ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 3, Fall 2004

WHEN ARE ASSOCIATES UNIT MULTIPLES?

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ABSTRACT. Let R be a commutative ring with identity. For $a, b \in R$ define a and b to be *associates*, denoted $a \sim b$, if $a|b$ and $b|a$, to be *strong associates*, denoted $a \approx b$, if $a = ub$ for some unit u of R, and to be *very strong associates*, denoted by $a \cong b$, if $a \sim b$ and further when $a \neq 0$, $a = rb$ implies that r is a unit. Certainly $a \cong b \Rightarrow a \approx b \Rightarrow a \sim b$. In this paper we study commutative rings R, called *strongly associate rings*, with the property that for $a, b \in R$, $a \sim b$ implies $a \approx b$ and commutative rings R, called *présimplifiable* rings, with the property that for $a, b \in R$, $a \sim b$ (or $a \approx b$) implies that $a \cong b$.

Let R be a commutative ring with identity and let $a, b \in R$. Then a and b are said to be *associates*, denoted $a \sim b$, if $a|b$ and $b|a$, or equivalently, if $Ra = Rb$. Thus if $a \sim b$, there exist $r, s \in R$ with $ra = b$ and $sb = a$ and hence $a = sra$. So if a is a regular element (i.e., nonzero divisor), $sr = 1$ and hence r and s are units. Hence if a and b are regular elements of a commutative ring R with $a \sim b$, then $a = ub$ for some $u \in U(R)$, the group of units of R. For $a, b \in R$, let us write $a \approx b$ if $a = ub$ for some $u \in U(R)$. Of course, $a \approx b$ implies $a \sim b$ for elements a and b of any commutative ring R and for an integral domain the converse is true. In [**14**], Kaplansky raised the question of when a commutative ring R satisfies the property that for all $a, b \in R$, $a \sim b$ implies $a \approx b$. He remarked that Artinian rings, principal ideal rings, and rings with $Z(R) \subseteq J(R)$ satisfy this property. (Here $Z(R)$ and $J(R)$ denote the set of zero divisors and Jacobson radical of a ring R, respectively.) But he gave two examples of commutative rings that fail to satisfy this property. Let us recall these two examples and give a third example. (1) Let $R = C([0,3])$, the ring of continuous functions on $[0,3]$. Define $a(t), b(t) \in R$ by $a(t) = b(t) = 1 - t$ on $[0,1]$, $a(t) = b(t) = 0$ on $[1,2]$, and $a(t) = -b(t) = t - 2$ on $[2,3]$. Then $a(t) \sim b(t)$ (for $c(t)a(t) = b(t)$ and $c(t)b(t) = a(t)$ where $c(t) = 1$ on $[0,1]$, $c(t) = 3 - 2t$ on $[1,2]$, and $c(t) = -1$ on $[2,3]$), but $a(t) \not\approx b(t)$.

2000 AMS *Mathematics Subject Classification.* 13A05, 13A15, 13F99.
Received by the editors on September 3, 2002.

(2) Let $R = \{(n, f(X)) \in \mathbf{Z} \times GF(5)[X] \mid f(0) \equiv n \bmod 5\}$, a subring of $\mathbf{Z} \times GF(5)[X]$. Then $(0, X) \sim (0, \bar{2}X)$, but $(0, X) \not\approx (0, \bar{2}X)$. Here it is interesting to note that, while $\mathbf{Z} \times GF(5)[X]$ has the property that $a \sim b \Rightarrow a \approx b$, its subring R does not. (3) (Fletcher [**12**]). Let K be a field and $R = K[X, Y, Z]/(X - XYZ)$. Then $\bar{X} \sim \bar{X}\bar{Y}$, but $\bar{X} \not\approx \bar{X}\bar{Y}$. Actually, we later show (Theorem 9) that K can be replaced by any commutative ring.

The purpose of this paper is to study commutative rings R with the property that, for all $a, b \in R$, $a \sim b$ implies $a \approx b$. Let us call such rings *strongly associate*. These rings, called "associate rings," were introduced and studied by Spellman et al. [**16**]. The basis for our choice of the word "strongly associate" will become apparent from the next paragraph.

A general study of various associate relations was begun by Anderson and Valdes-Leon in [**5**]. Let R be a commutative ring and let $a, b \in R$. There a and b were defined to be *associates*, denoted $a \sim b$, if $a|b$ and $b|a$, *strong associates*, denoted $a \approx b$, if $a = ub$ for some $u \in U(R)$, and *very strong associates*, denoted $a \cong b$, if $a \sim b$ and further when $a \neq 0$, $a = rb$ ($r \in R$) implies $r \in U(R)$. Clearly $a \cong b \Rightarrow a \approx b$ and $a \approx b \Rightarrow a \sim b$, but examples were given to show that neither of these implications could be reversed. Thus it is of interest to study commutative rings R where for all $a, b \in R$ (i) $a \sim b \Rightarrow a \approx b$, (ii) $a \approx b \Rightarrow a \cong b$, or (iii) $a \sim b \Rightarrow a \cong b$. We have already defined a ring R satisfying (i) to be strongly associate. Following Bouvier [**7**], we define a commutative ring R to be *présimplifiable* if, for $x, y \in R$, $xy = x$ implies $x = 0$ or $y \in U(R)$. We first note that (ii) and (iii) are equivalent to R being présimplifiable and give some other conditions equivalent to R being présimplifiable. Note that while $\sim$ and $\approx$ are both equivalence relations on R, the relation $\cong$ is an equivalence relation on R if and only if R is présimplifiable. While the various implications of Theorem 1 are known, we include a proof for the convenience of the reader.

Theorem 1. *For a commutative ring R, the following conditions are equivalent.*

(1) *For all $a, b \in R$, $a \sim b \Rightarrow a \cong b$.*

(2) *For all $a, b \in R$, $a \approx b \Rightarrow a \cong b$.*

(3) *For all* $a \in R$, $a \cong a$.

(4) R *is présimplifiable.*

(5) $Z(R) \subseteq 1 - U(R) = \{1 - u \mid u \in U(R)\}$.

(6) $Z(R) \subseteq J(R)$.

(7) *For* $0 \neq r \in R$, $sRr = Rr \Rightarrow s \in U(R)$.

Proof. (1) $\Rightarrow$ (2) Clear. (2) $\Rightarrow$ (3) For $a \in R$, $a \approx a$ and so $a \cong a$. (3) $\Rightarrow$ (4) Assume that $x = yx$. Now $x \cong x$. Hence $x = 0$ or $y \in U(R)$. (4) $\Rightarrow$ (5) Let $x \in Z(R)$. Suppose that $zx = 0$ where $z \neq 0$. Then $z = z(1-x)$, so $1 - x \in U(R)$ and the result follows. (5) $\Rightarrow$ (6) Let $x \in Z(R)$. For $r \in R$, $-rx \in Z(R)$ and hence $1 + rx \in U(R)$. Thus $x \in J(R)$. (6) $\Rightarrow$ (7) Suppose that $0 \neq r \in R$ and $sRr = Rr$. Then $r = str$ for some $t \in R$. Thus $r(1 - st) = 0$, so $1 - st \in Z(R) \subseteq J(R)$. Then $st = 1-(1-st)$ is a unit and so s itself is a unit. (7) $\Rightarrow$ (1) Suppose that $a \sim b$ and $a \neq 0$. Suppose that $a = rb$. Then $Ra = rRb = rRa$. Hence $r \in U(R)$. So $a \cong b$. □

Corollary 2 (Kaplansky [**14**]). *A présimplifiable ring R is strongly associate.*

Présimplifiable rings have been investigated by Bouvier in a series of papers [**7**]–[**11**] and by Anderson and Valdes-Leon [**5, 6**]. Examples of présimplifiable rings include integral domains and quasilocal rings. For a commutative ring R, R is présimplifiable $\Leftrightarrow$ $R[[X]]$ is présimplifiable, while $R[X]$ is présimplifiable $\Leftrightarrow$ R is présimplifiable and 0 is a primary ideal of R [**5**, pp. 471–472] (of course, if 0 is primary, then R is présimplifiable). For a Noetherian ring R, R is présimplifiable $\Leftrightarrow$ $\bigcap_{n=1}^{\infty} I^n = 0$ for each proper (principal) ideal of R $\Leftrightarrow$ for each $0 \neq a \in R$, there exists a natural number $N(a)$ so that if $a = a_1 \cdots a_n$ where each a_i is a nonunit, then $n \leq N(a)$ [**5**, Theorem 3.9]. Condition (7) of Theorem 1 was introduced and used by Fletcher [**12, 13**] in his study of unique factorization in commutative rings with zero divisors. He called a ring satisfying (7) a "pseudo-domain." Our main interest in présimplifiable rings is Corollary 2: a présimplifiable ring is strongly associate. It is easily seen that a présimplifiable ring is indecomposable. Since a direct product of strongly associate rings is strongly associate

(Theorem 3), a strongly associate ring need not be présimplifiable. Note that $\mathbf{Z}(\mathbf{Z}_4)$, the idealization of $\mathbf{Z}$ by $\mathbf{Z}_4$, is strongly associate, but not présimplifiable (Corollary 16). Since $\mathbf{Z}(\mathbf{Z}_4)$ is indecomposable, it is not even a direct product of présimplifiable rings.

We first note how strongly associate and présimplifiable rings behave with respect to direct products, direct limits, inverse limits, and ultraproducts. Parts (1) and (2) of Theorem 3 for strongly associate rings are given in Spellman et al. [**16**] while parts (2) and (3) for présimplifiable rings are given by Bouvier [**10**]. Recall that $(\Lambda, \leq)$ is a *directed quasi-ordered set* if $\leq$ is a reflexive and transitive relation on Λ and for $\alpha, \beta \in \Lambda$, there exists $\gamma \in \Lambda$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$. For the definition of a direct and inverse system and their limits, see Rotman [**15**].

Theorem 3. (1) *Let $\{R_\alpha\}_{\alpha\in\Lambda}$ be a nonempty family of commutative rings. Then $R = \prod_{\alpha\in\Lambda} R_\alpha$ is strongly associate if and only if each R_α is strongly associate. However, R is not présimplifiable whenever $|\Lambda| > 1$.*

(2) *Let $(\Lambda, \leq)$ be a directed quasi-ordered set and let $\{R_\alpha\}_{\alpha\in\Lambda}$ be a direct system of rings. If each R_α is strongly associate, respectively, présimplifiable, then the direct limit $R = \varinjlim R_\alpha$ is strongly associate, respectively, présimplifiable.*

(3) *Let $(\Lambda, \leq)$ be a directed quasi-ordered set and let $\{R_\alpha\}_{\alpha\in\Lambda}$ be an inverse system of rings. If each R_α is présimplifiable, then the inverse limit $R = \varprojlim R_\alpha$ is présimplifiable.*

(4) *Let $\mathcal{F}$ be an ultrafilter on Λ. Then the ultraproduct $\prod R_\alpha/\mathcal{F}$ is strongly associate, respectively, présimplifiable, $\Leftrightarrow$ $\{\alpha \in \Lambda \mid R_\alpha$ is strongly associate, respectively, présimplifiable$\} \in \mathcal{F}$. Hence an ultraproduct of strongly associate, respectively, présimplifiable, rings is strongly associate, respectively, présimplifiable.*

Proof. (1) Note that, for $(a_\alpha), (b_\alpha) \in \prod R_\alpha$, $(a_\alpha) \sim (b_\alpha)$, respectively, $(a_\alpha) \approx (b_\alpha)$, $\Leftrightarrow$ each $a_\alpha \sim b_\alpha$, respectively, $a_\alpha \approx b_\alpha$. Thus $\prod R_\alpha$ is strongly associate if and only if each R_α is strongly associate. The second statement follows from the fact that a présimplifiable ring is indecomposable.

(2) We give the proof for the présimplifiable case. The proof for the strongly associate case is similar. Let $x, y \in R$ with $x = xy$ and $y \neq 0$. For $\alpha \in \Lambda$, let λ_α: $R_\alpha \to R$ be the natural map. Now there exists $\alpha_0 \in \Lambda$ and $x_{\alpha_0}, y_{\alpha_0} \in R_{\alpha_0}$ with $\lambda_{\alpha_0}(x_{\alpha_0}) = x$, $\lambda_{\alpha_0}(y_{\alpha_0}) = y$, and $x_{\alpha_0} = x_{\alpha_0} y_{\alpha_0}$. Now $y_{\alpha_0} \neq 0$ since $\lambda_{\alpha_0}(y_{\alpha_0}) = y \neq 0$, so $y_{\alpha_0} \in U(R_{\alpha_0})$. Hence $y = \lambda_{\alpha_0}(y_{\alpha_0}) \in U(R)$. Thus R is présimplifiable.

(3) The proof of (3) is given by Bouvier [**10**]. His proof which assumes that $(\Lambda, \leq)$ is totally ordered may easily be modified to the case where $(\Lambda, \leq)$ is a directed quasi-ordered set.

(4) Observe that the property of a ring R being strongly associate, respectively, présimplifiable, can be expressed in terms of a first-order sentence: $\sigma = \forall x \; \forall y \; \exists z \; \exists w \; \exists u \; \exists v \; \exists k \; \forall l[((xz = y) \wedge (yw = x)) \Rightarrow ((kl = l) \wedge (uv = k) \wedge (xu = y))]$, respectively, $\sigma = \forall x \, \forall y \, \exists w \, \exists v \, \forall z[(xy = x) \Rightarrow (((x = w) \wedge (wz = w)) \vee ((yu = v) \wedge (uz = z)))]$. Thus (4) follows from Łos's theorem. □

Corollary 4 (Kaplansky [**14**]). *A principal ideal ring or Artinian ring is strongly associate.*

Proof. A principal ideal ring is a finite direct product of PID's and SPIR's (recall that a *special principal ideal ring* is a principal ideal ring with one prime ideal and that prime ideal is nilpotent) while an Artinian ring is a finite direct product of (0-dimensional) local rings. In either case, the ring is a product of strongly associate rings and hence is strongly associate. □

We have observed that a quasilocal ring is présimplifiable and hence strongly associate. We next show that a semi-quasilocal ring is strongly associate. But since a finite direct product of quasilocal rings is semi-quasilocal, a semi-quasilocal ring need not be présimplifiable.

Theorem 5. *A semi-quasilocal ring* $(R, M_1, \ldots, M_n)$ *is strongly associate.*

Proof. Let $a, b \in R$ with $a \sim b$. We show that $a \approx b$. We may assume that $a \neq 0$. Choose $r \in R$ with $ra = b$. Since $a \neq 0$, some $R_{M_i} a \neq 0$, say $R_{M_1}a \neq 0$. Then in R_{M_1}, $a/1 \sim b/1$; so $a/1 \cong b/1$. Hence $(r/1)(a/1) = b/1$ gives that $r/1$ is a unit in R_{M_1}. Suppose that r has been chosen so that $r/1$ is a unit in R_{M_i} for $i = 1, \ldots, s$. Suppose that $r/1$ is not a unit in $R_{M_{s+1}}$. Then $R_{M_{s+1}}a = 0$. (For, as shown above, if $R_{M_{s+1}}a \neq 0$, then $r/1$ is a unit in $R_{M_{s+1}}$.) Hence $\operatorname{ann}(a)_{M_{s+1}} = R_{M_{s+1}}$, i.e., $\operatorname{ann}(a) \not\subseteq M_{s+1}$. Choose $t \in \operatorname{ann}(a) \cap M_1 \cap \cdots \cap M_s - M_{s+1}$. Then $(r+t)a = b$, $(r+t)/1$ is a unit in R_{M_i} for $i = 1, \ldots, s$ since $r/1 \in U(R_{M_i})$ and $t/1 \in M_{i_{M_i}}$ and $(r+t)/1$ is a unit in $R_{M_{s+1}}$ since $r/1 \in M_{s+1_{M_{s+1}}}$ and $t/1 \in U(R_{M_{s+1}})$. Thus replacing r by $r+t$ we have $ra = b$ and $r/1$ is a unit in R_{M_i} for $i = 1, \ldots, s+1$. Continuing, we get an $r \in R$ with $ra = b$ where $r/1$ is a unit in each R_{M_i}, i.e., r is a unit of R. □

Other classes of rings that are strongly associate are rings with "good factorization properties." A nonzero nonunit of a commutative ring R is *irreducible* if $a = bc$ implies $a \sim b$ or $a \sim c$ and R is said to be *atomic* if every (nonzero) nonunit of R is a finite product of irreducibles. A ring R is called a *bounded factorization ring* (*BFR*) if, for each nonzero nonunit $a \in R$, there exists a natural number $N(a)$ so that if $a = a_1 \cdots a_n$ where each a_i is a nonunit, then $n \leq N(a)$. We have already remarked that a BFR is présimplifiable and hence is strongly associate. An atomic ring with only finitely many nonassociate atoms that are not prime is called a *generalized Cohen-Kaplansky* (*CK*) *ring*. Aḡargün, Anderson, and Valdes-Leon [**2**] showed that a generalized CK ring is a finite direct product of finite local rings, SPIR's, and integral domains. Hence a generalized CK ring is strongly associate.

Various notions of unique factorization rings with zero divisors have been given. Perhaps the most natural is the following. A commutative ring R is a *unique factorization ring* (*UFR*) if (1) R is atomic and (2) if $0 \neq a_1 \cdots a_n = b_1 \cdots b_m$ where each a_i, b_j is irreducible, then $n = m$ and after re-ordering, if necessary, each $a_i \sim b_i$. Since a UFR is clearly présimplifiable, a UFR is strongly associate. (It turns out that a UFR R is either a UFD, an SPIR, or a quasilocal ring (R, M) with $M^2 = 0$, see, for example, Anderson and Valdes-Leon [**5**].) Fletcher [**12, 13**] defined a second type of unique factorization ring. His type of unique factorization ring turns out to be a finite direct product of UFD's and SPIR's and hence is strongly associate. A third type of unique factorization ring called a weak UFR was defined by Aḡargün, Anderson, and Valdes-Leon [**1**]. It turns out to be either a finite direct product of UFD's and SPIR's or a quasilocal ring (R, M) with $M^2 = 0$. Thus it too is strongly associate.

Kaplansky's second example $\{(n, f(X)) \in \mathbf{Z} \times GF(5) \mid f(0) \equiv n \bmod 5\}$ is the pullback of $\mathbf{Z} \to GF(5) \leftarrow GF(5)[X]$. We next consider conditions under which a pullback is (not) strongly associate or présimplifiable.

Proposition 6. *Let R_1, R_2, and R_3 be commutative rings with homomorphisms $p_i : R_i \to R_3$ for $i = 1, 2$. Suppose that $\ker p_1 \nsubseteq Z(R_1)$ and that there exists $u \in U(R_1)$ with $p_1(u)$, $p_1(u^{-1}) \in p_2(R_2) - p_2(U(R_2))$. Then the pullback P of $R_1 \to R_3 \leftarrow R_2$ is not strongly associate.*

Proof. Choose $a \in \ker p_1 - Z(R_1)$. Let $u \in U(R_1)$ with $p_1(u), p_1(u^{-1}) \in p_2(R_2) - p_2(U(R_2))$. Suppose that $p_1(u) = p_2(b_1)$ and $p_1(u^{-1}) = p_2(b_2)$ where $b_1, b_2 \in R_2$. Now $(a, 0), (u, b_1), (u^{-1}, b_2) \in P$ and $(a, 0) \sim (ua, 0)$ in P since $(ua, 0) = (u, b_1)(a, 0)$ and $(u^{-1}, b_2)(ua, 0) = (a, 0)$. But suppose that $(a, 0) \approx (ua, 0)$. Then there exists $(u_1, u_2) \in U(P)$ with $(u_1, u_2)(a, 0) = (ua, 0)$. Hence $u_i \in U(R_i)$ and $u_1 a = ua$. Since $a \notin Z(R_1)$, $u_1 = u$. But then $p_1(u) = p_1(u_1) = p_2(u_2) \in p_2(U(R_2))$, a contradiction. $\square$

Theorem 7. *Let R_1, R_2, and R_3 be commutative rings with surjective homomorphisms $p_i : R_i \to R_3$, $i = 1, 2$, which are not isomorphisms. Suppose that R_1 and R_2 are integral domains. Let P be the pullback of $R_1 \to R_3 \leftarrow R_2$. Then P is strongly associate, respectively, présimplifiable, if and only if $p_1(U(R_1)) = p_2(U(R_2))$, respectively, $p_i^{-1}(1) \subseteq U(R_i)$, or equivalently, $p_i^{-1}(U(R_3)) = U(R_i)$, for $i = 1, 2$.*

Proof. (1) The strongly associate case.

($\Rightarrow$) Suppose that P is strongly associate. Assume that $p_1(U(R_1)) \neq p_2(U(R_2))$, say $p_1(U(R_1)) \nsubseteq p_2(U(R_2))$. Choose $u \in U(R_1)$ with $p_1(u) \notin p_2(U(R_2))$. Since p_1 is not injective, $\ker p_1 \nsubseteq Z(R_1)$. By Proposition 6, P is not strongly associate, a contradiction.

($\Leftarrow$) Suppose that $p_1(U(R_1)) = p_2(U(R_2))$. Let $(a_1, a_2), (b_1, b_2) \in P$

with $(a_1, a_2) \sim (b_1, b_2)$. We may assume that $(a_1, a_2) \neq (0, 0)$, say $a_1 \neq 0$. So $(r_1, r_2)(a_1, a_2) = (b_1, b_2)$ and $(s_1, s_2)(b_1, b_2) = (a_1, a_2)$ where $(r_1, r_2), (s_1, s_2) \in P$. Hence $s_1 r_1 a_1 = a_1$ and $a_1 \neq 0$ gives $s_1 r_1 = 1$. So $r_1 \in U(R_1)$. If $a_2 \neq 0$, then likewise $r_2 \in U(R_2)$. In this case $(r_1, r_2) \in U(P)$ and hence $(a_1, a_2) \approx (b_1, b_2)$. So assume $a_2 = 0$. Then $b_2 = 0$. Choose $\tilde{r}_2 \in U(R_2)$ with $p_2(\tilde{r}_2) = p_1(r_1)$. Then $(r_1, \tilde{r}_2) \in U(P)$ and $(r_1, \tilde{r}_2)(a_1, a_2) = (r_1, \tilde{r}_2)(a_1, 0) = (r_1 a_1, 0) = (b_1, 0) = (b_1, b_2)$. So $(a_1, a_2) \approx (a_2, b_2)$.

(2) The présimplifiable case. Note that

$$p_i^{-1}(1) \subseteq U(R_i) \Leftrightarrow p_i^{-1}(U(R_3)) = U(R_i).$$

$(\Rightarrow)$ Suppose that P is présimplifiable. Let $r \in R_1$ with $p_1(r) = 1$; so $(r, 1) \in P$. Choose $0 \neq a \in \ker p_2$; so $(0, a) \in P$. Then $(0, a) = (r, 1)(0, a)$. Since P is présimplifiable, $(r, 1) \in P$. Hence $r \in U(R_1)$.

$(\Leftarrow)$ Suppose that $(x_1, x_2) = (y_1, y_2)(x_1, x_2)$ where $(x_1, x_2), (y_1, y_2) \in P$. Assume that $(x_1, x_2) \neq (0, 0)$; say $x_1 \neq 0$. Then $x_1 y_1 = x_1$; so $y_1 = 1$. Thus $p_2(y_2) = p_1(y_1) = 1$. So by hypothesis, $y_2 \in U(R_2)$. Then $(y_1, y_2) \in U(P)$. Hence P is présimplifiable. □

Corollary 8. *Let D be an integral domain with prime ideal M. Let $p_1 : D \to D/M$ be the natural map and $p_2 : (D/M)[X] \to D/M$ be defined by $p_2(f(X)) = f(0)$. Then the pullback P of $D \to D/M \leftarrow (D/M)[X]$ is strongly associate, respectively, présimplifiable, if and only if $p_1(U(D)) = U((D/M))$, respectively, $M = 0$.*

Proof. First, suppose that $M = 0$. Then it is easily checked that P is présimplifiable and hence strongly associate. So assume $M \neq 0$. By Theorem 7, P is strongly associate $\Leftrightarrow p_1(U(D)) = p_2(U((D/M)[X]))$. Since $p_2(U((D/M)[X])) = U(D/M)$, the first result follows. Also, by Theorem 7, P is présimplifiable $\Rightarrow p_2^{-1}(1) \subseteq U((D/M)[X])$. But since $1 + X \in p_2^{-1}(1)$, this is a contradiction. Thus P is not présimplifiable if $M \neq 0$. □

Note that for $D = \mathbf{Z}$ and $M = (p)$, p a prime, $U(Z) \to U(\mathbf{Z}/(p))$ is surjective only for $p = 0, 2, 3$. Then the pullback of $\mathbf{Z} \to \mathbf{Z}/(p) \leftarrow$

$(\mathbf{Z}/(p))[X]$ is strongly associate if and only if $p = 0$, 2, and 3 and is présimplifiable if and only if $p = 0$. Thus we recover Kaplansky's second example.

However, the homomorphic image or subring of a strongly associate ring or présimplifiable ring need not be so. The third example $R = K[X,Y,Z]/(X - XYZ)$, K a field, shows that the homomorphic image of a présimplifiable ring need not even be strongly associate. The second example $R = \{(n, f(X)) \in \mathbf{Z} \times GF(5)[X] \mid n \equiv f(0) \bmod 5\} \subseteq \mathbf{Z} \times GF(5)[X]$ shows that a subring of a strongly associate ring need not be strongly associate. Also, $K[X,Y,Z]/(X - XYZ) = K[X,Y,Z]/((X) \cap (1 - YZ))$ naturally embeds into the strongly associate ring $K[X,Y,Z]/(X) \times K[X,Y,Z]/(1 - YZ) \approx K[Y,Z] \times K[X,Y,Y^{-1}]$. In fact, $K[X,Y,Z]/(X-XYZ)$ is a subdirect product of two integral domains. Actually, any reduced ring is a subdirect product of strongly associate rings: $R \hookrightarrow \prod\{R/P \mid P$ is a minimal prime ideal of $R\}$. Thus if R is reduced with a finite number of minimal primes (e.g., R is Noetherian and reduced), R is a finite subdirect product of strongly associate rings. And the first example of a non-strongly associate ring $R = C([0,3])$ is a subring of a direct product of copies of $\mathbf{R}$ which is strongly associate. Next, suppose that R is a présimplifiable ring in which 0 is not primary, e.g., $R = K[[X,Y]]/(X)(X,Y)$, K a field. By a previously mentioned result, the présimplifiable ring $R[[X]]$ has $R[X]$ as a non-présimplifiable subring. In fact, we next observe that any ring R is both a subring of a strongly associate ring and of a non-strongly associate ring. The first observation is due to Spellman et al. [**16**].

Theorem 9. (1) *Every ring is a subring of a strongly associate ring, namely* $R \hookrightarrow \prod_{M \in \operatorname{Max}(R)} R_M$ *where* $\operatorname{Max}(R)$ *is the set of maximal ideals of* R.

(2) *Every ring is a subring of a non-strongly associate ring. Indeed, for any commutative ring* R, R *is a subring of the non-strongly associate ring* $R[X,Y,Z]/(X - XYZ)$.

Proof. (1) Each R_M being quasilocal is présimplifiable and hence strongly associate. Thus $\prod R_M$ is strongly associate by Theorem 3. And, of course, R naturally embeds into $\prod R_M$ by $r \to (r/1)$.

(2) Denoting the images of X, Y, and Z in $R[X,Y,Z]/(X - XYZ)$ by x, y, and z, we have $x = xyz$, *so* $x \sim xy$. But $x \not\approx xy$. For suppose that $\bar{f}x = xy$ where $f \in R[X,Y,Z]$. Then $fX - YX \in X(1-YZ)$, so $f - Y \in (1-YZ)$ and hence $f = Y + h(1-YZ)$ for some $h \in R[X,Y,Z]$. To show that $\bar{f}$ cannot be a unit, it suffices to show that $(Y + h(1-YZ), X) \neq R[X,Y,Z]$. Suppose that $(Y+h(1-YZ), X) = R[X,Y,Z]$. Setting $Y = Z$ and $X = 0$, we get that $Y + h(1-Y^2)$ is a unit in $R[Y]$ where now $h \in R[Y]$. Setting $h = a_0 + a_1Y + \cdots + a_nY^n$, $Y + h(1-Y^2) = a_0 + (a_1+1)Y + (a_2 - a_0)Y^2 + \cdots + (a_n - a_{n-2})Y^n - a_{n-1}Y^{n+1} - a_nY^{n+2}$. Since $Y + h(1-Y^2)$ is a unit, a_0 must be a unit and $a_1+1, a_2-a_0, \ldots, a_n - a_{n-2}, a_{n-1}, a_n$ must be nilpotent. Working backwards gives that $a_n, a_{n-1}, \ldots, a_2$ are nilpotent. But then $a_2 - a_0$ must be a unit, a contradiction. $\square$

Having observed that a subring of a strongly associate ring need not be strongly associate, Spellman et al. [**16**] dubbed a ring R "superassociate" if every subring of R is strongly associate. However, we prefer the following terminology. A commutative ring R is *hereditarily strongly associate*, respectively, *hereditarily présimplifiable* if each subring of R is strongly associate, respectively, présimplifiable. Also, in Spellman et al., [**16**] a commutative ring R was defined to be *domainlike* provided every zero divisor is nilpotent, that is, 0 is a primary ideal of R. Thus a domain-like ring is présimplifiable since the nilradical is always contained in the Jacobson radical. The converse is false as $K[[X,Y]]/(X^2, XY)$ is présimplifiable but not domain-like. Note that $R[X]$ is présimplifiable $\Leftrightarrow$ $R[X]$ is domain-like $\Leftrightarrow$ R is domain-like. Also, if $S \subseteq R$ is a pair of commutative rings with R domain-like, then $0_S = 0_R \cap S$ is primary, so S is again domain-like. Thus a domain-like ring is hereditarily présimplifiable and hence hereditarily strongly associate. However, we have already noted that a présimplifiable ring need not be hereditarily présimplifiable.

Since an integral domain is hereditarily strongly associate (even domain-like), previous examples show that a homomorphic image of a hereditarily strongly associate ring or direct product of a family of hereditarily strongly associate rings need not be hereditarily strongly associate. Also, note that $\mathbf{Z} \times \mathbf{Z}$ and $\mathbf{Z}_2 \times \mathbf{Z}_2$ are hereditarily strongly associate, but are not domain-like. By Theorem 11 a Boolean ring is strongly associate. Since a subring of a Boolean ring is again Boolean, a

Boolean ring is hereditarily strongly associate. But a Boolean ring with more than two elements is not présimplifiable and hence not domain-like.

Let R be a commutative ring. We have already observed that if r and s are regular elements of R with $r \sim s$, then $r \approx s$. (In fact, $r \cong s$.) We next give a slight extension.

Lemma 10. *Suppose that* $\mathrm{re} \sim sf$ *where* r *and* s *are regular elements of a commutative ring* R *and* e *and* f *are idempotents of* R. *Then* $e = f$ *and* $\mathrm{re} \approx sf$. *Hence for* $a \in R$, $a \sim \mathrm{re}$ *implies* $a \approx \mathrm{re}$. *However,* $\mathrm{re} \cong sf \Leftrightarrow e = 0$ *or* 1.

Proof. Now $1 - f \in (0 : f) = (0 : sf) = (0 : re) = (0 : e)$, so $(1 - f)e = 0$ and hence $e = ef$. Likewise, $f = ef$, and hence $e = f$. So $re \sim se$. Then in the ring Re, re and se are regular; so $\mathrm{re} = (ue)se$ for some $u \in R$ with ue a unit in Re. Then $v = ue + (1 - e)$ is a unit in R and $\mathrm{re} = v(se)$. Hence $\mathrm{re} \approx se = sf$. Suppose that $a \sim \mathrm{re}$. Then $a(1 - e) = 0$, so $a = ae$. Hence $ae \sim re$. Since ae is a regular element of Re, as above we can replace a by a regular element of R. So $a = ae \approx \mathrm{re}$. The last statement is obvious. $\square$

A commutative ring R is called a *P.P. ring* if principal ideals of R are projective. Equivalently, R is a P.P. ring if each element $x \in R$ can be written in the form $x = \mathrm{re}$ where r is regular and e is idempotent. This together with Lemma 10 gives Theorem 11 below. Certainly, a von Neumann regular ring is a P.P. ring (as von Neumann regular rings are characterized by the property that each element is the product of a unit and an idempotent) and hence is strongly associate by the next theorem. However, a P.P. ring, respectively, von Neumann regular ring, is présimplifiable if and only if it is a domain, respectively, field.

Theorem 11. *A P.P. ring is strongly associate.*

We next give some examples using the method of idealization. Recall that, if R is a commutative ring and M is an R-module, then $R(M) = R \oplus M$ under the usual addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ is a commutative ring with identity. Here we have $U(R(M)) = U(R) \oplus M$. In Anderson and Valdes-Leon [**6**], the relations $\sim$, $\approx$, and $\cong$ were extended to modules. We recall these definitions. Let $m_1, m_2 \in M$. We define $m_1 \sim m_2 \Leftrightarrow Rm_1 = Rm_2$, $m_1 \approx m_2 \Leftrightarrow m_1 = um_2$ for some $u \in U(R)$, and $m_1 \cong m_2 \Leftrightarrow m_1 \sim m_2$ and further when $m_1 \neq 0$, $m_1 = rm_2 \Rightarrow r \in U(R)$. We call M *strongly associate* if $m_1 \sim m_2 \Rightarrow m_1 \approx m_2$ and M *présimplifiable* if $m_1 \sim m_2 \Rightarrow m_1 \cong m_2$. Theorem 1 carries over to modules mutatis mutandis. For example, M is présimplifiable if and only if $Z(M) \subseteq J(R)$ or $M = 0$. Thus every R-module is présimplifiable if and only if R is quasilocal. Note that M is strongly associate, respectively, présimplifiable, if and only if each cyclic submodule of M is strongly associate, respectively, présimplifiable. Hence a submodule of a strongly associate, respectively, présimplifiable, module is again strongly associate, respectively, présimplifiable. Unlike the ring case, a direct sum or direct product of R-modules is présimplifiable if and only if each direct summand (factor) is, but a direct sum or product of strongly associate modules need not be strongly associate (see Theorem 15 for an example).

Let R be a commutative ring and I a proper ideal of R. Then R/I is both a ring and an R-module. We next show the relationship between R/I being strongly associate as a ring and as an R-module.

Theorem 12. *Let R be a commutative ring and I a proper ideal of R. Put $\bar{R} = R/I$. Then the following two conditions are equivalent.*

(1) *$\bar{R}$ is a strongly associate R-module.*

(2) *$\bar{R}$ is a strongly associate ring and the natural map $U(R) \to U(\bar{R})$ is surjective.*

Proof. (1) $\Rightarrow$ (2). Let $a, b \in \bar{R}$ with $\bar{R}a = \bar{R}b$. Then $Ra = Rb$ so there is a $u \in U(R)$ with $ua = b$. Then $\bar{u} \in U(\bar{R})$ and $\bar{u}a = b$. Thus $\bar{R}$ is strongly associate as a ring. Let $v \in U(\bar{R})$. So $R\bar{1} = Rv$. Hence there exists $u \in U(R)$ with $u\bar{1} = v$. Then $\bar{u} = v$ so $U(R) \to U(\bar{R})$ is surjective.

(2) $\Rightarrow$ (1). Let $a, b \in \bar{R}$ with $Ra = Rb$. Then $\bar{R}a = \bar{R}b$ so there is a $v \in U(\bar{R})$ with $va = b$. Choose $u \in U(R)$ with $\bar{u} = v$. Then $ua = \bar{u}a = va = b$. So $\bar{R}$ is strongly associate as an R-module. $\square$

The implication (1) $\Rightarrow$ (2) of the previous theorem remains true if "strongly associate" is replaced by "présimplifiable." However, the implication (2) $\Rightarrow$ (1) does not remain valid if "strongly associate" is replaced by "présimplifiable." For example, if we take $R = \mathbf{Z}$ and $I = (2)$, then certainly $\mathbf{Z}/(2)$ is présimplifiable as a ring and $U(\mathbf{Z}) \to U(\mathbf{Z}/(2))$ is surjective, but $\mathbf{Z}/(2)$ is not présimplifiable as a $\mathbf{Z}$-module (see Theorem 15).

Corollary 13. *Let R be a commutative ring. Then every R-module is strongly associate if and only if for each proper ideal I of R, R/I is a strongly associate ring and the natural map $U(R) \to U(R/I)$ is surjective. Hence if R is semi-quasilocal, every R-module is strongly associate.*

Proof. The first statement follows immediately from the previous theorem. Suppose that R is semi-quasilocal. Let I be a proper ideal of R. Then R/I is again semi-quasilocal and hence is strongly associate as a ring by Theorem 5. Also, it is well known that for R semi-quasilocal the natural map $U(R) \to U(R/I)$ is surjective. Thus by the previous theorem, the R-module R/I is strongly associate. Since every cyclic R-module is strongly associate, every R-module is strongly associate.

However, the converse of the second part of the previous corollary is not true, i.e., if R is a commutative ring with every R-module strongly associate, R need not be semi-quasilocal. As an example, let R be any Boolean ring. For I a proper ideal of R, R/I is a Boolean ring and hence is strongly associate by Theorem 11 and certainly $U(R) \to U(R/I)$ is surjective since the only unit of a Boolean ring is the identity. So every R-module is strongly associate, but a Boolean ring need not be semi-quasilocal.

Theorem 14. *Let R be a commutative ring and M an R-module.*

(1) *If $R(M)$ is strongly associate, then R and M are strongly associate.*

(2) *$R(M)$ is présimplifiable if and only if R and M are présimplifiable.*

(3) *Suppose that R is présimplifiable. Then $R(M)$ is strongly associate if and only if M is strongly associate.*

Proof. (1) Suppose that $R(M)$ is strongly associate. Let $r_1, r_2 \in R$ with $r_1 \sim r_2$. Then $(r_1, 0) \sim (r_2, 0)$ in $R(M)$. So there is a unit $(u, m) \in R(M)$ with $(r_2, 0) = (u, m)(r_1, 0)$. Since u is a unit of R, $r_2 = ur_1$, so $r_1 \approx r_2$. Hence R is strongly associate. The proof that M is strongly associate is similar.

(2) Anderson and Valdes-Leon [**6**].

(3) ($\Rightarrow$) This follows from (1). ($\Leftarrow$) If $(0, m_1) \sim (0, m_2)$, then $m_1 \sim m_2$ and so $m_2 = um_1$ where $u \in U(R)$. Then $(u, 0) \in U(R, M)$ and $(0, m_2) = (u, 0)(0, m_1)$, so $(0, m_1) \approx (0, m_2)$. (This part did not require R to be présimplifiable.) Next suppose that $(a, m_1) \sim (b, m_2)$ where a and b are nonzero. Then $a \sim b$, so $a \cong b$. Suppose that $(b, m_2) = (c, n)(a, m_1)$. Then $b = ca$ so $c \in U(R)$. But then $(c, n) \in U(R(M))$. Hence $(a, m_1) \cong (b, m_2)$ and so $(a, m_1) \approx (b, m_2)$. $\square$

We next determine when an abelian group is strongly associate or présimplifiable.

Theorem 15. *Let G be an abelian group.*

(1) *G is présimplifiable $\Leftrightarrow$ G is torsion-free.*

(2) *G is strongly associate $\Leftrightarrow$ $G = F \oplus T$ where F is torsion-free and T is torsion with either $4T = 0$ or $6T = 0$.*

Proof. (1) For any integral domain a torsion-free module is présimplifiable. Conversely, suppose that G is a présimplifiable abelian group. Suppose that $0 \neq a \in G$ has finite order n. Then $a = (n+1)a$, so $n+1$ is a unit of $\mathbf{Z}$, a contradiction. Hence G is torsion-free. Alternatively, observe that $Z(G) \subseteq J(\mathbf{Z}) = 0$.

(2) Note that a nonzero cyclic group $\mathbf{Z}_n$ is strongly associate $\Leftrightarrow$ $n = 2$, 3, 4, or 6. For suppose that $\mathbf{Z}_n$ is strongly associate. Then for $1 \leq l \leq n-1$ with $(n, l) = 1$, $\bar{1} \sim \bar{l}$. Hence $\bar{1} \approx \bar{l}$, and so $\bar{l} = \pm\bar{1}$. Hence $\phi(n) \leq 2$ and thus $n = 2$, 3, 4, or 6. Conversely, it is easily checked that $\mathbf{Z}_n$ is strongly associate for $n = 2, 3, 4$ and 6.

Suppose that G is strongly associate. Then the torsion subgroup T of G is also strongly associate. For $0 \neq g \in T$, $\langle g \rangle$ is strongly associate and hence has order 2, 3, 4, or 6. Moreover, G can not have elements of order both 3 and 4 for then G has an element a of order 12 and $\langle a \rangle$ is not strongly associate. Hence either $4T = 0$ or $6T = 0$. Since T is of bounded order, it is a direct summand of G. Conversely, suppose that $G = F \oplus T$ where F is torsion-free and $4T = 0$ or $6T = 0$. It suffices to show that each nonzero cyclic subgroup $\langle a \rangle$ of G is strongly associate. If a has infinite order, this is clear. So assume that a has finite order. Then $4a = 0$ or $6a = 0$, so $\langle a \rangle \cong \mathbf{Z}_n$ for $n = 2, 3, 4$ or 6. By the first paragraph of (2), $\langle a \rangle$ is strongly associate. □

Corollary 16. *Let G be an abelian group and let $\mathbf{Z}(G)$ be the idealization of $\mathbf{Z}$ and G.*

(1) $\mathbf{Z}(G)$ *is présimplifiable* $\Leftrightarrow$ G *is torsion-free.*

(2) $\mathbf{Z}(G)$ *is strongly associate* $\Leftrightarrow$ $G = F \oplus T$ *where F is torsion-free and* $4T = 0$ *or* $6T = 0$.

Hence $\mathbf{Z}(\mathbf{Z}_5)$ is not strongly associate. This example is also given in Allenby [**3**]. We next show that every ideal of $\mathbf{Z}(\mathbf{Z}_5)$ can be generated by two elements.

Lemma 17. *Suppose that p is prime. Then every ideal of the idealization $\mathbf{Z}(\mathbf{Z}_p)$ can be generated by two elements.*

Proof. Let I be a nonzero proper ideal of $\mathbf{Z}(\mathbf{Z}_p)$. If some $(0,0) \neq (0,a) \in I$, then $(0,\bar{1}) \in I$, so $I/((0,\bar{1}))$ is principal. So assume no $(0,0) \neq (0,a) \in I$. Now $(n,0) \in I$ for some $n > 0$. Let n_1 be the least positive integer with $(n_1,0) \in I$. Then $(n,0) \in I \Rightarrow (n,0) \in ((n_1,0))$. If $I = ((n_1,0))$, we are done. So assume that some $(n,\bar{a}) \in I$ where $n \neq 0$ and $a \in \mathbf{Z}$ with $a \neq 0$. We can choose n_2 minimal with $(n_2,\bar{1}) \in I$. Then $I = ((n_1,0),(n_2,\bar{1}))$. For if $(n,\bar{a}) \in I$, $(n - an_2, 0) = (n,\bar{a}) - a(n_2,\bar{1}) \in ((n_1,0))$ and hence $(n,\bar{a}) \in ((n_1,0),(n_2,\bar{1}))$. □

We have observed that a principal ideal ring is strongly associate. However, we have the following example.

Example 18. Let $p \geq 5$ be prime. Then every ideal of the idealization $\mathbf{Z}(\mathbf{Z}_p)$ can be generated by two elements, but $\mathbf{Z}(\mathbf{Z}_p)$ is not strongly associate.

Likewise, it is not hard to determine which $K[X]$-modules, K a field, are présimplifiable or strongly associate. Since $J(K[X]) = 0$, a $K[X]$-module is présimplifiable if and only if it is torsion-free. We leave it for the reader to verify that for $f(X) \in K[X]$, the cyclic $K[X]$-module $K[X]/(f(X))$ is strongly associate if and only if $\deg f(X) \leq 1$. Thus a $K[X]$-module M is strongly associate if and only if either M is torsion-free or there is a fixed linear polynomial $f(X) \in K[X]$ with each nonzero torsion cyclic submodule of M isomorphic to $K[X]/(f(X))$, or equivalently, $M = F \oplus T$ where F is torsion-free (possibly 0) and T is either 0 or is isomorphic to a direct sum of copies of $K[X]/(f(X))$ for some fixed linear polynomial $f(X) \in K[X]$.

In Spellman et al. [**16**] the question is raised whether R being strongly associate implies that $R[X]$ is strongly associate. (Of course, $R[X]$ strongly associate implies that R is strongly associate.) Our next example, [**5**, Example 6.1], shows that this need not be the case.

Example 19. Let $R = \mathbf{Z}_{(2)}(\mathbf{Z}_4)$, the idealization of $\mathbf{Z}_{(2)}$ and $\mathbf{Z}_4$, so R is a one-dimensional local ring. Hence R is présimplifiable and thus strongly associate. However, $R[X]$ is not strongly associate. For let $a = (0, \bar{1}) \in R$ and $f = (1, 0) + (2, 0)X \in R[X]$. Then $a \sim af$, but $a \not\approx af$.

We have seen that a subring of a strongly associate ring need not be strongly associate. We next show that an overring of a strongly associate ring need not be strongly associate.

Example 20. (A local ring with a regular ring of quotients that is not strongly associate.) Let (A, M) be a two-dimensional regular local ring, and let $0 \neq f \in M^2$ be a principal prime (e.g., $A = K[[X, Y]]$, K a field, and $f = X^2 + Y^3$). Then A_f is a non-Euclidean PID [**4**]. Hence there is a proper ideal I of A_f with $U(A_f) \to U(A_f/I)$ not surjective. (For it is easily proved that if $U(B) \to U(B/J)$ is surjective for each proper ideal J of a PID B, then B is a Euclidean domain with (smallest) algorithm $\theta : B \to \mathbf{N}$ given by $\theta(0) = 0$, $\theta(a) = 1$ for $u \in U(B)$, and $\theta(p_1 \cdots p_n) = n + 1$ where $p_1, \dots, p_n$ are principal primes of B.) Let $R = A(A_f/I)$, the idealization of A and A_f/I. Then R is a local ring and $f(= (f, 0))$ is a regular principal prime of R. Now $R_f = A_f(A_f/I)$. Since A_f/I is not a strongly associate A_f-module (Theorem 12), R_f is not a strongly associate ring (Theorem 14).

Example 19 shows that a polynomial ring over a strongly associate ring need not be strongly associate. This raises the following question.

Question 21. *Let R be a strongly associate ring. Is the power series ring $R[[X]]$ strongly associate?*

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