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INTERPOLATION SPACES FOR PDE-PRESERVING PROJECTORS ON HILBERT-SCHMIDT ENTIRE FUNCTIONS

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ABSTRACT. Let Π be a projector on a vector space H and I a set of functionals such that $\ker \Pi = I^{\perp}$. Then Πf interpolates f on I, in the sense that $f - \Pi f \in I^{\perp}$, for all $f \in H$. We say that I is an interpolation space (for Π). We establish a one-to-one correspondence between the PDE-preserving projectors and sequences of functionals on the space of Hilbert-Schmidt entire functions on a Hilbert space. In this way we characterize the PDE-preserving projectors and the corresponding interpolation spaces. A PDE-preserving projector is a projector that preserves homogeneous solutions to homogeneous convolution equations.

1. Introduction. Let $f \in \mathcal{H}(C^n)$ be an entire function in n variables and let $p_0, \dots, p_d$ be points in C^n. Then there is a polynomial F of degree $\leq d$ that interpolates f at the points p_j, i.e., $F(p_j) = f(p_j)$ for all j. We may write this $f - F \in \{\delta_{p_j}\}^{\perp}$ and this suggests the following: Let H be a vector space and $I \subseteq H^*$ a set of linear functionals, we say that $F \in H$ interpolates $f \in H$ on I if $f - F \in I^{\perp}$. Thus the Taylor polynomial of order d at the point a for $f \in \mathcal{H}(C^n)$ is the unique polynomial of degree $\leq d$ that interpolates f on $I = I[a : d] \equiv \{D_y^j \delta_a : j \leq d,\ y \in C^n\}$, where D_y^j denotes the j:th directional derivative along y.

Let $I \subseteq H^*$ be a set of functionals and assume we can find a projector $\Pi : H \to \Pi H$ such that $\ker \Pi = I^{\perp}$. Then Πf interpolates f on I for every $f \in H$. We say that I is an interpolation space if such a projector exists and that it is an interpolation space for Π. In particular, every finite-dimensional subspace is an interpolation space. Since a projector is uniquely determined by its image and kernel ($H = \ker \Pi \oplus \operatorname{Im} \Pi$), a set of functionals I can be an interpolation space for at most one projector onto a given space. For example, the Taylor projector on

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$\mathcal{H}(C^n)$ of order d and at the point a, obtained by mapping a function onto the corresponding Taylor polynomial, is the unique projector onto the set of polynomials of degree at most d for which $I[a : d]$ is an interpolation space.

In [**4**] and [**8**, Paper II], the PDE-preserving projectors on $\mathcal{H}(C^n)$, and the corresponding interpolation spaces, are characterized. A PDE-preserving projector is a continuous projector onto polynomial spaces that preserves homogeneous solutions to homogeneous convolution equations, i.e., $P(D)f = 0$ implies $P(D)\Pi f = 0$ for every homogeneous polynomial P. It turns out that the interpolation spaces are finite-dimensional.

In this note we make a similar study for the space of entire functions of Hilbert-Schmidt holomorphy type $\mathcal{H}_H(E)$ on a (separable) Hilbert space E. A similar, but different, type of holomorphy is studied in [**6**]. We establish one-to-one correspondence between the PDE-preserving projectors and sequences of functionals on $\mathcal{H}_H(E)$. In this way we characterize the PDE-preserving projectors and the corresponding interpolation spaces. The entire functionals, $\mathcal{H}'_H(E)$, can be identified with the exponential type functions of Hilbert-Schmidt type by virtue of the Fourier-Borel transform (Proposition (2)). Our results imply that we can define the derivative of a projector in the way suggested by Calvi and Filipsson [**4**] when E is finite-dimensional. For example, the derivative of the Taylor projector of order d and at the point a is the corresponding Taylor projector of order $d-1$. The Kergin projector is PDE-preserving and acts very naturally on holomorphic functions. For more on the Kergin operator we refer to [**1, 2, 8, 10, 11**]. We derive an error formula for approximating a function with its Kergin polynomial (Theorem (7)). The result shows, in particular, that given any $f \in \mathcal{H}_H(E)$ and bounded sequence of points in E, the corresponding sequence of Kergin polynomials converges to f (see [**3, 8**] for more on such type of results).

The article is organized as follows. In Section 2 we introduce the space $\mathcal{H}_H(E)$. We characterize the continuous convolution operators on $\mathcal{H}_H(E)$ and the entire functionals. We recall that a convolution operator is an operator that commutes with all translation operators. In Section 3 we prove our main results Theorems 5 and 6. In the last section we derive the error formula for the PDE-preserving Kergin projector.

Most of the results that we obtain in Section 3 for $\mathcal{H}_H(E)$, can presumably be extended to other types of holomorphy, such as holomorphy of nuclear type on a Banach space (see [**9**] or [**5**, Section 4]) and holomorphy on nuclear sequence spaces (see [**11**]), where we have a similar characterization of the convolution operators and the entire functionals.

2. Entire functions of Hilbert-Schmidt type. In this section we introduce the Hilbert-Schmidt entire functions. We shall be quite brief and a more comprehensive presentation can be found in [**12**]. We also refer to [**6**] where a similar type of holomorphy is studied.

We denote by $\mathcal{H}_G(X)$ the space of complex-valued Gateaux holomorphic functions on a complex vector space X. If $f \in \mathcal{H}_G(X)$, we denote by $D_y^n f(x)$ the n:th directional derivative at x along y. Throughout this paper, E denotes a separable complex Hilbert space. We denote by $\mathcal{P}_F({}^nE)$ the space of n-homogeneous polynomials on E of finite type. That is, $\mathcal{P}_F({}^nE)$ is the subspace of the space of n-homogeneous polynomials $\mathcal{P}({}^n E)$ on E, spanned by the elements $(\cdot, y)^n$, $y \in E$, where $(\cdot,\cdot)$ denotes the inner product on E. We endow $\mathcal{P}_F({}^nE)$ with the inner product defined by $((\cdot,y)^n,(\cdot,z)^n)_n \equiv n!(z,y)^n$ (we can identify the space of symmetric tensors $\otimes_{n,s}E$ with $\mathcal{P}_F({}^nE)$ and $(\cdot,\cdot)_n$ is the inner product induced from the inner product space $\otimes_{n,s}E$ in this way). The space of n-homogeneous Hilbert-Schmidt polynomials, denoted by $\mathcal{P}_H({}^n E)$, is the completion of $\mathcal{P}_F({}^nE)$ with respect to the inner product $(\cdot,\cdot)_n$. We use the symbol $\|\cdot\|_n$ for the corresponding norm. In view of our purposes, it is convenient to note that

$$(P,(\cdot,y)^n/n!)_n = P(y), \quad y \in E,\ P \in \mathcal{P}_H({}^nE), \tag{1}$$

and, by the symmetrization formula, [**9**, Lemma 1, page 9],

$$\|(\cdot,y_1)\cdots(\cdot,y_n)\|_n \le \sqrt{n!}(er)^n, \quad y_j \in E, \|y_j\| \le r. \tag{2}$$

Let (e_j) be an orthonormal basis in E. For a given multi-index $\alpha \in N_\infty \equiv \oplus_{j=1}^\infty N$, let $e_\alpha \equiv \prod_{\operatorname{supp}\alpha}(\cdot,e_j)^{\alpha_j} \in \mathcal{P}_H({}^{|\alpha|}E)$. Here $\operatorname{supp}\alpha \equiv \{j : \alpha_j \neq 0\}$, $|\alpha| \equiv \sum \alpha_j$ and we shall also use the notation $l(\alpha) \equiv \max\{j : \alpha_j \neq 0\}$. The elements e_α, $|\alpha| = n$, form an orthogonal basis in $\mathcal{P}_H({}^n E)$ and $\|e_\alpha\|_n^2 = \alpha! \equiv \alpha_1! \cdots$ (see [**6**, Lemma 1]). Thus $\mathcal{P}_H({}^n E)$ can be identified with the space of all sequences (P_α) such that $\sum_{|\alpha|=n}|P_\alpha|^2\alpha! < \infty$ and in this way we have

$$\|P\|_n^2 = \sum_{|\alpha|=n}|P_\alpha|^2\alpha!, \quad P \in \mathcal{P}_H({}^nE). \tag{3}$$

$\mathcal{P}_H({}^n E)$ is continuously embedded into the Banach space $\mathcal{P}_C({}^n E)$, of the n-homogeneous continuous polynomials, with norm not greater than $1/\sqrt{n!}$.

Lemma 1. *Let E be a Hilbert space and let $P \in \mathcal{P}_H({}^n E)$, $Q \in \mathcal{P}_H({}^n E)$. Then $PQ \in \mathcal{P}_H({}^{n+m}E)$ and*

$$\|P\|_m\|Q\|_n \le \|PQ\|_{n+m} \le 2^{n+m}\|P\|_m\|Q\|_n. \tag{4}$$

Thus, multiplication by P defines a continuous operator with closed range between $\mathcal{P}_H({}^n E)$ and $\mathcal{P}_H({}^{n+m}E)$.

Proof. The left hand side inequality in (4) follows by [**6**, Lemma 3] and the rest of the lemma is established in [**12**]. $\square$

We denote by $\mathfrak{A}_H(E)$ the space of all formal power series $f = \sum f_n$, $f_n \in \mathcal{P}_H({}^nE)$, i.e., $\mathfrak{A}_H(E) \equiv \prod_{n=0}^{\infty}\mathcal{P}_H({}^nE)$ $(\mathcal{P}_H({}^0E) \equiv C)$. $\mathfrak{A}_H(E)$ is a ring by virtue of Lemma 1. The space of Hilbert-Schmidt polynomials, denoted by $\mathcal{P}_H(E)$, is the subring $\oplus_{n=0}^{\infty}\mathcal{P}_H({}^n E)$, or alternatively, the space spanned by $\cup_n\mathcal{P}_H({}^n E)$ in $\mathcal{H}_G(E)$. The polynomials of degree less than or equal to d, is the space $\mathcal{P}_H^d(E) \equiv \prod_n^d \mathcal{P}_H({}^nE)$ $(\simeq \sum_n^d \mathcal{P}_H({}^nE))$.

If E is a Hilbert space, the space of entire functions of Hilbert-Schmidt type on E, denoted by $\mathcal{H}_H(E)$, is the space defined as follows. $\mathcal{H}_H(E)$ is the space of all $f = \sum f_n \in \mathfrak{A}_H(E)$ such that

$$\|f\|_{H:r} \equiv \sum r^n\|f_n\|_n/\sqrt{n!} < \infty, \quad r > 0, \tag{5}$$

endowed with the semi-norms thus defined. $\mathcal{H}_H(E)$ is a Fréchet space, a subring of $\mathfrak{A}_H(E)$ and, in particular, $\mathcal{H}_H(C^n)$ is the space of entire functions endowed with the compact-open topology. The series $\sum f_n$ converges absolutely in $\mathcal{H}_H(E)$ and, by our discussion on the injection $\mathcal{P}_H({}^nE) \to \mathcal{P}_C({}^nE)$, uniformly on bounded sets for every $f = \sum f_n \in \mathcal{H}_H(E)$. Thus, $\mathcal{H}_H(E)$ is separable and every element in $\mathcal{H}_H(E)$ defines an entire function of bounded type, and so $\mathcal{H}_H(E)$ can also be described as the space of all $f \in \mathcal{H}_G(E)$ such that $f_n \equiv D^n_{(\cdot)} f(0)/n! \in \mathcal{P}_H({}^n E)$, $n = 0, \dots$, and (5) holds.

Given $r > 0$ we denote by $\mathrm{EXP}_r(E)$ the (Banach) space of all $\varphi = \sum \varphi_n \in \mathfrak{A}_H(E)$ such that for some $M > 0$, $\|\varphi_n\|_n \le M r^n/\sqrt{n!}$, $n = 0, \dots$ equipped with the norm $|||\varphi|||_{H:r} \equiv \sup_n \sqrt{n!} r^{-n} \|\varphi_n\|_n$. The symbol $\mathrm{EXP}_H(E)$ denotes the union $\cup_{r>0} \mathrm{EXP}_r(E)$ equipped with the corresponding inductive locally convex topology. Thus $\mathrm{EXP}_H(E)$ is given by all $\varphi = \sum \varphi_n \in \mathfrak{A}_H(E)$ such that $\overline{\lim}(\sqrt{n!}\|\varphi_n\|_n)^{1/n} < \infty$ and is a subring of $\mathfrak{A}_H(E)$. Every $\varphi \in \mathrm{EXP}_H(E)$ defines an exponential type function, i.e. a Gateaux holomorphic function with $\|\varphi(y)\| \le M e^{r\|y\|}$ for some $M, r \ge 0$, and its power series converges in $\mathrm{EXP}_H(E)$.

If $y \in E$ we put $e_y \equiv e^{(\cdot, y)} \in \mathrm{EXP}_H(E) \subseteq \mathcal{H}_H(E)$. We note that, for any n, the topology on $\mathcal{P}_H({}^n E)$ is the topology induced by $\mathcal{H}_H(E)$ as well as by $\mathrm{EXP}_H(E)$. Based on this, one can prove the following (details are given in [**12**]).

Proposition 2. *$\mathcal{H}_H(E)$ is reflexive and the map $\mathcal{F} : \lambda \mapsto \mathcal{F}\lambda$, $\mathcal{F}\lambda(y) \equiv \overline{\lambda(e_y)}$, defines an anti-linear isomorphism between $\mathcal{H}'_H(E)$ (strong topology) and* $\mathrm{EXP}_H(E)$.

We put $\mathcal{H}_H(E)$ and $\mathrm{EXP}_H(E)$ into duality by the sesqui-linear form $\langle f, \varphi \rangle \equiv \mathcal{F}^{-1}\varphi(f) = \sum_n (f_n, \varphi_n)_n$. Multiplication $\psi \mapsto \varphi\psi$ is continuous on $\mathrm{EXP}_H(E)$ for every $\varphi \in \mathrm{EXP}_H(E)$. Thus, by Proposition 2, the transpose $\bar{\varphi}(D) \equiv {}^t\varphi : \mathcal{H}_E(E) \to \mathcal{H}_H(E)$ exists for the duality between $\mathcal{H}_H(E)$ and $\mathrm{EXP}_H(E)$. For example, $\bar{P}(D) = \sum_j \bar{a}_j D^n_{y_j}$ if $P = \sum_j a_j (\cdot, y_j)^n \in \mathcal{P}_F({}^n E)$. Every operator $\bar{\varphi}(D)$ is continuous and, in particular, $\bar{e}_a(D)$ is translation by $a \in E$, i.e., $\bar{e}_a(D) f(x) = f(x + a)$. Thus $\mathcal{H}_H(E)$ is stable under translations and every operator $\bar{\varphi}(D)$ is a continuous convolution operator. It is not difficult to prove that we obtain all the continuous convolution operators in this way. The homogeneous convolution operators, $\bar{P}(D)$, $P \in \mathcal{P}_H({}^n E)$, will play a central role in the sequel and for such operators we have:

Theorem 3. *Let $0 \neq P \in \mathcal{P}_H({}^mE)$. Then $\bar{P}(D) \circ P$ is a bijection on $\mathcal{H}_H(E)$ and hence $\mathcal{H}_H(E) = \ker \bar{P}(D) \oplus \operatorname{Im} P$.*

Proof. By Lemma 1, the multiplication operator $P : \mathcal{P}_H({}^nE) \to \mathcal{P}_H({}^{n+m}E)$ is a continuous one-to-one map with closed image. Its Hilbert adjoint is the restriction of $\bar{P}(D)$ to $\mathcal{P}_H({}^{n+m}E)$ and hence $\bar{P}(D) \circ P$ is a bijection on $\mathcal{P}_H({}^nE)$ for all n. By Lemma 1 we obtain

$$(6) \quad \|P\|_m^2 \|Q\|_n^2 \leq \|PQ\|_{n+m}^2 = (Q, \bar{P}(D)PQ)_n \leq \|Q\|_n \|\bar{P}(D)PQ\|_n,$$

for all $Q \in \mathcal{P}_H({}^nE)$. Let $g = \sum g_n \in \mathcal{H}_H(E)$ be arbitrary. For every n there is a unique $f_n \in \mathcal{P}_H({}^nE)$ with $\bar{P}(D)Pf_n = g_n$ and, by (6), $\|f_n\|_n \leq \|g_n\|_n / \|P\|_m^2$. Hence $f \equiv \sum f_n \in \mathcal{H}_H(E)$ and we deduce that $\bar{P}(D)Pf = g$. Thus $\bar{P}(D) \circ P$ maps $\mathcal{H}_H(E)$ onto $\mathcal{H}_H(E)$. If $\bar{P}(D)Pf = 0$, we deduce that $\bar{P}(D)Pf_n = 0$ for all n and hence $f = \sum f_n = 0$. Thus $\bar{P}(D) \circ P$ is a bijection on $\mathcal{H}_H(E)$ and, as an easy consequence, $\mathcal{H}_H(E) = \ker \bar{P}(D) \oplus \operatorname{Im} P$. □

Theorem 3 shows that the (closed) ideal $\operatorname{Im} P = \{P\varphi : \varphi \in \mathrm{EXP}_H(E)\}$ in $\mathrm{EXP}_H(E)$ is an interpolation space for the projector $\mathcal{H}_H(E) \to \operatorname{Im} P$ defined by the decomposition $\mathcal{H}_H(E) = \ker \bar{P}(D) \oplus \operatorname{Im} P$.

3. PDE-preserving projectors.

Definition 1. A PDE-preserving projector on $\mathcal{H}_H(E)$ is a continuous projector Π onto some space $\mathcal{P}_H^d(E)$, $d \geq 0$, such that $\ker \bar{P}(D) \subseteq \ker \bar{P}(D)\Pi$ for every homogeneous polynomial $P \in \mathcal{P}_H({}^nE)$, $n = 0, \ldots,$ compare [**8**, p. 51].

A sequence $\Phi = (\varphi_0, \ldots, \varphi_d)$, $\varphi_j \in \mathrm{EXP}_H(E)$, of length $d+1$ is nondegenerate if $\varphi_j(0) \neq 0$ for all j. Every nondegenerate sequence can be normalized so that $\varphi_j(0) = 1$ for all j, and, in view of this, it suffices to consider such normalized sequences. We let Σ_d denote the set of all normalized sequences in $\mathrm{EXP}_H(E)$ of length $d+1$. If $\Phi = (\varphi_0, \ldots, \varphi_d)$ and $\Psi = (\psi_0, \ldots, \psi_d)$ belong to Σ_d, we say that Ψ is related to Φ if for each $j \leq d$ there are homogeneous polynomials

$P_i^j \in \mathcal{P}_H({}^iE)$, $i = 1, \ldots, d-j$ such that

$$\psi_j = \varphi_j + P_1^j \varphi_{j+1} + \cdots + P_{d-j}^j \varphi_d. \tag{7}$$

We obtain an equivalence relation $\sim$ on Σ_d in this way. If $\Phi \in \Sigma_d$ we denote by

$$\begin{aligned} I(\Phi) &\equiv \operatorname{span}\{P_j\varphi_j : P_j \in \mathcal{P}_H({}^jE),\ j \leq d\} \\ &= \{P_0\varphi_0 + \cdots + P_d\varphi_d : P_j \in \mathcal{P}_H({}^jE)\}. \end{aligned}$$

Note that if $\Phi \sim \Psi$ then $I(\Phi) = I(\Psi)$. Below we prove that the converse holds true, Theorem 6.

Lemma 4. *For every $\Phi = (\varphi_0, \ldots, \phi_d) \in \Sigma_d$ there is a sequence $\Psi = (\psi_0, \ldots, \psi_d) \sim \Phi$ such that*

$$\psi_{ij} = 0, \quad j = 1, \ldots, d-i, \quad i = 0, \ldots, d-1, \tag{8}$$

where $\psi_i = \sum_j \psi_{ij}$, $\psi_{ij} \in \mathcal{P}_H({}^jE)$.

Proof. Put $\psi_d \equiv \varphi_d$. For the general j:th step, we put $\psi_{d-j} \equiv \phi_j + \cdots + \phi_0$ where ϕ_i are defined recursively as follows:

$$\begin{aligned} \phi_j &\equiv \varphi_{d-j} \\ \phi_{j-1} &\equiv -H_1(\phi_j)\varphi_{d-j+1} \\ \vdots\ &\quad\ \vdots \\ \phi_0 &\equiv -[H_j(\phi_j) + \cdots + H_j(\phi_1)]\varphi_d. \end{aligned}$$

Here H_i denotes the projector of $\mathrm{EXP}_H(E)$ onto the corresponding homogeneous part, $\sum_n \xi_n \mapsto \xi_i \in \mathcal{P}_H({}^iE)$. We see that $\Psi \sim \Phi$ and it is easily checked that Ψ satisfies (8). $\square$

A sequence $\Phi \in \Sigma_d$ is said to be a diagonal sequence when the homogeneous parts vanish in the sense of (8). Thus every equivalence class contains a diagonal sequence and it is easily checked that it is unique.

When we establish the second part of the following theorem, we follow a proof in [**11**, Section 9] quite closely.

Theorem 5. *Let* $\Phi = (\varphi_0, \dots, \varphi_d) \in \Sigma_d$. *Then there is a* (*unique*) PDE-*preserving projector* Π_Φ *of* $\mathcal{H}_H(E)$ *onto* $\mathcal{P}^d_H(E)$ *such that* $\ker \Pi_\Phi = I(\Phi)^\perp$.

Conversely, if Π *is a* PDE-*preserving projector onto* $\mathcal{P}^d_H(E)$, *then there is a* $\Phi = (\varphi_j) \in \Sigma_d$ *such that* $\ker \Pi = I(\Phi)^\perp$, *i.e.* $\Pi = \Pi_\Phi$.

Proof. In view of Lemma 4 we can assume that Φ is diagonal so that (8) holds true for the elements $\varphi_i = \sum_j \varphi_{ij}$. Consider the map Π on $\mathcal{H}_H(E)$ defined by

$$\Pi f \equiv H_0 \bar{\varphi}_0(D) f + \cdots + H_d \bar{\varphi}_d(D) f \in \mathcal{P}^d_H(E), \tag{9}$$

where H_j denotes the projector of $\mathcal{H}_H(E)$ onto the corresponding homogeneous part. The map $H_j \bar{\varphi}_j(D)$ is continuous for all j and hence Π is continuous.

We prove that Π preserves homogeneous solutions. Let $P \in \mathcal{P}_H({}^nE)$. We note that $\bar{P}(D) H_j = H_{j-n} \bar{P}(D)$ for all $j \geq n$. Hence if $\bar{P}(D) f = 0$, $n \leq d$,

$$\bar{P}(D) H_j \bar{\varphi}_j(D) f = H_{j-n} \bar{P}(D) \bar{\varphi}_j(D) f = H_{j-n} \bar{\varphi}_j(D) \bar{P}(D) f = 0 \tag{10}$$

for all $j \geq n$. From this we deduce that $\bar{P}(D)\Pi f = 0$. Since $\bar{P}(D) \circ \Pi = 0$ if $n > d$, Π preserves homogeneous solutions.

We prove that $\ker \Pi = I(\Phi)^\perp$. Assume $f \in \ker \Pi$ and consider $(\cdot, y)^j \varphi_j \in I(\Phi)$. We have

$$\langle f, (\cdot, y)^j \varphi_j \rangle = D^j_y \bar{\varphi}_j(D) f(0) = j! H_j \Pi f(y). \tag{11}$$

Hence $\langle f, (\cdot, y)^j \varphi_j \rangle = 0$ and, since the elements $(\cdot, y)^j$, $y \in E$, form a total set in $\mathcal{P}_H({}^jE)$ we deduce that $f \in I(\Phi)^\perp$. Conversely, if $f \in I(\Phi)^\perp$, (11) shows that $H_j \bar{\varphi}_j(D) f = 0$ for all j and hence $f \in \ker \Pi$.

Next we prove that Π is a projector onto $\mathcal{P}^d_H(E)$. Since Π is continuous, it suffices, by totality, to prove that $\Pi(\cdot, y)^n = (\cdot, y)^n$, $n \leq d$. But we note that

$$H_j \bar{\varphi}_j(D)(\cdot, y)^n = \frac{n!}{j!} \varphi_{j(n-j)}(y)(\cdot, y)^j$$

if $j \leq n$ and $H_j \bar{\varphi}_j(D)(\cdot, y)^n = 0$ if $j > n$. Since Φ is diagonal

$$\varphi_{j0} = 1, \qquad \varphi_{j(n-j)} = 0, \quad j < n,$$

and hence $\Pi(\cdot, y)^n = (\cdot, y)^n$.

Conversely, let Π be a PDE-preserving projector. Let (e_j) be an orthonormal basis in E and consider the functionals $\pi_\alpha \in \mathcal{H}'_H(E)$ defined by

$$\Pi f = \sum_{|\alpha| \leq d} \pi_\alpha(f) e_\alpha,$$

i.e., $\pi_\alpha(f) \equiv D^\alpha \Pi f(0)/\alpha!$ where D^α denotes the directional derivative along the vectors e_j corresponding to the multi-index $\alpha \in N_\infty$. Since Π is a projector, $\pi_\alpha \circ \Pi = \pi_\alpha$ and $\pi_\alpha(e_\beta) = \delta^\alpha_\beta$, $|\beta| \leq d$. It follows that $\Pi_\alpha \equiv \mathcal{F}\pi_\alpha \in \operatorname{Im} e_\alpha$, i.e., by Theorem 3, $\Pi_\alpha \in (\ker D^\alpha)^\perp$ for all α. Indeed, since Π is PDE-preserving, $D^\alpha f = 0$ implies $D^\alpha \Pi f = 0$ and hence $\langle f, \Pi_\alpha \rangle = \pi_\alpha(f) = 0$. Thus there are (unique) $\varphi_\alpha \in \mathrm{EXP}_H(E)$, $|\alpha| \leq d$, such that

$$e_\alpha \varphi_\alpha = \alpha! \Pi_\alpha, \quad \Pi_\alpha \equiv \mathcal{F}\pi_\alpha \in \mathrm{EXP}_H(E). \tag{12}$$

Further, we note that $\varphi_\alpha(0) = \pi_\alpha(e_\alpha) = 1$ and since $\pi_\alpha \circ \Pi = \pi_\alpha$

$$\langle \Pi f, e_\alpha \varphi_\alpha \rangle = \alpha! \pi_\alpha(\Pi f) = \alpha! \pi_\alpha(f) = \langle f, e_\alpha \varphi_\alpha \rangle. \tag{13}$$

We prove that

$$e_\alpha \varphi_\alpha = e_\alpha \varphi_{\alpha'} \tag{14}$$

if $|\alpha| = |\alpha'| \leq d$. In view of (13) and since $\varphi_\alpha(0) = 1$, this will imply that for each $j \leq d$ we may choose φ_j as any element φ_α with $|\alpha| = j$. Note that, by continuity, it suffices to show that (14) holds on $Y^{(n)} \equiv \{y = \sum y_j e_j \in E : y_j = 0,\ j > n\}$ for all n.

Let $y \in E$ and put $u(y) = (\cdot, y)^j \in \mathcal{P}_H({}^jE)$ and $v(y) = H_j \Pi e_y \in \mathcal{P}_H({}^jE)$. Let $P \in \mathcal{P}_H({}^jE)$ and assume $P \in \{u(y)\}^\perp$. Then $\bar{P}(D) e_y = 0$ and hence, since Π is PDE-preserving, $0 = \bar{P}(D)\Pi e_y = (P, v(y))_j$. Thus $\{u(y)\}^\perp \subseteq \{v(y)\}^\perp$ and hence there is a constant $c(y)$ such that $cu = v$. This means that

$$y^\alpha \varphi_\alpha(y) = c(y) y^\alpha, \quad |\alpha| = j. \tag{15}$$

We note that c is uniquely determined except for its value at the origin, and $y \mapsto c(y)$ is an analytic function on $Y^{(n)} \setminus \{0\} \simeq C^n \setminus \{0\}$ for all n. Indeed, (15) implies that, for given n, c is analytic on $C^n \setminus \{y_i = 0\}$ for all $i \leq n$ and hence on the union $C^n \setminus \{0\}$. Now let $|\alpha| = |\alpha'| = j$ and $y \in Y^{(n)}$. If $y^\alpha = 0$ both sides in (14) vanish, hence we can assume $y^\alpha \neq 0$. If $m \equiv \max\{l(\alpha), l(\alpha'), n\}$, there is a sequence $y(i)$ in $Y^{(m)} \simeq C^m$ such that $y(i) \to y$ as $i \to \infty$ and $y(i)^{\alpha'} \neq 0$ for all i. Hence, by (15),

$$\varphi_{\alpha'}(y) = \lim c(y(i)) = c(y) = \varphi_\alpha(y).$$

This completes the proof. $\square$

Remark 1. Let $\Phi \in \Sigma_d$ be any sequence, i.e., not necessarily diagonal. The proof shows that the operator Π defined by (9) is a continuous operator and that it preserves homogeneous solutions. Moreover, $\ker \Pi = I(\Phi)^\perp$ and $\operatorname{Im} \Pi \cap \ker \Pi = \{0\}$. However, Π may fail to be a projector onto $\mathcal{P}_H^d(E)$.

We have noted that $\Phi \sim \Psi$ implies $I(\Phi) = I(\Psi)$ and by Theorem 5, $I(\Phi) = I(\Psi)$ implies $\Pi_\Phi = \Pi_\Psi$. We now prove that we have equivalencies.

Theorem 6. *Let $\Phi, \Psi \in \Sigma_d$. Then the following are equivalent*:

(1) $\Phi \sim \Psi$

(2) $I(\Phi) = I(\Psi)$

(3) $\Pi_\Phi = \Pi_\Psi$,

and every interpolation space $I(\Phi)$ is closed.

Proof. We only have to prove that (3) implies (1) and that $I(\Phi)$ is closed.

Assume $\Pi_\Phi = \Pi_\Psi$. Then $I(\Phi)^\perp = I(\Psi)^\perp$ and hence $\overline{I(\Phi)} = \overline{I(\Psi)}$ for the duality between $\mathrm{EXP}_H(E)$ and $\mathcal{H}_H(E)$. We prove that $\Psi = (\psi_j)$ is related to $\Phi = (\varphi_j)$. In view of Lemma 4 we may assume that Φ satisfies (8), i.e., Φ is a diagonal sequence.

By hypothesis

$$\psi_0 = \lim_\alpha \phi^\alpha = \lim_\alpha (P_0^\alpha \varphi_0 + \cdots + P_d^\alpha \varphi_d), \quad P_j^\alpha \in \mathcal{P}_H({}^jE),$$

for some net (ϕ^α) in $I(\Phi)$. The objective is to prove that each term $\phi_j^\alpha \equiv P_j^\alpha \varphi_j$ converges to some $P_j \varphi_j$ where $P_j \in \mathcal{P}_H({}^jE)$ and $P_0 = 1$.

Let us start with the first term $\phi_0^\alpha \equiv P_0^\alpha \varphi_0$. Since P_0^α are complex numbers, it suffices to prove that $P_0^\alpha \to 1$ in C. But since $\bar{P}_j^\alpha(D)1 = 0$, $j \geq 1$, we have

$$1 = \langle 1, \psi_0 \rangle = \lim \langle 1, P_0^\alpha \varphi_0 \rangle = \lim P_0^\alpha \varphi_0(0) = \lim P_0^\alpha.$$

Next we prove that $\phi_1^\alpha \equiv P_1^\alpha \varphi_1$ converges to $\psi_{01}\varphi_1$, $\psi_i = \sum_j \psi_{ij}$, $\psi_{ij} \in \mathcal{P}_H({}^jE)$. First we show that P_1^α converges to ψ_{01} in $\mathcal{P}_H^1(E)$ for the topology $\sigma(\mathcal{P}_H({}^1E), \mathcal{P}_H({}^1E))$. Let $P \in \mathcal{P}_H({}^1E)$ be arbitrary. Since $\bar{P}_j^\alpha(D)P = 0$, $j \geq 2$ we deduce that $\langle P, \phi_j^\alpha \rangle$ for all $j \geq 2$. Moreover, by our assumption that Φ is diagonal, $\varphi_{01} = 0$, and hence $\langle P, \phi_0^\alpha \rangle = 0$. Hence

$$\begin{aligned}(P, \psi_{01})_1 &= \langle P, \psi_0 \rangle \\ &= \lim \langle P, P_1^\alpha \varphi_1 \rangle = \lim \langle \bar{\varphi}_1(D) P, P_1^\alpha \rangle \\ &= \lim (H_1 \bar{\varphi}_1(D) P, P_1^\alpha)_1 = \lim (P, P_1^\alpha)_1\end{aligned}$$

which proves the weak convergence in $\mathcal{P}_H({}^1E)$. This implies that $\phi_1^\alpha \to \psi_{01}\varphi_1$ for the topology $\sigma(\mathrm{EXP}_H(E), \mathcal{H}_H(E))$. Indeed, let $f \in \mathcal{H}_H(E)$ be arbitrary. Then

$$\langle f, P_1^\alpha \varphi_1 \rangle = (H_1 \bar{\varphi}_1(D) f, P_1^\alpha)_1 \longrightarrow (H_1 \bar{\varphi}_1(D) f, \psi_{01})_1 = \langle f, \psi_{01}\varphi_1 \rangle.$$

Continuing in this fashion we obtain that $\psi_0 = \varphi_0 + \psi_{01}\varphi_1 + \cdots + \psi_{0d}\varphi_d$. Our arguments also show that $I(\Phi)$ is closed. To prove that (7) holds for $j \geq 1$ we take $y \neq 0$ arbitrary and repeat the arguments above for $(\cdot, y)^j \psi_j$. □

Theorems 5 and 6 show that there is a one-to-one correspondence between the PDE-preserving projectors onto $\mathcal{P}_H^d(E)$ and the equivalence classes in Σ_d. Hence, since every equivalence class contains a unique diagonal sequence, we can identify PDE-preserving projectors with diagonal sequences in a one to one way. Moreover, given the diagonal sequence, which we obtain from the procedure in the proof of Lemma 4, the projector is given by (9).

By virtue of Theorem 6, we can define the derivative of a PDE-preserving projector, in the way suggested by Calvi and Filipsson [**4**] when E is finite-dimensional. If $\Phi = (\varphi_0, \dots, \varphi_d) \in \Sigma_d$ and $n \le d$, we let $\Phi^{(n)} \equiv (\varphi_n, \dots, \varphi_d) \in \Sigma_{d-n}$. Let us note that if $\Phi \sim \Psi$ in Σ_d, then $\Phi^{(n)} \sim \Psi^{(n)}$ for all $n \le d$. If $\Pi = \Pi_\Phi$ is a PDE-preserving projector, we define the derivative $D^n\Pi = \Pi^{(n)}$ of order n by $\Pi^{(n)} \equiv \Pi_{\Phi^{(n)}}$ if $n \le d$ and otherwise $\Pi^{(n)} \equiv 0$. By formula (9) we have the identity $\bar{P}(D)\Pi = \Pi^{(n)}\bar{P}(D)$ for any homogeneous polynomial $P \in \mathcal{P}_H({}^nE)$, which indicates the PDE-preserving property of Π.

Similarly, $(\Psi, \Phi) \sim (\Psi', \Phi')$ in Σ_{d+l+1} whenever $\Phi \sim \Phi'$ and $\Psi \sim \Psi'$ in Σ_d and Σ_l respectively. Here $(\Psi, \Phi) \equiv (\psi_0, \dots, \psi_l, \varphi_0, \dots, \varphi_d)$. Thus, for any equivalence class $[\Psi]$ in Σ_l, we can define $\int_{[\Psi]} \Pi \equiv \Pi_{(\Psi,\Phi)}$, where $\Pi = \Pi_\Phi$, and note that $(\int_{[\Psi]} \Pi)^{(l+1)} = \Pi$.

Example 1. Let $p = (p_0, \dots, p_d)$ be a finite sequence in E. The projector corresponding to the sequence $\Phi = (e_{p_0}, \dots, e_{p_d}) \in \Sigma_d$ is an Abel-Gontcharrof projector $G[p]$. Thus if $f \in \mathcal{H}_H(E)$, then $\bar{P}(D)(f - G[p]f)(p_j) = 0$ for all $j \le d$ and every homogeneous polynomial $P \in \mathcal{P}_H({}^jE)$. If the points in p all are equal, $p_j = a$, we obtain the Taylor projector $T[a : d]$.

Example 2. Let $p = (p_0, \dots, p_d)$ be a sequence of points in E. If $f \in \mathcal{H}_H(E)$ we let

$$\int_{[p]} f \equiv \int_{S_d} f\big(p_0 + s_1(p_1 - p_0) + \cdots + s_d(p_d - p_0)\big)\, ds, \tag{16}$$

where $S_d \equiv \{s = (s_k \ge 0) \in R^d : \sum s_k \le 1\}$ denotes the simplex in R^d. If $p = (p_0)$, i.e., if p consists only of one point, we put $\int_{[p]} f \equiv f(p_0)$. The Kergin projector, with respect to p, is the projector $K[p] \equiv \Pi_\Phi$ where $\Phi = (\varphi_j) \in \Sigma_d$, $\varphi_j \equiv \mathcal{F}\lambda_j$, and where $\lambda_j \in \mathcal{H}'_H(E)$ is defined by $\lambda_j(f) \equiv j! \int_{[p^j]} f$, $p^j = (p_0, \dots, p_j)$. If the points in p are distinct we obtain the Lagrange projector $L[p]$ and if they all are equal, $K[p]$ is the Taylor projector for the corresponding point. The derivatives of the Kergin projector are the so called Mean-Value projectors, studied in [**7**] when E is finite-dimensional.

4. Kergin interpolation. In Example 2 we defined the Kergin projector $K[p]$ for any sequence of points $p = (p_0, \dots, p_d)$ in E. For more on the Kergin operator we refer to [**1, 2, 8, 10, 11**]. The integral (16) does not depend on the ordering of the points. From this we deduce that $K[p]$ is independent of the ordering of the points and $I(\Phi) = I$ where

$$I \equiv \operatorname{span}\{Pe_{p_j} : P \in \mathcal{P}_H({}^nE), \quad n \le |p|_j,\ j \le d\}$$

and where $|p|_j$ denotes the number of repetitions of the point p_j in p. Hence I is an interpolation space for $K[p]$ and $K[p]$ interpolates function values in the sense that $\bar{P}(D)(f - K[p]f)(p_j) = 0$ for all j and $P \in \mathcal{P}_H({}^nE)$, $n \le |p|_j$.

One can show that the Kergin projector is given by the following formula

(17)
$$K[p]f(x) \equiv \int_{[p_0]} f + \int_{[p_0,p_1]} D_{x-p_0} f + \cdots + \int_{[p_0,\dots,p_d]} D_{x-p_0} \cdots D_{x-p_{d-1}} f.$$

Theorem 7. *Let $f \in \mathcal{H}_H(E)$. Then for any sequence of points $(p_0, \dots)$ in E we have for any j and $\rho > 0$,*

(18)
$$\begin{aligned} \left\| f - K[p^j]f \right\|_{H:\rho} &\le \left\| f - \sum_{n=0}^{j} f_n \right\|_{H:\rho_j} \\ &\left(= \sum_{n=j+1}^{\infty} \frac{[4((3+e)r_j + \rho)]^n}{\sqrt{n!}} \|f_n\|_n \right), \end{aligned}$$

where $p^j = (p_0, \dots, p_j)$, $r_j \equiv \max\{\|p_i\| : 0 \le i \le j\}$ and $\rho_j \equiv 4((3+e)r_j + \rho)$.

Proof. Let j be arbitrary and put $r = r_j$. Since the Kergin polynomial for f, with respect to the points (p^j, x), interpolates f at the point x, we deduce from formula (17)

$$
\begin{aligned}
\big(f - K[p^j]f\big)(x) &= \int_{[p^j,x]} D_{x-p_0} \cdots D_{x-p_j} f \\
&= \sum_{n=j+1}^{\infty} \int_{[p^j,x]} D_{x-p_0} \cdots D_{x-p_j} f_n \\
&= \sum_{n=j+1}^{\infty} F_n(x).
\end{aligned}
\tag{19}
$$

It is clear that F_n is a finite sum of terms of the form

$$
\int_{[p^j,x]} D_x^m D_{q_0} \cdots D_{q_{j-m}} f_n, \quad m \le j+1, \tag{20}
$$

where $q_i \in \{p_0, \ldots, p_j\}$. We shall derive an estimate for an arbitrary such type of term F_n^m.

Now $F_n^m(x) = \int_{S_{j+1}} D_x^m D_{q_0} \cdots D_{q_{j-m}} f_n(\xi(s) + s_{j+1}x)\, ds$ where

$$
\xi(s) \equiv p_0 + s_1(p_1 - p_0) \cdots s_j(p_j - p_0) + s_{j+1}(-p_0), \quad s \in S_{j+1}.
$$

We note that $\|\xi(s)\| \le 3r$ and the Binomial formula gives

$$
(\cdot, \xi(s) + s_{j+1}x)^{n-j-1} = \sum_{i=0}^{n-j-1} \binom{n-j-1}{i} s_{j+1}^i (\cdot, x)^i (\cdot, \xi(s))^{n-j-1-i}.
$$

In view of this and (1), the expression (20) for F_n^m can be written

$$
\begin{aligned}
F_n^m(x) &= \int_{S_{j+1}} \sum_{i=0}^{n-j-1} \Big\langle D_{q_0} \cdots D_{q_{j-m}} f_n, (\cdot, x)^m \frac{(\cdot, \xi(s) + s_{j+1}x)^{n-j-1}}{(n-j-1)!} \Big\rangle\, ds \\
&= \sum_{i=0}^{n-j-1} \binom{n-j-1}{i} \frac{(m+i)!}{(n-j-1)!} \int_{S_{j+1}} \bar{Q}^i_{nm}(s, D) f_n(x)\, ds \\
&= \sum_{i=0}^{n-j-1} F_n^{mi},
\end{aligned}
$$

where

$$Q^i_{nm}(s,\cdot) \equiv s^i_{j+1}(\cdot,q_0)\cdots(\cdot,q_{j-m})(\cdot,\xi(s))^{n-j-1-i} \in \mathcal{P}_H({}^{n-m-i}E).$$

The integral for $\bar{Q}^i_{nm}(s,D)f_n$ exists in $\mathcal{P}_H({}^{m+i}E)$ and by (2) and Lemma 1 we deduce

$$\begin{aligned}\|Q^i_{nm}(s,\cdot)\|_{n-m-i} &\le 2^{n-m-i}\|(\cdot,q_0)\cdots(\cdot,q_{j-m})\|_{n-m+1}\\ &\quad\times \|(\cdot,\xi(s))^{n-j-1-i}\|_{n-j-1-i}\\ &\le 2^{n-m-i}\sqrt{(j-m+1)!}(er)^{j-m+1}\\ &\quad\times \sqrt{(n-j-1-i)!}(3r)^{n-j-1-i}\\ &\le 2^{n-m-i}\sqrt{(n-m-i)!}(er)^{j-m+1}(3r)^{n-j-1-i}\end{aligned}$$

and

$$\begin{aligned}\|\bar{Q}^i_{nm}(s,D)f_n\|^2_{m+i} &= (f_n, Q^i_{nm}(s,\cdot)\bar{Q}^i_{nm}(s,D)f_n))_n \le\\ &\le 2^n\|f_n\|_n\|Q^i_{nm}(s,\cdot)\|_{n-m-i}\|\bar{Q}^i_{nm}(s,D)f_n\|_{m+i},\end{aligned}$$

and hence

$$\begin{aligned}\|F^{mi}_n\|_{m+i} &\le \binom{n-j-1}{i}\frac{(m+i)!}{(n-j-1)!}\frac{1}{(j+1)!}\sup_{s\in S_{j+1}}\|\bar{Q}^i_{nm}(s,D)f_n\|_{m+i}\\ &\le \binom{n-j-1}{i}\binom{n}{j+1}\sqrt{\frac{(m+i)!}{n!}}\left(\frac{1}{2}\right)^m\\ &\quad\times\left(\frac{1}{2}\right)^i(3r)^{n-j-1-i}(er)^{j-m+1}4^n\|f_n\|_n.\end{aligned}$$

Thus

$$\begin{aligned}\|F^m_n\|_{H:\rho} &= \sum_{i=0}^{n-j-1}\frac{\rho^{m+i}}{\sqrt{(m+i)!}}\|F^{mi}_n\|_{m+i}\\ &\le \frac{4^n}{\sqrt{n!}}\|f_n\|_n\binom{n}{j+1}(\rho/2)^m(er)^{j-m+1}\\ &\quad\times\sum_{i=0}^{n-j-1}\binom{n-j-1}{i}(3r)^{n-j-1-i}(\rho/2)^i\\ &= \frac{4^n}{\sqrt{n!}}\|f_n\|_n\binom{n}{j+1}(\rho/2)^m(er)^{j-m+1}(3r+\rho/2)^{n-j-1}.\end{aligned}$$

Now, F_n is a finite sum $\sum_{m=0}^{j+1} F_{nm}$ where each F_{nm} is a sum of $\binom{j+1}{m}$ terms of the form F_n^m. In view of this the Binomial formula gives

$$\begin{aligned}\|F_n\|_{H:\rho} &\le \frac{4^n}{\sqrt{n!}}\|f_n\|_n\binom{n}{j+1}(3r+\rho/2)^{n-j-1}\\ &\quad\times\sum_{m=0}^{j+1}\binom{j+1}{m}(er)^{j-m+1}(\rho/2)^m\\ &= \frac{4^n}{\sqrt{n!}}\|f_n\|_n\binom{n}{j+1}(3r+\rho/2)^{n-j-1}(er+\rho/2)^{j+1}\\ &\le \frac{4^n}{\sqrt{n!}}[(3+e)r+\rho)]^n\|f_n\|_n.\end{aligned}$$

This estimate and (19) completes the proof. □

The estimate (18) shows that for any bounded sequence of interpolation points and $f \in \mathcal{H}_H(E)$, the corresponding sequence of Kergin polynomials converges to f. More generally, a sufficient condition for a function f, in order that its sequence of Kergin polynomials converges, is that the (increasing) sequence $\sum_{n=0}^{j} r_j^n \rho^n \|f_n\|_n/\sqrt{n!}$ is bounded for every $\rho > 0$, compare [**8**, Theorem 7.1].

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