is not true, as the following example shows:

Example 3.4. Let $X = l_\infty^2$ and $Y = \{(x, 0) : x \in \mathbf{R}\}$. Then Y obviously is a U-subspace. For $n \geq 1$, let $B_n = B((n, 0), n + 1/2)$. Then $\{B_n\}$ is a nested sequence of balls in X with centers in Y. Let $x = (1, -1)$. Then $r_n - \|y_n\| = 1/2 < d(x, Y) = 1$. Nevertheless, (e) holds. Indeed, for $n = 1$ and $y = (1, 0)$,

$$\|x - y \pm y_1\| = \|(\pm 1, -1)\| = 1 < \frac{3}{2} = r_1.$$

As observed in [**11**], the conditions can be strengthened if Y is an ideal in X.

Definition 3.5 [**7**]. A subspace $Y \neq \{0\}$ of a Banach space X is said to be an ideal in X if there exists a norm one projection P on X^* with $\ker(P) = Y^\perp$.

We will recall the following facts from [**11**]. Firstly, every Banach space X is an ideal in X^{**} via the canonical projection on X^{***}. Secondly, if Y Is an ideal in X, then for every $x^* \in X^*$, $Px^* \in X^*$ is a Hahn-Banach extension of the restriction $x^*|_Y \in Y^*$. Therefore, we can and will identify Px^* and $x^*|_Y$ for all $x^* \in X^*$. This makes it possible to identify Y^* with the range of P and to consider the, generally non-Hausdorff, topology $\sigma(X, Y^*)$, which we will denote simply by σ. Then $B(Y)$ is σ-dense in $B(X)$. Thirdly, if Y is an ideal as well as a U-subspace of X, then the projection P is unique.

Some of the statements in the following theorem were first proved in [**11**, Theorem 3]. We have, however, included the proofs as ours are somewhat different and give some additional criteria, especially statement (c). Note that a special case of this theorem will be needed in Section 5.

Theorem 3.6. *Let Y be an ideal in a Banach space X. Then the following are equivalent*:

(a) *Y is a U-subspace of X.*

(b) *If $\{B(y_n, n)\}$ is a nested sequence of balls in X with centers in Y, $\|y_1\| < 1$ and $\|x\| \le 1$, and U is a convex σ-neighborhood of x, then*

$$\Delta_K \cap \left[\bigcup_n B((x + y_n, x - y_n), n)\right] \neq \varnothing,$$

where $K = U \cap B(Y)$ and $\Delta_K = \{(y, y) : y \in K\}$.

(c) *If $\{B(y_n, n)\}$ is a nested sequence of balls in X with centers in Y, $\|y_1\| < 1$ and $\|x\| \le 1$, then for all $\alpha \in [0, 1]$,*

$$\inf_n \frac{d_\alpha(x - y_n, x + y_n)}{n} < 1,$$

where $d_\alpha(x_1, x_2) = \inf_n \{\alpha\|x_1 - y\| + (1 - \alpha)\|x_2 - y\| : y \in B(Y)\}$.

(d) *If $\{B(y_n, n)\}$ is a nested sequence of balls in X with centers in Y, $\|y_1\| < 1$ and $\|x\| \le 1$ and U is a convex σ-neighborhood of x, then for $K = U \cap B(Y)$,*

$$K \cap \left[\bigcup_n B(x + y_n, n)\right] \neq \varnothing.$$

Proof. (a) $\Rightarrow$ (b). We follow the reasoning of [**11**, Theorem 1 (a) $\Rightarrow$ (b)]. Suppose (b) doesn't hold. Then there is a nested sequence

$\{B(y_n, n)\}$ of balls in X with centers in Y, $\|y_1\| < 1$ and $\|x\| \leq 1$, such that

$$\Delta_K \cap \bigcup_{n \geq 1} B((x - y_n, x + y_n), n) = \varnothing.$$

Now consider the l_∞ norm on the product space and separate the convex set Δ_K from the open convex set $\cup_{n \geq 1} B((x - y_n, x + y_n), n)$. That is, there exist $x_1^*, x_2^* \in X^*$ and $\gamma \in \mathbf{R}$ such that

$$\|x_1^*\| + \|x_2^*\| = 1$$

and

$$x_1^*(k) + x_2^*(k) \leq \gamma \leq x_1^*(x + y_n) + x_2^*(x - y_n) - n \tag{3.1}$$

for all $k \in K$ and $n \geq 1$. From (3.1) it follows that

$$1 + \frac{\gamma}{n} \leq \frac{x_1^*(x) + x_2^*(x)}{n} + (x_1^* - x_2^*)\left(\frac{y_n}{n}\right)$$

whence,

$$\begin{aligned} 1 &\leq \limsup (x_1^* - x_2^*)\left(\frac{y_n}{n}\right) \\ &\leq \limsup x_1^*\left(\frac{y_n}{n}\right) + \limsup (-x_2^*)\left(\frac{y_n}{n}\right) \\ &\leq \|x_1^*|_Y\| + \|x_2^*|_Y\| \leq \|x_1^*\| + \|x_2^*\| \leq 1. \end{aligned} \tag{3.2}$$

Therefore $\|x_i^*|_Y\| = \|x_i^*\|$ and hence, by (a), $x_i^* = Px_i^*$, $i = 1, 2$. Further, $P(x_1^* - x_2^*) = (x_1^* - x_2^*)$, so that $\|x_1^* - x_2^*\| = \|x_1^*|_Y - x_2^*|_Y\| = 1$, by (3.2). Since $x_i^* = Px_i^*$, $i = 1, 2$ and x is in the σ-closure of K, $x_1^*(x) + x_2^*(x) \leq \gamma$, by the first inequality in (3.1). And, from the second inequality in (3.1), it follows that

$$\gamma \leq x_1^*(x) + x_2^*(x) + (x_1^* - x_2^*)(y_n) - n < x_1^*(x) + x_2^*(x),$$

as $\|x_1^* - x_2^*\| = 1$ and $\|y_n\| < n$. Thus we have a contradiction.

(b) $\Rightarrow$ (c). Let $\{B(y_n, n)\}$ be a nested sequence of balls in X with centers in Y, $\|y_1\| < 1$ and $\|x\| \leq 1$. By the special case of (b) with $U = X$, there exist $y \in B(Y)$ and $m \geq 1$ such that

$$\|x - y \pm y_m\| < m.$$

Thus for any $\alpha \in [0,1]$, we get

$$\alpha\|x-y+y_m\| + (1-\alpha)\|x-y-y_m\| < m.$$

Hence, (c) follows.

(c) $\Rightarrow$ (a). By Remark 2.13, we can consider nested sequences of balls of the type $\{B(y_n, n)\}$ in all the statements of Theorem 3.1. And, clearly, Theorem 3.1(b) with $r_n = n$ follows from (c) for $\alpha = 1/2$.

(b) $\Rightarrow$ (d). Obvious.

(d) $\Rightarrow$ (a). We adapt the proof of [**11**, Theorem 3, (c) $\Rightarrow$ (a)]. If (a) doesn't hold, there exists $x^* \in X^*$ such that $x^* = Px^* + z^*$, $\|x^*\| = \|Px^*\|$ and $z^* \neq 0$. Choose $x \in S(X)$ such that $z^*(x) > 0$. Choose $0 < \varepsilon < z^*(x)$. Let $y^* = Px^*$ and $Z = \ker(y^*)$ in Y. Find $\{y_n\} \subseteq Z$ such that $\|y_1\| < 1$¡ $\|y_{n+1} - y_n\| < 1$ and $y^*(y_n) = n + \varepsilon/(n+2) - \varepsilon/2$. Then $\{B(y_n, n)\}$ is a nested sequence of balls and we claim that if $U = \{z \in X : y^*(x-z) > -\varepsilon/2\}$ and $K = U \cap B(Y)$, then

$$K \cap \left[\bigcup_n B(x+y_n, n)\right] = \varnothing.$$

Suppose there exist $y \in K$ and $n_0 \geq 1$ such that $\|x + y_{n_0} - y\| < n_0$. Then

$$\begin{aligned} x^*(x-y) + n_0 + \frac{\varepsilon}{n_0+2} - \frac{\varepsilon}{2} &\leq x^*(x) + x^*(y_{n_0}) - x^*(y) \\ &\leq \|x + y_{n_0} - y\| < n_0. \end{aligned}$$

It follows that

$$\begin{aligned} &x^*(x-y) < \frac{\varepsilon}{2} \quad \text{or} \quad y^*(x-y) + z^*(x) < \frac{\varepsilon}{2} \\ &\Longrightarrow z^*(x) < \frac{\varepsilon}{2} - y^*(x-y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Contradiction. $\square$

4. Some applications and examples.

Example 4.1. Let K be a compact Hausdorff space. For $f \in C(K)$, let $p(f) = \sup\{f(t) : t \in K\}$. p is a sublinear functional on $C(K)$. Fix $t_0 \in K$ and let

$$Y = \{f \in C(K) : f(t_0) = 0\} = \ker(\delta_{t_0}).$$

Let $\Lambda \in Y_p^*$ and $C = N_Y(\Lambda)$. Then $C\inf\{f(t) : t \in K\} \leq \Lambda(f) \leq C\sup\{f(t) : t \in K\}$ for all $f \in Y$. It follows that Λ is a nonnegative functional. Let $\hat{\Lambda}$ be an extension of Λ to $C(K)$ dominated by Cp. Then $\hat{\Lambda}$ is a nonnegative linear functional on $C(K)$ and $\hat{\Lambda}(1) = C$. It follows that $\hat{\Lambda}$ is actually norm continuous on $C(K)$ and hence is represented by a nonnegative measure μ with $\mu(1) = C$.

Note that $Y^\perp = \mathbf{R}\delta_{t_0}$, in $C(K)^*$. And hence, if μ_1 and μ_2 are extensions of some $\Lambda \in Y_p^*$, then $\mu_1 - \mu_2 = \alpha\delta_{t_0}$ for some $\alpha \in \mathbf{R}$. It follows that $\alpha = \mu_1(1) - \mu_2(1) = C - C = 0$. That is, $\mu_1 = \mu_2$. Therefore, Y is a p-U-subspace of $C(K)$.

So by Theorem 2.14, if $\{B_p(f_n, r_n)\}$ is a nested sequence of p-balls in $C(K)$ with $\{f_n\} \subseteq Y$, $0 \in B_p(f_1, r_1)$ and $p(f) \leq 1$, then there exists $g \in Y$ such that

$$p(f_n \pm (f - g)) < r_n \quad \text{for some} \quad n \geq 1.$$

Let us see this directly. Observe that there is nothing to prove if $f \in Y$. So $|f(t_0)| > 0$. Choose r_n so large that $|f(t_0)| < r_n$. Let r'_n be such that $|f(t_0)| < r'_n < r_n$. Now

$$\lim_{t \to t_0} [f_n(t) \pm f(t)] = f_n(t_0) \pm f(t_0) = \pm f(t_0) < r'_n.$$

So there is a neighborhood U of t_0 such that for all $t \in U$, $f_n(t) \pm f(t) < r'_n$ and $f(t)$ has the same sign as $f(t_0)$.

Use Urysohn's lemma to get $h \in C(K)$ such that $0 \leq h \leq 1$, $h(t_0) = 0$ and $h|_{U^c} \equiv 1$ and let $g = fh$. Note that $g \in Y$ and $p(g) \leq p(f) \leq 1$. Now if $t \in U^c$, then

$$f_n(t) \pm (f - g)(t) = f_n(t) \leq p(f_n).$$

And if $t \in U$, suppose $f(t_0) > 0$. Then $f(t) > 0$ and

$$f_n(t) + (f - g)(t) = f_n(t) + (1 - h(t))f(t) \leq f_n(t) + f(t) < r'_n,$$

and

$$f_n(t) - (f - g)(t) = f_n(t) - f(t) + h(t)f(t) \leq f_n(t) + f(t) < r'_n.$$

And if $f(t_0) < 0$, then the arguments are interchanged. It follows that

$$p(f_n \pm (f - g)) \leq \max\{r'_n, p(f_n)\} < r_n.$$

Example 4.2. On ℓ^∞, $p((x_n)) = \limsup x_n$ is a sublinear functional. On the subspace c, the functional $\Lambda((x_n)) = \lim x_n$ is dominated by p. It does not have unique extension to ℓ^∞ dominated by p.

To see this, let $x_0 = (1, 0, 1, 0, \dots , 1, 0, \dots) \in \ell^\infty$. Let $y = (y_n) \in c$. Let $\alpha = \Lambda(y)$. Then, since $\alpha > \alpha - 1$,

$$p(y - x_0) = p(y_1 - 1, y_2, y_3 - 1, y_4, \dots) = \alpha$$

and

$$p(x_0 - y) = p(1 - y_1, -y_2, 1 - y_3, -y_4, \dots) = 1 - \alpha.$$

Therefore,

$$0 = \sup\{\Lambda(y) - p(y - x_0) : y \in c\} < \inf\{\Lambda(y) + p(x_0 - y) : y \in c\} = 1.$$

To interpret this in terms of nested sequences of p-balls (cf. Theorem 2.12 (c)), note that $Z = \ker(\Lambda) = c_0$. Let $x = (1, -1, 1, -1, \dots , 1, -1, \dots) \in \ell^\infty$, let $z_n \in Z = c_0$ and define $y_n = \alpha_n(1, 1, \dots , 1, \dots) + z_n \in c$, $y = \alpha(1, 1, \dots , 1, \dots) + z \in c$, where $|\alpha| \leq 1$, $\alpha_n \geq 0$, $\alpha_1 < 1$ and $\alpha_n \uparrow \infty$. Let $r_n = 1 + \alpha_n$ for $n \geq 1$. Then $\{B_{\widetilde{p}}(y_n + c_0, r_n)\}$ is a nested sequence of $\widetilde{p}$-balls in ℓ^∞/c_0 and

$$\widetilde{p}(y_n + y - x + c_0) = \max\{(\alpha_n + \alpha - 1), (\alpha_n + \alpha + 1)\} \geq r_n, \quad \text{if } \alpha \geq 0;$$
$$\widetilde{p}(y_n - y + x + c_0) = \max\{(\alpha_n - \alpha - 1), (\alpha_n - \alpha + 1)\} \geq r_n, \quad \text{if } \alpha < 0.$$

It may also be noted that, in terms of Theorem 2.12 (b),

$$\inf_n \frac{d_1((y_n - x + c_0, y_n + x + c_0), \Delta_1(c/c_0))}{r_n} = 2.$$

Therefore, c is not a p-U-subspace of ℓ^∞. Moreover, as $p(x) \leq \|x\|$, even Hahn-Banach extensions are not unique, that is, there is no unique way of extending the notion of limits to ℓ^∞. Observe that to verify this directly is computationally more complicated than the above argument. □

Example 4.3. We have just observed that c is not a U-space of ℓ^∞. But c_0 is known to be a U-subspace of ℓ^∞ (c_0 is Hahn-Banach smooth). It follows that a linear functional of norm 1 on c has unique Hahn-Banach extension to ℓ^∞ if its restriction to c_0 is already of norm 1. Here we would like to observe that the converse is also true for positive functionals. Recall that $c^* = l_1$ and that $\mathbf{s} = (s_0, s_1, \dots, s_n, \dots) \in l_1$ acts on $\mathbf{x} = (x_1, \dots, x_n, \dots) \in c$ as

$$\langle \mathbf{s}, \mathbf{x} \rangle = s_0 \lim x_n + \sum_{n=1}^{\infty} s_n x_n.$$

We want to show that the functional $\mathbf{s}$ with $s_n \geq 0$ and $\|\mathbf{s}\|_1 = 1$ has unique Hahn-Banach extension to ℓ^∞ if and only if $s_0 = 0$. It clearly suffices to prove that for any $\mathbf{u} \in \ell^\infty$,

$$\sup\{\langle \mathbf{s}, \mathbf{x} \rangle - \|\mathbf{x} - \mathbf{u}\| : \mathbf{x} \in c\} = s_0(\liminf_n u_n) + \sum_{n=1}^{\infty} s_n u_n,$$

and

$$\inf\{\langle \mathbf{s}, \mathbf{x} \rangle + \|\mathbf{x} - \mathbf{u}\| : \mathbf{x} \in c\} = s_0(\limsup_n u_n) + \sum_{n=1}^{\infty} s_n u_n.$$

Now for any $\mathbf{x} \in c$ and $\mathbf{u} \in \ell^\infty$, and for any $n \geq 1$, $x_n - \|\mathbf{x} - \mathbf{u}\| \leq u_n$ and, hence, $\lim_n x_n - \|\mathbf{x} - \mathbf{u}\| \leq \liminf_n u_n$. Since $\|\mathbf{s}\|_1 = 1$, we have

$$\begin{aligned} \langle \mathbf{s}, \mathbf{x} \rangle - \|\mathbf{x} - \mathbf{u}\| &= s_0 \lim x_n + \sum_{n=1}^{\infty} s_n x_n - \|\mathbf{x} - \mathbf{u}\| \\ &= s_0(\lim x_n - \|\mathbf{x} - \mathbf{u}\|) + \sum_{n=1}^{\infty} s_n (x_n - \|\mathbf{x} - \mathbf{u}\|) \\ &\leq s_0(\liminf u_n) + \sum_{n=1}^{\infty} s_n u_n. \end{aligned}$$

And, hence,

$$\sup\{\langle \mathbf{s}, \mathbf{x} \rangle - \|\mathbf{x} - \mathbf{u}\| : \mathbf{x} \in c\} \leq s_0(\liminf_n u_n) + \sum_{n=1}^{\infty} s_n u_n.$$

To show equality, let $\varepsilon > 0$. For notational simplicity, let $\lambda_1 = \liminf u_n$ and $\lambda_2 = \limsup u_n$ and put $\lambda = (\lambda_1 + \lambda_2)/2$. Choose

$N \geq 1$ such that, for all $n \geq N+1$, $\lambda_1 - \varepsilon < u_n < \lambda_2 + \varepsilon$ and $\sum_{n=N+1}^{\infty} s_n < \min\{\varepsilon, \varepsilon/\|\mathbf{u}\|\}$. Let $d = \sup_{n \geq N+1} |\lambda - u_n|$. Observe that, by the last condition, $d \leq (\lambda_2 - \lambda_1)/2 + \varepsilon$ and, therefore, $\lambda - d \geq \lambda_1 - \varepsilon$. Also $|\lambda - d| \leq \|\mathbf{u}\|$. Define $\mathbf{x} \in c$ by

$$x_n = \begin{cases} u_n + d & \text{if } 1 \leq n \leq N, \\ \lambda & \text{if } n \geq N+1. \end{cases}$$

Then $\|\mathbf{x} - \mathbf{u}\| = d$ and

$$\begin{aligned} \langle \mathbf{s}, \mathbf{x} \rangle - \|\mathbf{x} - \mathbf{u}\| &= s_0 \lambda + \sum_{n=1}^{\infty} s_n x_n - d \\ &= s_0(\lambda - d) + \sum_{n=1}^{N} s_n(x_n - d) + \sum_{n=N+1}^{\infty} s_n(x_n - d) \\ &= s_0(\lambda - d) + \sum_{n=1}^{N} s_n u_n + \sum_{n=N+1}^{\infty} s_n(\lambda - d) \\ &\geq s_0(\lambda_1 - \varepsilon) + \sum_{n=1}^{\infty} s_n u_n - \varepsilon - \varepsilon \\ &\geq s_0 \lambda_1 + \sum_{n=1}^{\infty} s_n u_n - \varepsilon(s_0 + 2), \end{aligned}$$

and we are done.

The other identity can also be proved similarly. □

We now note that what we observed in Example 4.2 is, in fact, a special case of a more general phenomenon. The following simple yet useful result should be known, but we are unable to cite a reference. The closest one we could find was [**4**, Corollary 21.3], where it is noted that if $Y = \{0\}$, then f has a unique extension dominated by p if and only if p is linear on X.

Proposition 4.4. *Let X be a vector space and Y a subspace. Let p be a sublinear functional such that p restricted to Y is linear, and let $f = p|_Y$. Then f has a unique extension dominated by p if and only if p is linear on X.*

Proof. If p is linear, the uniqueness of the extension is easily seen.

Conversely, by Lemma 2.6, if $\hat{f}$ is the unique extension of f, then for any $x_0 \notin Y$,

$$\hat{f}(x_0) = \sup\{f(y) - p(y - x_0) : y \in Y\} = \inf\{f(y) + p(x_0 - y) : y \in Y\}.$$

Moreover, $\hat{f} \leq p$. Now since $f = p|_Y$,

$$\begin{aligned}\hat{f}(x_0) &= \inf\{f(y) + p(x_0 - y) : y \in Y\}\\ &= \inf\{p(y) + p(x_0 - y) : y \in Y\} \geq p(x_0).\end{aligned}$$

Thus $p = \hat{f}$ is linear. □

Example 4.5 (The Banach Limit). In Example 4.2 we saw that the functional $\Lambda((x_n)) = \lim x_n$ on c does not have unique extension to ℓ^∞ dominated by the sublinear functional $\limsup x_n$.

However, the usual definition of Banach limit on ℓ^∞ requires the functional to be translation invariant, see [**15**]. For this, one has to consider the sublinear functional

$$p(x_n)) = \limsup_n \frac{1}{n}\sum_{k=1}^{n} x_k.$$

On c, the functional $\Lambda((x_n)) = \lim x_n$ is still dominated by p. Again it does not have unique extension to ℓ^∞ dominated by p, that is, there is no uniquely defined Banach limit on ℓ^∞.

This follows from Proposition 4.4. Clearly, on c, p coincides with Λ and therefore Λ would have a unique extension dominated by p if and only if p were linear on ℓ^∞. However, if p is linear, then for every $(x_n) \in \ell^\infty$, $\limsup_n (1/n)\sum_{k=1}^n x_k = \liminf_n (1/n)\sum_{k=1}^n x_k$, that is, $\lim_n (1/n)\sum_{k=1}^n x_k$ exists. It is rather easy to see that this is false. For example, let

$$x_0 = (1, -1, -1, \underbrace{1, \dots, 1}_{2^2 \text{ terms}}, \underbrace{-1, \dots, -1}_{2^3 \text{ terms}}, \dots, \underbrace{(-1)^k, \dots, (-1)^k}_{2^k \text{ terms}}, \dots) \in \ell^\infty.$$

Let

$$\sigma_m((x_n)) = \frac{1}{m}\sum_{k=1}^{m} x_k.$$

Then, for $m = \sum_{i=0}^{2k} 2^i = 2^{2k+1} - 1$,

$$\sigma_m(x_0) = \frac{\sum_{i=0}^{k} 2^{2i} - \sum_{i=0}^{k-1} 2^{2i+1}}{\sum_{i=0}^{2k} 2^i} = \frac{(2^{2k+2}-1)-2(2^{2k}-1)}{3(2^{2k+1}-1)} = \frac{2^{2k+1}+1}{3(2^{2k+1}-1)},$$

while for $m = \sum_{i=0}^{2k-1} 2^i = 2^{2k} - 1$,

$$\sigma_m(x_0) = \frac{\sum_{i=0}^{k-1} 2^{2i} - \sum_{i=0}^{k-1} 2^{2i+1}}{\sum_{i=0}^{2k-1} 2^i} = \frac{(2^{2k}-1)-2(2^{2k}-1)}{3(2^{2k}-1)} = \frac{-2^{2k}+1}{3(2^{2k}-1)}.$$

Clearly the value of $\sigma_m(x_0)$ for other values of m lies between the above values and, therefore,

$$p(x_0) = \limsup \sigma_m(x_0) = \frac{1}{3} \quad \text{and} \quad -p(-x_0) = \liminf \sigma_m(x_0) = -\frac{1}{3}.$$

Again, to interpret this in terms of nested sequences of p-balls, note that, as before, $Z = \ker(\Lambda) = c_0$. Let x_0 be as above, let $z_n \in c_0$ and define $y_n = \alpha_n(1,1,\dots,1,\dots) + z_n \in c$, $y = \alpha(1,1,\dots,1,\dots) + z \in c$, where $\alpha_n \geq 0$, $\alpha_1 < 1$ and $\alpha_n \uparrow \infty$. Let $r_n = \alpha_n + 1/2$ for $n \geq 1$. Then $\{B_{\widetilde{p}}(y_n + c_0, r_n)\}$ is a nested sequence of $\widetilde{p}$-balls in ℓ^∞/c_0. Then, as above, for $m = 2^{2k+1} - 1$,

$$\begin{aligned}
\sigma_m(y_n + y - x_0) &= \frac{(\alpha_n + \alpha - 1)\sum_{i=0}^{k} 2^{2i} + (\alpha_n + \alpha + 1)\sum_{i=0}^{k-1} 2^{2i+1}}{\sum_{i=0}^{2k} 2^i} \\
&= \alpha_n + \alpha - \frac{2^{2k+1}+1}{3(2^{2k+1}-1)}, \\
\sigma_m(y_n - y + x_0) &= \alpha_n - \alpha + \frac{2^{2k+1}+1}{3(2^{2k+1}-1)},
\end{aligned}$$

while, for $m = 2^{2k} - 1$,

$$\sigma_m(y_n + y - x_0) = \alpha_n + \alpha + \frac{2^{2k+1} - 1}{3(2^{2k} - 1)},$$
$$\sigma_m(y_n - y + x_0) = \alpha_n - \alpha - \frac{2^{2k+1} - 1}{3(2^{2k} - 1)}.$$

It follows that

$$\begin{aligned}
\widetilde{p}(y_n + y - x_0 + c_0) &= \max\left\{\alpha_n + \alpha - \frac{1}{3}, \alpha_n + \alpha + \frac{2}{3}\right\} \\
&= \alpha_n + \alpha + \frac{2}{3} = r_n + \alpha + \frac{1}{6} \geq r_n, \text{ if } \alpha \geq -\frac{1}{6} \\
\widetilde{p}(y_n - y + x_0 + c_0) &= \max\left\{\alpha_n - \alpha + \frac{1}{3}, \alpha_n - \alpha - \frac{2}{3}\right\} \\
&= \alpha_n - \alpha + \frac{1}{3} = r_n - \alpha - \frac{1}{6} \geq r_n, \text{ if } \alpha \leq -\frac{1}{6}.
\end{aligned}$$

It may also be noted that, in this case,

$$\frac{d_1((y_n - x_0 + c_0, y_n + x_0 + c_0), \Delta_1(c/c_0))}{r_n} = 2, \quad \text{for all } n.$$

Example 4.6 (Uniqueness of extensions of positive linear functionals). Let $Y \subseteq X$ be a subspace of an ordered linear space $(X, \geq)$. Say that Y is cofinal in X if given any $x \in x$, there exists $y \in Y$ such that $x \leq y$ (this is assured if X has an order unit $e \in Y$). Then any $f \in Y^\#$, $f \geq 0$ has an extension $\hat{f} \in X^\#$, $\hat{f} \geq 0$, see [**6**]. Briefly, this is seen as follows: define

$$q(x) = \inf\{f(y) : x \leq y, y \in Y\}.$$

Then q is finite-valued, sublinear and $f(y) = q(y)$ for all $y \in Y$. Extend f to $\hat{f}$ such that $\hat{f} \leq q$. Then $\hat{f} \geq 0$ (if $x \geq 0$, then $-x \leq 0$, so $\hat{f}(-x) \leq q(-x) \leq f(0) = 0$, thus $\hat{f}(x) \geq 0$). On the other hand, let F be an extension of f with $F \geq 0$. For $x \in X$, there exists $y \in Y$ such that $x \leq y$. Thus $F(x) \leq F(y) = f(y)$ and, hence, $F(x) \leq q(x)$. Thus an extension of a positive functional is positive if and only if it is an extension dominated by q.

In this case our theorem takes the following form:

Theorem 4.7. *Let X be an ordered vector space and Y a cofinal subspace. Let $f \in Y^{\#}$ be positive. Let*

$$q(x) = \inf\{f(y) : x \leq y, y \in Y\}.$$

Then the following are equivalent:

(a) *f has a unique positive extension $\hat{f}$ to $X^{\#}$.*

(b) *q is linear.*

(c) *If $\{B_q(y_n, r_n)\}$ is a nested sequence of q-balls in X such that the centers $\{y_n\} \subseteq Y$, $0 \in B_q(y_1, r_1)$ and $q(x) \leq 1$, then there exist $y \in Y$ and $n_0 \geq 1$ such that*

$$q(y_{n_0} \pm (x - y)) < r_{n_0}.$$

(d) *Y is a q-U-subspace of X.*

Proof. (a) $\Leftrightarrow$ (b). This follows from Proposition 4.4.

(a) $\Leftrightarrow$ (c). It clearly suffices to show that, for $Z = \ker(f) \subseteq Y$, $\widetilde{q}(x + Z) = q(x)$ for all $x \in X \setminus Z$.

By definition of $\widetilde{q}$, $\widetilde{q}(x + Z) \leq q(x)$. Now for any $z \in Z$ and $\varepsilon > 0$, there exists $y \in Y$ such that $x + z \leq y$ and $f(y) < q(x + z) + \varepsilon$. Then $x \leq y - z$. Therefore, $q(x) \leq f(y - z) = f(y) < q(x + z) + \varepsilon$. And, hence, $q(x) \leq \widetilde{q}(x + Z)$.

(c) $\Leftrightarrow$ (d) follows from Theorem 2.14. $\square$

Remark 4.8. It can be easily verified that $B_q(y_0, r) = \{x \in X : f(y_0) - r < -q(-x)\}$. In particular, if q is linear, the q-balls are actually half-spaces.

From the proof of [**6**, Theorem 2.6.3] (which is often called the Krein-Rutman theorem), one can see that the conditions for unique extension of f as a *continuous* positive functional are the same as in the linear space situation.

Example 4.9. Returning to Example 4.3, observe that $\mathbf{s} = (s_0, s_1, \dots, s_n, \dots) \in l_1$ with $s_n \geq 0$ acts as a positive functional on c. By Proposition 4.4, it has a unique extension as a positive functional if and only if $q(\mathbf{x}) = \inf\{\langle \mathbf{s}, \mathbf{y}\rangle : \mathbf{x} \leq \mathbf{y}, \mathbf{y} \in c\}$ is linear on ℓ^∞. Observe that for any $\mathbf{u} \in \ell^\infty$, $q(\mathbf{u}) \leq \|\mathbf{u}\|$. Thus, from definition and Example 4.3, it follows that if $\|\mathbf{s}\|_1 = 1$, then for any $\mathbf{u} \in \ell^\infty$,

$$\begin{aligned}
-q(-\mathbf{u}) &\leq s_0(\liminf_n u_n) + \sum_{n=1}^{\infty} s_n u_n = \sup\{\langle \mathbf{s}, \mathbf{x}\rangle - \|\mathbf{x} - \mathbf{u}\| : \mathbf{x} \in c\} \\
&\leq \sup\{\langle \mathbf{s}, \mathbf{x}\rangle - q(\mathbf{x} - \mathbf{u}) : \mathbf{x} \in c\} \\
&\leq \inf\{\langle \mathbf{s}, \mathbf{x}\rangle + q(\mathbf{u} - \mathbf{x}) : \mathbf{x} \in c\} \\
&\leq \inf\{\langle \mathbf{s}, \mathbf{x}\rangle + \|\mathbf{x} - \mathbf{u}\| : \mathbf{x} \in c\} \\
&= s_0(\limsup_n u_n) + \sum_{n=1}^{\infty} s_n u_n \leq q(\mathbf{u}).
\end{aligned}$$

Thus, if q is linear, then $-q(-\mathbf{u}) = q(\mathbf{u})$ and therefore $s_0 = 0$. Conversely, if $s_0 = 0$, then from the above inequalities,

$$\sup\{\langle \mathbf{s}, \mathbf{x}\rangle - q(\mathbf{x} - \mathbf{u}) : \mathbf{x} \in c\} = \inf\{\langle \mathbf{s}, \mathbf{x}\rangle + q(\mathbf{u} - \mathbf{x}) : \mathbf{x} \in c\},$$

and this clearly implies that the uniqueness of the extension dominated by q.

Combining Example 4.3 with this, we conclude that the condition $s_0 = 0$ is necessary and sufficient for $\mathbf{s}$ to have a unique Hahn-Banach extension as well as a unique extension as a positive functional to ℓ^∞. $\square$

Example 4.10 (Uniqueness in Strassen's theorem). We now deal with the uniqueness question in the theorem of Strassen mentioned in the introduction. This theorem has applications in results about the existence of probability measures with given marginals, about dilation of measures, etc., see [**10, 14, 17**] for details.

Let X be a Banach space and S the collection of all sublinear functionals on X. Let (Ω, Σ) be a measurable space.

Definition 4.11. A mapping $q : w \rightsquigarrow q_w$ from Ω to S is

(i) weakly measurable if the real-valued function $w \rightsquigarrow q_w(x)$ is measurable for every $x \in X$;

(ii) bounded if there exists $K > 0$ such that $|q_w(x)| \leq K\|x\|$ for every $x \in X$ and $w \in \Omega$.

If q is bounded by $K > 0$, it follows easily that $|q_w(x) - q_w(y)| \leq K\|x - y\|$ and, hence, q_w is continuous on X.

Theorem 4.12 [**17**]. *Let X be a separable Banach space and (Ω, Σ, μ) be a complete probability space. Let $p : w \rightsquigarrow p_w$ be a bounded weakly measurable mapping from Ω into S. Denote by s the sublinear function*

$$s(x) = \int_\Omega p_w(x)\, d\mu(w), \quad x \in X.$$

For $x^* \in X^*$, the following are equivalent:

(a) *x^* is dominated by s on X;*

(b) *there exists a bounded weakly measurable mapping $w \rightsquigarrow g_w$ from Ω to X^* such that g_w is dominated μ-almost everywhere by p_w and*

$$x^*(x) = \int_\Omega g_w(x)\, d\mu(w), \quad \textit{for every } x \in X.$$

Clearly, (b) $\Rightarrow$ (a). Briefly the converse is proved as follows (see [**10**] for the details): identify X with the constant functions in $L^1(\mu, X)$. For any $f \in L^1(\mu, X)$, the map $w \rightsquigarrow p_w(f(w))$ is obviously integrable and, therefore, we can extend the sublinear functional s from X to $L^1(\mu, X)$ by defining

$$S(f) = \int_\Omega p_w(f(w))\, d\mu(w), \quad f \in L^1(\mu, X).$$

Extend x^* to F on $L^1(\mu, X)$ such that $F \leq S$. By the representation of $L^1(\mu, X)^*$, there exists a bounded weakly measurable map $w \rightsquigarrow g_w$ from Ω to X^* such that

$$F(f) = \int_\Omega g_w(f(w))\, d\mu(w), \quad \text{for every } f \in L^1(\mu, X).$$

Further, it can be shown that g_w is dominated μ-almost everywhere by p_w.

We are interested in exploring the question of when x^* has a unique Hahn-Banach extension to $L^1(\mu, X)$ dominated by S.

Definition 4.13. Let $n \geq 1$, and let $\{\Omega_1, \Omega_2, \dots, \Omega_n\}$ be a partition of Ω. Let $\alpha_i = \mu(\Omega_i)$ and $\mu_i = (1/\alpha_i)\mu|_{\Omega_i}$. Let

$$s_i(x) = \int_{\Omega_i} p_w(x)\, d\mu(w), \quad x \in X.$$

By Theorem 4.12 then, there exists $x_i^* \in X^*$ dominated by s_i on X. We will call the representation $\mu = \sum_{i=1}^n \alpha_i \mu_i$ an n-decomposition of μ and the corresponding representation $x^* = \sum_{i=1}^n \alpha_i x_i^*$, the induced decomposition of x^*.

Theorem 4.14. *The following are equivalent*:

(a) *x^* has a unique Hahn-Banach extension to $L^1(\mu, X)$ dominated by S.*

(b) *For each 2-decomposition of $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$, the induced decomposition of $x^* = \alpha_1 x_1^* + \alpha_2 x_2^*$ is unique with x_i^* dominated by s_i.*

(c) *For every $n \geq 1$ and each n-decomposition of $\mu = \sum_{i=1}^n \alpha_i\mu_i$, the induced decomposition of $x^* = \sum_{i=1}^n \alpha_i x_i^*$ is unique with x_i^* dominated by s_i.*

(d) *For every $n \geq 1$, each n-decomposition of $\mu = \sum_{i=1}^n \alpha_i\mu_i$ and, for each choice of $x_1, x_2, \dots, x_n \in X$,*

$$\sup\Big\{x^*(x) - \sum_{i=1}^n \alpha_i s_i(x - x_i) : x \in X\Big\} = \inf\Big\{x^*(x) + \sum_{i=1}^n \alpha_i s_i(x_i - x) : x \in X\Big\}.$$

Proof. (a) $\Rightarrow$ (b). Suppose $x^* = \alpha_1 y_1^* + \alpha_2 y_2^*$, $y_i^* \in X^*$ and $y_i^* \leq s_i$. By Theorem 4.12,

$$y_i^* = \int_{\Omega_i} g_i(w)\, d\mu_i.$$

Put

$$h(w) = \begin{cases} g_1(w) & \text{if } w \in \Omega_1, \\ g_2(w) & \text{if } w \in \Omega_2. \end{cases}$$

Then

$$x^* = \alpha_1 y_1^* + \alpha_2 y_2^* = \alpha_1 \int_{\Omega_1} h(w)\, d\mu_1 + \alpha_2 \int_{\Omega_2} h(w)\, d\mu_2 = \int_{\Omega} h(w)\, d\mu.$$

By uniqueness, $g(w) = h(w)$, a.e., and hence, $x_i^* = y_i^*$.

$(b) \Rightarrow (c)$. Given any n-decomposition of $\mu = \sum_{i=1}^n \alpha_i \mu_i$, the expression

$$\mu = \alpha_1 \mu_1 + (1 - \alpha_1) \left[\frac{1}{1-\alpha_1} \sum_{i=2}^n \alpha_i \mu_i \right]$$

is a 2-decomposition of μ and therefore, in the induced decomposition $x^* = \sum_{i=1}^n \alpha_i x_i^*$, x_i^* is unique; similarly for all other x_i^*s.

$(c) \Rightarrow (d)$. Define a sublinear functional P on X^n by

$$\begin{aligned} P(x_1, x_2, \dots, x_n) &= \sum_{i=1}^n \alpha_i s_i(x_i) = \sum_{i=1}^n \alpha_i \int_{\Omega_i} p_w(x_i)\, d\mu_i \\ &= \sum_{i=1}^n \int_{\Omega_i} p_w(x_i)\, d\mu. \end{aligned}$$

Let $F = \{(x, x, \dots, x) : x \in X\} \subseteq X^n$ and define Λ on F by $\Lambda(x, x, \dots, x) = x^*(x)$. Observe that, since $x^* \leq s$ on X, $\Lambda \leq P$ on F. By Lemma 2.6, for any $x_1, x_2, \dots, x_n \in X$,

$$\begin{aligned} &\sup \Big\{ x^*(x) - \sum_{i=1}^n \alpha_i s_i(x - x_i) : x \in X \Big\} \\ &\qquad \leq \inf \Big\{ x^*(x) + \sum_{i=1}^n \alpha_i s_i(x_i - x) : x \in X \Big\}. \end{aligned}$$

And, if the inequality above is strict, the extension of Λ to X^n is not unique. Let $\mathbf{L}$ be an extension of Λ. Then there exist $y_1^*, y_2^*, \dots, y_n^* \in X^*$ such that $\mathbf{L}(x_1, x_2, \dots, x_n) = \sum_{i=1}^n y_i^*(x_i)$. Moreover, since $\mathbf{L} \leq P$, $y_i^* \leq \alpha_i s_i$. Further, $x^*(x) = \Lambda(x, x, \dots, x) = \sum_{i=1}^n y_i^*(x)$. Therefore,

$x^* = \sum_{i=1}^n \alpha_i(y_i^*/\alpha_i)$ is an induced decomposition of x^* and, clearly, different extensions of Λ gives rise to different induced decompositions.

(d) $\Rightarrow$ (a). Let $f = \sum_{i=1}^n x_i \chi_{\Omega_i} \in L^1(\mu, X)$ be a simple function, where $\{\Omega_1, \Omega_2, \dots, \Omega_n\}$ is a partition of Ω. Let μ_i and α_i be as before. Then $\mu = \sum_{i=1}^n \alpha_i \mu_i$ is an n-decomposition of μ and, by (d),

$$\sup\Big\{x^*(x) - \sum_{i=1}^n \alpha_i s_i(x - x_i) : x \in X\Big\} = \inf\Big\{x^*(x) + \sum_{i=1}^n \alpha_i s_i(x_i - x) : x \in X\Big\}.$$

It follows that

$$\sup\{x^*(x) - S(x - f) : x \in X\} = \inf\{x^*(x) + S(f - x) : x \in X\},$$

i.e., the extension of x^* to the space of simple functions dominated by S is unique.

To show that the extension of x^* to $L^1(\mu, X)$ dominated by S is unique, we use Theorem 2.12.

Let $\{B_S(x_n, r_n)\}$ be a nested sequence of S-balls in $L^1(\mu, X)$ such that the centers $\{x_n\} \subseteq \{x \in X : x^*(x) \geq 0\}$, $0 \in B_S(x_1, r_1)$ and $f \in L^1(\mu, X)$ such that $S(f) \leq 1$.

Fix $\varepsilon > 0$. Let $K > 0$ be such that $|p_w(x)| \leq K\|x\|$ for every $x \in X$. Choose a simple function $f_1 \in L^1(\mu, X)$ such that$\|f - f_1\|_1 < \varepsilon/K$. It follows that $S(f_1) \leq S(f_1 - f) + S(f) < 1 + \varepsilon$. And, hence, $S(f_1/(1+\varepsilon)) < 1$. By the uniqueness of extension to simple functions and Theorem 2.12 (d) applied to the nested sequence $\{B_S(x_n/(1+\varepsilon), (r_n - \varepsilon)/(1+\varepsilon))\}$ of S-balls, there exist $x \in X$ and $n_0 \geq 1$ such that for $Z = \ker x^* \subseteq X$,

$$\widetilde{S}\bigg(\frac{x_{n_0}}{1+\varepsilon} \pm \bigg(\frac{f_1}{1+\varepsilon} - x\bigg) + Z\bigg) < \frac{r_{n_0} - \varepsilon}{1+\varepsilon}.$$

Therefore,

$$\widetilde{S}(x_{n_0} \pm (f_1 - (1+\varepsilon)x) + Z) < r_{n_0} - \varepsilon.$$

It follows that

$$\widetilde{S}(x_{n_0} \pm (f - (1+\varepsilon)x) + Z) < r_{n_0}.$$

And this completes the proof. □

One special case of Theorem 4.14 that is of particular importance is when $X = C(K)$ for some compact convex set K in a locally convex Hausdorff topological vector space. In this case, for a suitably defined sublinear functional on X, we obtain [**14**, Theorem 3.2] as a corollary.

5. The Vlasov Property. We begin by recalling the definition.

Definition 5.1 [**16**]. A Banach space X is said to have the Vlasov Property, if there do not exist a nested sequence of balls $\{B_n\}$, and x^* and $y_k^* \in S(X^*)$ such that, for some constant c,

$$\begin{aligned} x^*(b) &> c \quad \text{for all } b \in \cup B_n, \\ y_k^*(b) &> c \quad \text{for all } b \in B_n, \quad n \le k, \end{aligned}$$

and $\operatorname{dist}(\operatorname{co}(y_1^*, y_2^*, \dots), x^*) > 0$.

Observe that since the balls $\{B_n\}$ are nested, the above conditions are same as

$$x^*(b) > c \quad \text{for all } b \in \cup B_n, \tag{5.1}$$

$$y_n^*(b) > c \quad \text{for all } b \in B_n, \tag{5.2}$$

and we will use this form in the sequel.

Since the definition is rather difficult to handle, it is desirable to have a more workable form.

Proposition 5.2. *A Banach space X has the Vlasov Property if and only if for every nested sequence $\{B_n\}$ of balls and $x^*, y_n^* \in S(X^*)$, if* (5.1) *and* (5.2) *are satisfied for some $c \in \mathbf{R}$, then $y_n^* \to x^*$ weakly.*

Proof. From the contrapositive of the definition, it is clear that X has the Vlasov Property if and only if for every nested sequence $\{B_n\}$ of balls and $x^*, y_n^* \in S(X*)$, if there exists $c \in \mathbf{R}$ such that (5.1) and (5.2) are satisfied, then $x^* \in \overline{\operatorname{co}}(y_1^*, y_2^*, \dots)$.

The sufficiency is thus immediate. And, once we observe that any subsequence $\{y_{n_k}^*\}$ of $\{y_n^*\}$ also satisfies all the given conditions, and hence allows us to conclude $x^* \in \overline{\text{co}}\,(y^*_{n_1}, y^*_{n_2}, \dots)$, necessity also follows. $\square$

We also recall the definitions and results about the ANPs.

Definition 5.3. (a) A subset Φ of $B(X^*)$ is called a norming set for X if $\|x\| = \sup\{x^*(x) : x^* \in \Phi\}$ for all $x \in X$.

(b) A sequence $\{x_n\}$ in $S(X)$ is said to be asymptotically normed by Φ if, for any $\varepsilon > 0$, there exist an $x^* \in \Phi$ and $N \geq 1$ such that $x^*(x_n) > 1 - \varepsilon$ for all $n \geq N$.

(c) For $\kappa =$ I, II, II$'$ or III, a sequence $\{x_n\}$ in X is said to have the property κ if

I. $\{x_n\}$ is convergent.

II. $\{x_n\}$ has a convergent subsequence.

II$'$. $\{x_n\}$ is weakly convergent.

III. $\{x_n\}$ has a weakly convergent subsequence.

(d) For $\kappa =$ I, II, II$'$ or III, X is said to have the asymptotic norming property κ with respect to Φ, Φ-ANP-κ, if every sequence in $S(X)$ that is asymptotically normed by Φ has property κ.

(e) For $\kappa =$ I, II, II$'$ or III, X is said to have the w^*-ANP-κ, if X^* has $B(X)$-ANP-κ.

Remark 5.4. The original definition of Φ-ANP-III was different. The equivalence with the one above was established in [**8**, Theorem 2.3]. The Φ-ANP-II$'$ and w^*-ANP-II$'$ were introduced and studied in [**1**].

We recall the following result from [**8**, Theorem 3.1] and [**1**, Theorem 3.1].

Theorem 5.5. *A Banach space X*

(a) *has* w^*-ANP-I *if and only if X^* is strictly convex and* $(S(X^*), w^*) = (S(X^*), \|\cdot\|)$.

(b) *has* w^*-ANP-II *if and only if* $(S(X^*), w^*) = (S(X^*), \|\cdot\|)$.

(c) *has* w^*-ANP-II$'$ *if and only if X^* is strictly convex and* $(S(X^*), w^*)$

$= (S(X^*), w)$.

(d) *has* w^*-ANP-III *if and only if* $(S(X^*), w^*) = (S(X^*), w)$ *if and only if* X *is Hahn-Banach smooth.*

We need the following.

Lemma 5.6. *Given* x^*_α, $x^* \in S(X^*)$ *such that* $x^*_\alpha \to x^*$ *in the* w^*-*topology,* $\delta_n > 0$ *and* $\{x_n\} \subseteq B(X)$, *there exist a sequence* $\{y^*_n = x^*_{\alpha_n}\}$ *and an increasing sequence* $\{F_n\}$ *of finite subsets of* $B(X)$ *satisfying*

(i) $x_n \in F_n$, $\alpha_n \leq \alpha_{n+1}$.

(ii) $F_n(1-\delta_n)$-*norms* $\operatorname{span}\{x^*, y^*_1, y^*_2, \dots, y^*_n\}$, *i.e., for any* $y^* \in \operatorname{span}\{x^*, y^*_1, y^*_2, \dots, y^*_n\}$, *i.e., for any* $y^* \in \operatorname{span}\{x^*, y^*_1, y^*_2, \dots, y^*_n\}$, $\sup\{y^*(x) : x \in F_n\} \geq (1-\delta_n)\|y^*\|$.

(iii) $|y^*_n(z) - x^*(z)| < \delta_k$ *for all* $z \in F_k$, $n \geq k$.

Moreover, if $\sum_{n=1}^\infty \delta_n < 1$, *there exists* $\{u_n\} \subseteq X$ *such that* $\{B_n = B(u_n, n)\}$ *is a nested sequence of balls with* (5.1) *and* (5.2) *satisfied for* $c = -2$.

Proof. The sequences $\{y^*_n\}$ and $\{F_n\}$ satisfying (i)–(iii) obtained by an inductive construction essentially as in the proof of [**8**, Lemma 2.1].

We will define $u_n = \sum_{i=1}^n v_i$ for a suitable choice of v_n with $\|v_n\| < 1$. By (ii), find $v'_n \in F_n$ such that $x^*(v'_n) > 1 - \delta_n$. If $\|v'_n\| < 1$, put $v_n = v'_n$. If $\|v'_n\| = 1$, find $0 < \lambda_n < 1$ such that $x^*(\lambda_n v'_n) > 1 - \delta_n$ and put $v_n = \lambda_n v'_n$. Note that though v_k need not belong to F_k, $|y^*_n(v_k) - x^*(v_k)| = \lambda_k |y^*_n(v'_k) - x^*(v'_k)| < \lambda_k\delta_k < \delta_k$, i.e., (iii) is satisfied.

The balls $\{B_n\}$ are then clearly nested. Moreover, for any $n \geq 1$,

$$\inf x^*(B_n) = x^*\left(\sum_{i=1}^n v_i\right) - n = \sum_{i=1}^n [x^*(v_i) - 1] > -\sum_{i=1}^n \delta_i > -2$$

and

$$\begin{aligned}\inf y^*_n(B_n) &= \sum_{i=1}^n [y^*_n(v_i) - 1] = \sum_{i=1}^n \{[y^*_n(v_i) - x^*(v_i)] + [x^*(v_i) - 1]\} \\ &> -2\sum_{i=1}^n \delta_i > -2\sum_{i=1}^\infty \delta_i > -2. \qquad \square\end{aligned}$$

Theorem 5.7. *X has the Vlasov Property if and only if X has the* w^*-ANP-II$'$.

Proof. We first note that if $\{y_n^*\}$ satisfies (5.2), it is asymptotically normed by $B(X)$. Indeed, let $B_n = B(x_n, r_n)$. We may assume, without loss of generality, that $0 \in B_1$. If we now put $y_n = x_n/r_n$, it follows that $\|y_n\| < 1$. Then (5.2) and the fact that $\{B_n\}$ is nested implies that $y_n^*(y_k) \geq 1 + c/r_k$ for all $n \geq k$. Since $r_n \to \infty$, we are done. $\square$

Since X has the w^*-ANP-II$'$, $\{y_n^*\}$ is weakly convergent. If $y_n^* \to y^*$ weakly, it follows that $y^* \in S(X^*)$ and

$$y^*(b) > c \quad \text{for all } b \in \cup B_n.$$

Since X^* is strictly convex, by Theorem 1.3 this implies that $x^* = y^*$, i.e., $y_n^* \to x^*$ weakly. The sufficiency thus follows from Proposition 5.2.

For the converse, we use Theorem 5.5 (c), i.e., we show that X^* is strictly convex and $(S(X^*), w^*) = (S(X^*), w)$.

To show X^* is strictly convex, we again use Theorem 1.3. Suppose $\{B_n\}$ is a nested sequence of balls such that $B = \cup B_n$ is neither whole of X nor a half-space. Since B is a proper open convex subset of X, there exist $x^* \in S(X^*)$ and $\alpha \in \mathbf{R}$ such that

$$\alpha = \inf x^*(B) > -\infty \quad \text{and} \quad B \subseteq \{x : x^*(x) > \alpha\} = H.$$

Since $B \neq H$, there exist $z \in H$ and $y^* \in S(X^*)$ such that $\inf y^*(B) > y^*(z) = \beta$, say. Then $x^* \neq y^*$. Otherwise,

$$\beta = y^*(z) = x^*(z) > \alpha = \inf x^*(B) = \inf y^*(B) > \beta.$$

Putting $y_n^* = y^*$, we see that (5.1) and (5.2) are satisfied with any $c < \min(\alpha, \beta)$, but $\{y_n^*\}$ cannot converge weakly to x^*.

Now, if $(S(X^*), w^*) \neq (S(X^*), w)$, there exist a net $\{x_\alpha^*\}$ and x^* in $S(X^*)$, $x^{**} \in X^{**}$ and $\varepsilon > 0$ such that $x_\alpha^* \to x^*$ in the w^*-topology and $|x^{**}(x_\alpha^* - x^*)| \geq \varepsilon$ for all α.

Choose a sequence $\{\delta_n\}$ such that $\delta_n > 0$ for all n and $\sum_{n=1}^{\infty} \delta_n < 1$. By Lemma 5.6, there is a sequence $\{y_n^* = x_{\alpha_n}^*\}$ with $\alpha_n \leq \alpha_{n+1}$ and a nested sequence $\{B_n\}$ of balls such that (5.1) and (5.2) are satisfied with $c = -2$. But clearly y_n^* cannot converge to x^* weakly. □

Replacing the weak topology by the norm topology in the above theorem, we immediately obtain

Corollary 5.8. *X has w^*-ANP-I if and only if for every nested sequence $\{B_n\}$ of balls and $x^*, y_n^* \in S(X^*)$ if there exists $c \in \mathbf{R}$ such that (5.1) and (5.2) are satisfied, then $y_n^* \to x^*$ in norm.*

From the proof of Proposition 5.2, it is clear that the analog of the above properties for II or III doesn't give us anything new. Indeed, the strict convexity of X^* remains. So we need some modification.

Definition 5.9. A Banach space X has property V-κ, $\kappa =$ I, II, II$'$ or III, if for every nested sequence $\{B_n\}$ of balls and $\{y_n^*\} \subseteq S(X^*)$ if (5.2) is satisfied for some $c \in \mathbf{R}$, then $\{y_n^*\}$ has property κ.

And here is the main theorem of this section.

Theorem 5.10. *For a Banach space X and $\kappa =$ I, II, II$'$ or III, X has w^*-ANP-κ if and only if X has V-κ.*

Proof. For necessity it suffices to note, as before, that if $\{y_n^*\}$ satisfies (5.2), it is asymptotically normed by $B(X)$.

As in [**8**], we will prove the converse in three steps.

Step 1. $\kappa =$ III.

By Theorem 5.5 (d) it suffices to show that X is a U-subspace of X^{**}. Since X is always an ideal in X^{**}, if X is not a U-subspace of X^{**}, by Theorem 3.6 there exists a nested sequence $\{B(x_n, n)\}$ of balls in X^{**} with centers in X, $\|x_1\| < 1$ and $\|x_0^{**}\| \leq 1$, and a convex w^*-neighborhood U of x_0^{**}, such that for $K = U \cap B(X)$,

$$K \cap \Big[\bigcup_n B(x_0^{**} + x_n, n)\Big] = \varnothing.$$

Now, separate the convex set K from the open convex set $\cup_{n\geq 1} B(x_0^{**}+x_n, n)$. That is, there exist $x^{***} \in S(X^{***})$ and $\gamma \in \mathbf{R}$ such that

$$\sup x^{***}(K) \leq \gamma \leq x^{***}(x_0^{**} + x_n) - n \tag{5.3}$$

for all $n \geq 1$. Let $x^* = x^{***}|_X$. As in the proof of (a) $\Rightarrow$ (b) in Theorem 3.6, we conclude from (5.3) that $\|x^*\| = 1$ and

$$x^*(x_n) - n \geq \gamma - x^{***}(x_0^{**}) = c, \quad \text{say.} \tag{5.4}$$

Moreover, since x_0^{**} is in the w^*-closure of K, it also follows from (5.3) that

$$x^*(x_0^{**}) \leq x^{***}(x_0^{**}) + x^*(x_n) - n$$

or

$$(x^{***} - x^*)(x_0^{**}) \geq n - x^*(x_n) \geq n - \|x_n\| \geq 1 - \|x_1\| > 0. \tag{5.5}$$

By Goldstein's theorem, choose a net $\{x_\alpha^*\} \subseteq S(X^*)$ such that $\hat{x}_\alpha^* \to x^{***}$ in w^*-topology on $B(X^{***})$. It follows that $x_\alpha^* \to x^*$ in w^*-topology on $B(X^*)$, but by (5.5),

$$\lim_\alpha (x_\alpha^* - x^*)(x_0^{**}) = (x^{***} - x^*)(x_0^{**}) \geq 1 - \|x_1\| > 0.$$

Let $\varepsilon = (1 - \|x_1\|)/2$. Without loss of generality, we may assume

$$(x_\alpha^* - x^*)(x_0^{**}) \geq \varepsilon \quad \text{for all } \alpha.$$

Note that $x_n/n \in B(X)$. Let $\delta_n = 1/n$. Then, by the first part of Lemma 5.6, there exist a sequence $\{y_n^* = x_{\alpha_n}^*\}$ and an increasing sequence $\{F_n\}$ of finite subsets of $B(X)$ satisfying

(i) $x_n/n \in F_n$, $\alpha_n \leq \alpha_{n+1}$.

(ii) $F_n(1 - 1/n)$-norms $\overline{\text{span}}\,\{x^*, y_1^*, y_2^*, \dots, y_n^*\}$.

(iii) $|y_n^*(z) - x^*(z)| < 1/k \quad$ for all $z \in F_k$, $n \geq k$.

Now, by (iii) and (5.4),

$$y_n^*(x_n) - n = n[y_n^*(x_n/n) - 1] > n[x_n^*(x_n/n) - 1/n - 1] > c - 1.$$

That is, (5.2) is satisfied.

By (iii), $y_n^*(z) \to x^*(z)$ for all $z \in \cup_n F_n$ which, by (ii), is a norming set for $\overline{\text{span}}\,\{x^*, y_n^*; n \geq 1\}$. Thus any weakly convergent subsequence of $\{y_n^*\}$ must converge to x^*, which is impossible.

Step 2. For $\kappa = \text{II}$, we combined the arguments of Theorem 5.7 and Step I.

If $(S(X^*), w^*) \neq (S(X^*), \|\cdot\|)$, there exist a net $\{x_\alpha^*\}$ and x^* in $S(X^*)$ and $\varepsilon > 0$ such that $x_\alpha^* \to x^*$ in the w^*-topology and $\|x_\alpha^* - x^*\| \geq \varepsilon$ for all α.

As before, choose a sequence $\{\delta_n\}$ such that $\delta_n > 0$ for all n and $\sum_{n=1}^\infty \delta_n < 1$. Since $\|x^*\| = 1$, there exists $\{x_n\} \subseteq S(X)$ such that $x^*(x_n) > 1 - \delta_n$.

By Lemma 5.6, there exist a sequence $\{y_n^* = x_{\alpha_n}^*\}$ and an increasing sequence $\{F_n\}$ of finite subsets of $B(X)$ satisfying

(i) $x_n \in F_n$, $\alpha_n \leq \alpha_{n+1}$.

(ii) $F_n(1-\delta_n)$-norms span $\{x^*, y_1^*, y_2^*, \ldots, y_n^*\}$.

(iii) $|y_n^*(z) - x^*(z)| < \delta_k$ for all $z \in F_k$, $n \geq k$,

and a nested sequence $\{B_n\}$ of balls such that (5.2) is satisfied with $c = -2$.

Therefore, by V-II, $\{y_n^*\}$ has a convergent subsequence. But, again, as in Step I, any convergent subsequence of $\{y_n^*\}$ must converge to x^*, which is impossible.

Step 3. By Theorem 5.5 it now suffices to show that X has V-II$'$ implies X^* is strictly convex.

Proceed as in the proof of Theorem 5.7 to obtain a nested sequence $\{B_n\}$ of balls, $x^*, y^* \in S(X^*)$ and $c \in \mathbf{R}$ such that $x^* \neq y^*$, $x^*(\cup B_n) > c$ and $y^*(\cup B_n) > c$. Putting

$$y_n^* = \begin{cases} x^* & \text{if } n \text{ is odd} \\ y^* & \text{if } n \text{ is even} \end{cases}$$

we see that (5.2) is satisfied, but $\{y_n^*\}$ cannot converge weakly. $\square$

Remark 5.11. From the proof, it follows that it suffices to define Property V-κ for nested sequences of balls of the type $\{B(x_n, n)\}$.

Acknowledgments. We thank the referee for a careful reading of our paper and for some detailed and helpful comments that have considerably improved the paper's readability.

REFERENCES

1. P. Bandyopadhyay and S. Basu, *On a new asymptotic norming property*, Indag. Math. (N.S.) **10** (1999), 15–23.

2. Sudeshna Basu and TSSRK Rao, *Some stability results for asymptotic norming properties of Banach spaces*, Colloq. Math. **75** (1998), 271–284.

3. Bernard Beauzamy, *Introduction to Banach spaces and their geometry*, North Holland Math. Stud. **68** (1985).

4. G. Choquet, *Lectures on analysis*, Vol. II, W.A. Benjamin, Inc., New York, 1969.

5. N. Dunford and J.T. Schwartz, *Linear operators, Part* 1: *General theory*, Interscience, NY, 1958.

6. R.E. Edwards, *Functional analysis. Theory and applications*, Holt, Rinehart and Winston, New York, 1965.

7. G. Godefroy, N.J. Kalton and P.D. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. **104** (1993), 13–59.

8. Z. Hu and B.L. Lin, *On the asymptotic norming properties of Banach spaces*, Proc. Conference on Function Spaces, SIUE, Lecture Notes in Pure and Appl. Math., vol. 136, 1992, Dekker, New York, pp. 195–210.

9. ———, *Smoothness and asymptotic norming properties in Banach spaces*, Bull. Austral. Math. Soc. **45** (1992), 285–296.

10. P.A. Meyer, *Probability and potentials*, Blaisdell Publishing Co., Waltham, MA, 1966.

11. E. Oja and M. Põldvere, *On subspaces of Banach spaces where every functional has a unique norm-preserving extension*, Studia Math. **117** (1996), 289–306.

12. ———, *Intersection properties of ball sequences and uniqueness of Hahn-Banach extensions*, Proc. Roy. Soc. Edinburgh Sect. A **129**, (1999), 1251–1262.

13. R.R. Phelps, *Uniqueness of Hahn-Banach extensions and unique best approximations*, Trans. Amer. Math. Soc. **95** (1960), 238–265.

14. M. Rajagopalan and A.K. Roy, *Some results on dilations of measures on compact convex sets*, J. Math. Pures Appl. **55** (1976), 173–188.

15. W. Rudin, *Functional analysis*, McGraw-Hill Book Co., New York, 1973.

16. W. Sullivan, *Geometric properties determined by the higher duals of Banach space*, Illinois J. Math. **21** (1977), 315–331.

17. V. Strassen, *The existence of probability measures with given marginals*, Ann. Math. Statist. **36** (1965), 423–439.

18. L.P. Vlasov, *Approximative properties of sets in normed linear spaces*, Russian Math. Surveys **28** (1973), 1–66.

Stat-Math Division, Indian Statistical Institute, 203, B.T. Road, Kolkata 700 108, India
E-mail address: pradipta@isical.ac.in

Stat-Math Division, Indian Statistical Institute, 203, B.T. Road, Kolkata 700 108, India
E-mail address: ashoke@isical.ac.in