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SHARPENING HÖLDER'S AND POPOVICIU'S INEQUALITIES VIA FUNCTIONALS

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ABSTRACT. We prove some inequalities involving positive isotonic linear functionals which generalize Hölder's inequality and its reverse version. We also sharpen Jensen's inequalities for positive isotonic linear functionals.

1. Introduction. In the articles [**1**] and [**2**] sharpenings of the integral versions of Hölder's and Jensen's inequalities were obtained. Here we improve these results using a positive isotonic functional leading to some new generalizations of Hölder's and Popoviciu's inequalities. The new results sharpen Hölder's and Popoviciu's inequalities and their reversed versions both in discrete and integral forms.

Let E be a nonempty set and L be a linear class of real-valued functions $f : E \to R$ having the properties:

L1. $f, g \in L \Rightarrow (af + bg) \in L$ for all $a, b \in R$;

L2. $1 \in L$, i.e., if $f(t) = 1$ for all $t \in E$, then $f \in L$.

Let A be a positive isotonic linear functional on L. That is, we assume that

A1. $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L$, $a, b \in R$ (linearity);

A2. $f \in L$, $f(t) \geq 0$ on $E \Rightarrow A(f) \geq 0$ (positive isotonic).

Functional versions of well-known inequalities and related results could be found in [**10**]. Here, we mention results related to Jensen's inequality.

Theorem A [**10**, p. 112] (Jensen's inequality). *Let L satisfy conditions* L1, L2 *and A satisfy conditions* A1 *and* A2. *Suppose that $k \in L$*

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with $k \geq 0$ on E, $A(k) > 0$, and that ϕ is a continuous convex function on an interval $I \subseteq R$. Then for an arbitrary function $g : E \to I$ such that $kg \in L$, $k\phi(g) \in L$ and $A(kg)/A(k) \in I$, we have

$$\phi\left(\frac{A(kg)}{A(k)}\right) \leq \frac{A(k\phi(g))}{A(k)}. \tag{1.1}$$

If ϕ is a concave function, then the reverse inequality in (1.1) *holds.*

Also in [**10**, p. 124] and [**9**] the following related result is proved.

Theorem B. *Let E, L, A be defined as in Theorem* A *and assume that $p \in L$ with $p \geq 0$ on E and $0 < A(p) < u \in R$, $(ua - A(pg))/(u - A(p)) \in I$, $a \in I$, $pg \in L$ and $p\phi(g) \in L$. If ϕ is convex continuous on an interval I, then*

$$\phi\left(\frac{ua - A(pg)}{u - A(p)}\right) \geq \frac{u\phi(a) - A(p\phi(g))}{u - A(p)}. \tag{1.2}$$

If ϕ is a concave continuous function, then the reverse inequality in (1.2) *holds.*

2. Generalized Hölder's and Popoviciu's inequality. Here we present some inequalities involving isotonic linear functionals, and convex and concave functions, which generalize Hölder's and Popoviciu's inequalities. We extensively research the properties of the functions $G_{i,j}$ which are defined as follows:

Definition 1. Let f_i, $i = 1, 2, \dots, m-1$ be positive functions on $(0, \infty)$ and let $x_i > 0$, $i = 1, \dots, m$. For $r \leq s$, $r, s \in \{1, 2, \dots, m-1\}$, we denote

$$G_{r,s}(x_r, x_{r+1}, \dots, x_{s+1}) = x_r f_r\left(\frac{x_{r+1}}{x_r} f_{r+1}\left(\frac{x_{r+2}}{x_{r+1}} \cdots f_s\left(\frac{x_{s+1}}{x_s}\right)\right)\right)$$

and

$$G_{s+1,s}(x) = x.$$

If any of the $x_i = 0$, then we define that $G_{r,s}(x_r, x_{r+1}, \dots, x_{s+1}) = 0$.

Definition 2. Denote S to be a set of positive convex functions on $(0,\infty)$ and T a set of positive concave functions on $(0,\infty)$.

Theorem 1. *Let L satisfy conditions* L1, L2*, and A satisfy conditions* A1 *and* A2*. Let $a_i \in L$, $i = 1,\dots,m$ be positive functions on E, $A(a_i) > 0$, $i = 1,\dots,m$, and let $f_i \in S$, $i = 1,\dots,m-1$ be such that $f_1,\dots,f_{m-2}$ are increasing. Suppose that $G_{i,m-1}(a_i,\dots,a_m) \in L$.*

Then

$$A(G_{1,m-1}(a_1,\dots,a_m)) \geq G_{1,m-1}(A(a_1),\dots,A(a_m)). \tag{2.1}$$

If $f_i \in T$, $i = 1,\dots,m-1$, and $f_1,\dots,f_{m-2}$ are increasing, then the reverse of (2.1) *holds.*

Proof. Using $m-1$ times Jensen's inequality (1.1) for the convex functions $f_i > 0$, $i = 1,\dots,m-1$, as $f_1,\dots,f_{m-2}$ are positive increasing and as A is positive isotonic, we obtain

$$\begin{aligned}
&A(G_{1,m-1}(a_1,\dots,a_m))\\
&\quad = A\left(a_1 f_1\left(\frac{G_{2,m-1}(a_2,\dots,a_m)}{a_1}\right)\right)\\
&\quad \geq A(a_1) f_1\left(\frac{A(G_{2,m-1}(a_2,\dots,a_m))}{A(a_1)}\right)\\
&\quad \geq A(a_1) f_1\left(\frac{A(a_2)}{A(a_1)} f_2\left(\frac{A(G_{3,m-1}(a_3,\dots,a_m))}{A(a_2)}\right)\right)\\
&\quad \geq \cdots\\
&\quad \geq A(a_1) f_1\left(\frac{A(a_2)}{A(a_1)} f_2\left(\frac{A(a_3)}{A(a_2)} \cdots f_{m-1}\left(\frac{A(a_m)}{A(a_{m-1})}\right)\right)\right)\\
&\quad = G_{1,m-1}(A(a_1),\dots,A(a_m)).
\end{aligned}$$

The proof of the second statement is similar and thus omitted. □

The following are some examples of positive isotonic linear functionals [**10**, p. 49], [**6**, p. 523], [**15**, p. 452]:

(a) The range of x is $\{1,2,\dots,m\}$ or $\{1,2,\dots\}$, so that $f(x)$ is a (finite or infinite) sequence $\{a_1,a_2,\dots\}$, $A(f) = \sum c_i a_i / \sum c_i$ where $c_i \geq 0$, and $0 < \sum c_i < \infty$.

(b) E is the interval $(0,1)$, L is the class of all bounded functions on E, $A(f)$ is the Banach integral of f over $(0,1)$.

(c) E is a set of all real numbers, L the class of all uniformly almost periodic functions, $A(f)$ is the mean value of f.

(d) More generally, E is any group, L is the class of all functions almost periodic on E, and $A(f)$ is von Neumann's mean value of f.

(e) $L = \{g : [0,1] \to R$ such that $\lim_{x\to 1} g(x)$ is finite$\}$, $A : L \to R$, $A(g) = \lim_{x\to 1} g(x)$.

(f) Let (Ω, Σ, μ) be a space with positive finite measure. Let $L = L_1(\Omega, \Sigma, \mu)$. For $f \in L_1$ define

$$A(f) = \frac{1}{\mu(\Omega)} \int_\Omega f(x)\, d\mu(x).$$

Remark 1. The domain of definition and the range of the functions f_i can be changed, i.e., Theorem 1 still holds if f_i are defined on different intervals $I_i \subseteq R$ and if the positivity of the functions f_i, $i = 1, \dots, m-1$ is omitted. In that case we must suppose a number of additional assumptions as follows:

$$(*) \qquad f_i \quad \text{are continuous on} \quad I_i, \qquad i = 1, \dots m-1,$$

$$\text{range}\left(\frac{1}{a_i} G_{i+1,m-1}(a_{i+1}, \dots, a_m)\right) \subseteq I_i, \quad i = 1, \dots m-1,$$

$$\text{range}\left(\frac{A(a_{i+1})}{A(a_i)} f_{i+1}\right) \subseteq I_i, \quad i = 1, 2, \dots, m-2,$$

$$\frac{A(G_{i+1,m-1}(a_{i+1}, \cdots, a_m))}{A(a_i)} \in I_i, \quad i = 1, \dots, m-1.$$

For simplicity, without loss of generality, all our following results will be stated for positive functions f_i, $i = 1, \dots, m-1$, defined on $(0, \infty)$, but their domain and range can be changed according to this remark.

Remark 2. Let $A_1, \dots, A_m$ be functionals satisfying A1 and A2 and

$$A_1(f) \geq A_2(f) \geq \cdots \geq A_m(f), \qquad \text{for all } f \in L.$$

If a_i, $i = 1, \dots, m$ and f_i, $i = 1, \dots, m-1$ satisfy the assumptions of Theorem 1 and if f_{m-1} is increasing, then the following holds

$$A_1(G_{1,m-1}(a_1, \dots, a_m)) \geq G_{1,m-1}(A_1(a_1), \dots, A_m(a_m)).$$

Remark 3. The conditions of convexity and monotonocity of the functions $f_1, \dots, f_{m-1}$ can be changed. If $f_1, \dots, f_{m-1}$ satisfy $(*)$ and if $f_1, \dots, f_{i-1}$ are convex increasing functions, f_i is decreasing convex, $f_{i+1}, \dots, f_{m-2}$ are increasing concave functions and f_{m-1} is concave, then inequality (2.1) holds. Similarly, if $f_1, \dots, f_{m-1}$ satisfy $(*)$ and if $f_1, \dots, f_{i-1}$ are concave increasing functions, f_i is decreasing concave, $f_{i+1}, \dots, f_{m-2}$ are increasing convex functions and f_{m-1} is convex, then the reverse of (2.1) holds.

Remark 4. If $E = \{1, 2, \dots, n\}$ and $A(f) = \sum_{i=1}^n f(k) = \sum_{i=1}^n f_k$, then inequality (2.1) transforms to inequality (2.1) from [**3**].

The next theorem is a consequence of Theorem 1 and it is a functional version of Hölder's inequality and its reverse.

Theorem 2. *Let A and a_i, $i = 1, \dots, m$ satisfy the assumptions of Theorem* 1. *If $p_i > 0$, $i = 1, \dots, m$ are such that $\sum_{i=1}^m 1/p_i = 1$, and $\prod_{i=1}^m a_i^{1/p_i} \in L$, then*

$$A\left(\prod_{i=1}^m a_i^{1/p_i}\right) \leq \prod_{i=1}^m A(a_i)^{1/p_i}. \tag{2.2}$$

If $p_1 > 0$ and $p_i < 0$, $i = 2, \dots, m$ are such that $\sum_{i=1}^m 1/p_i = 1$, then a reverse of (2.2) *holds.*

Proof. Let p_j, $j = 1, \dots, m$, $m \geq 2$, be positive numbers that satisfy $\sum_{j=1}^m 1/p_j = 1$. Let us define the numbers q_j, $j = 1, \dots, m-1$, as

$$\frac{1}{q_1} = 1 - \frac{1}{p_1}, \quad \frac{1}{q_j} = 1 - \frac{q_1 q_2 \cdots q_{j-1}}{p_j}, \quad j = 2, \dots, m-1. \tag{2.3}$$

It is easy to check, by induction, that

$$\frac{1}{q_1 q_2 \cdots q_j} = 1 - \frac{1}{p_1} - \frac{1}{p_2} - \cdots - \frac{1}{p_j},$$

and therefore, $q_1, q_2, \ldots, q_{m-1}$ are positive real numbers.

Let us define the functions f_i as $f_i(x) = x^{1/q_i}$, $i = 1, \ldots, m-1$. Then f_i, $i = 1, \ldots, m-1$, are concave increasing functions and the reverse of (2.1) holds, i.e.,

$$A\left(a_1\left(\frac{a_2}{a_1}\left(\frac{a_3}{a_2}\cdots\left(\frac{a_m}{a_{m-1}}\right)^{1/q_{m-1}}\right)^{1/q_2}\right)^{1/q_1}\right) \\ \leq A(a_1)\left(\frac{A(a_2)}{A(a_1)}\left(\frac{A(a_3)}{A(a_2)}\cdots\left(\frac{A(a_m)}{A(a_{m-1})}\right)^{1/q_{m-1}}\right)^{1/q_2}\right)^{1/q_1}$$

which, after a simple transformation, becomes (2.2). If $p_1 > 0$ and $p_i < 0$, $i = 2, \ldots, m$ such that $\sum_{i=1}^{m} 1/p_i = 1$, then f_1 is a decreasing convex function and the other functions are increasing concave and the reverse of (2.2) holds. □

Remark 5. If $A(f) = \int_E f\, d\mu$, where μ is a positive measure on E, then (2.2) is the well-known integral Hölder's inequality. The second part of Theorem 2 gives us an integral version of a result which is given in [**13**] and [**14**] by Pečarić and Vasić, see also [**8**, p. 102], and in [**12**] by Sun.

The following theorem is a generalization of Popoviciu's inequality [**11**].

Theorem 3. *Let E, L, A be defined as in Theorem 1. Let c_i, $i = 1, 2, \ldots, m$, be positive real numbers, a_i, $i = 1, 2, \ldots, m$ be positive functions on E, $A(a_i) > 0$, $f_i \in S$, $i = 1, \ldots, m-1$ and $f_1, \ldots, f_{m-2}$ are increasing. Furthermore, suppose that $c_{i-1} - A(a_{i-1}) > 0$, $G_{i,m-1}(a_i, \ldots, a_m) \in L$, $i = 1, \ldots m$, and $G_{i+1,m-1}(c_{i+1}, \ldots, c_m) - A(G_{i+1,m-1}(a_{i+1}, \ldots, a_m)) > 0$, $i = 1, \ldots, m-1$. Then*

$$\text{(2.4)} \quad G_{1,m-1}(c_1 - A(a_1), \ldots, c_m - A(a_m)) \\ \geq G_{1,m-1}(c_1, \ldots, c_m) - A(G_{1,m-1}(a_1, \ldots, a_m)).$$

If $f_i \in T$, $i = 1, \dots, m-1$, then a reverse of (2.4) *holds.*

Proof. The proof is similar to the previous one, only instead of Theorem A, we use Theorem B. □

Remark 6. If the functions f_i are defined as in the proof of Theorem 2 and satisfy the assumptions of Theorem 3, then if $p_i > 0$, $i = 1, \dots, m$, $\sum_{i=1}^{m} 1/p_i = 1$, the Popoviciu inequality in a functional form holds, i.e.,

$$c_1^{1/p_1} c_2^{1/p_2} \cdots c_m^{1/p_m} - A(a_1^{1/p_1} a_2^{1/p_2} \cdots a_m^{1/p_m}) \geq \prod_{i=1}^{m} (c_i - A(a_i))^{1/p_i} \tag{2.5}$$

and we can easily derive its integral version

$$c_1^{1/p_1} c_2^{1/p_2} \cdots c_m^{1/p_m} - \int_E a_1^{1/p_1} a_2^{1/p_2} \cdots a_m^{1/p_m} \, d\mu \geq \prod_{i=1}^{m} \left(c_i - \int_E a_i \, d\mu \right)^{1/p_i}. \tag{2.6}$$

The functional form (2.5) of the Popoviciu inequality was derived by Pečarić in [**9**]. The discrete version of inequality (2.6) was proved by Losonczi and Páles in [**5**].

But, if $p_1 > 0$ and $p_i < 0$, $i = 2, \dots, m$ with $\sum_{i=1}^{m} 1/p_i = 1$, then we get the reverse inequalities of (2.5) and (2.6). These are new inequalities of Popoviciu's type.

Theorem 4. *If $f_1, \dots, f_{m-2}$ are positive convex* (*concave*) *increasing functions and f_{m-1} is a positive convex* (*concave*) *function on* $(0, \infty)$, *then $G_{1,m-1}$ is a positively homogeneous convex* (*concave*) *function.*

Proof. Suppose that $f_1, \dots, f_{m-1}$ are convex and let α and β be nonnegative real numbers, $\alpha + \beta = 1$, $a_i = (a_{i,1}, a_{i,2})$, $a_{i,1}, a_{i,2} \geq 0$, $i = 1, 2, \dots, m$.

Define the functional A on $L = (0, \infty)^2$ as

$$A(f) = \alpha f_1 + \beta f_2, \quad \text{if } f = (f_1, f_2).$$

Then using Theorem 1 we have

$$\begin{aligned}G_{1,m-1}(&\alpha a_{1,1}+\beta a_{1,2},\dots,\alpha a_{m,1}+\beta a_{m,2})\\ &= G_{1,m-1}(A(a_1),\dots,A(a_m)) \le A(G_{1,m-1}(a_1,\dots,a_m))\\ &= \alpha G_{1,m-1}(a_1,\dots,a_m)_1+\beta G_{1,m-1}(a_1,\dots,a_m)_2\\ &= \alpha G_{1,m-1}(a_{1,1},\dots,a_{m,1})+\beta G_{1,m-1}(a_{1,1},\dots,a_{m,2}),\end{aligned}$$

i.e., $G_{1,m-1}$ is a convex function.

The proof of the concave case is similar. Finally, the homogeneity statement is obvious according to Definition 1, and thus the proof is complete. □

3. Sharpening inequalities via functionals.

Theorem 5. *Let E and F be functionals on L satisfying* A1, *and let D be a positive isotonic linear functional such that $F-E=D$.*

If $f_i \in S$, $i=1,\dots,m-1$, $f_1,\dots,f_{m-2}$ are increasing functions and $a_i>0$, $i=1,\dots,m$, then for $s=1,\dots,m$

$$\begin{aligned}&D(G_{1,m-1}(a_1,\dots,a_m))\\ &\quad\ge G_{1,s-1}(F(a_1),\dots,F(a_{s-1}),F(G_{s,m-1}(a_s,\dots,a_m)))\\ &\qquad - G_{1,s-1}(E(a_1),\dots,E(a_{s-1}),E(G_{s,m-1}(a_s,\dots,a_m)))\end{aligned} \tag{3.1}$$

and the function

$$\varphi : s \mapsto G_{1,s-1}(D(a_1),\dots,D(a_{s-1}),D(G_{s,m-1}(a_s,\dots,a_m)))$$

is decreasing.

If $f_i \in T$, $i=1,\dots,m-1$ and $f_1,\dots,f_{m-2}$ are increasing, then a reversed inequality holds in (3.1) *and the function φ is increasing.* (*We assume that all above-mentioned terms are well-defined.*)

Proof. Let us prove (3.1). First, we consider $2\le s\le m-1$. Denote

$$z = G_{s,m-1}(a_s,\dots,a_m).$$

Using Theorem 1 we get

$$\begin{aligned}D(G_{1,m-1}(a_1,\dots,a_m)) &= D(G_{1,s-1}(a_1,\dots,a_{s-1},z))\\ &\ge G_{1,s-1}(D(a_1),\dots,D(a_{s-1}),D(z)).\end{aligned} \tag{3.2}$$

As f_i, $i = 1, \dots, m-2$, are increasing using discrete Jensen's inequality several times, we get

(3.3)
$$\begin{aligned}
&G_{1,s-1}(D(a_1), \dots, D(a_{s-1}), D(z)) + G_{1,s-1}(E(a_1), \dots, E(a_{s-1}), E(z))\\
&= D(a_1) f_1\left(\frac{G_{2,s-1}(D(a_2), \dots, D(z))}{D(a_1)}\right)\\
&\quad + E(a_1) f_1\left(\frac{G_{2,s-1}(E(a_2), \dots, E(z))}{E(a_1)}\right)\\
&\geq F(a_1) f_1\left(\frac{G_{2,s-1}(D(a_2), \dots, D(z)) + G_{2,s-1}(E(a_2), \dots, E(z))}{F(a_1)}\right)\\
&\geq F(a_1) f_1\left(\frac{F(a_2)}{F(a_1)} f_2\left(\frac{G_{3,s-1}(D(a_3), \dots, D(z)) + G_{3,s-1}(E(a_3), \dots, E(z))}{F(a_2)}\right)\right)\\
&\geq \cdots \geq G_{1,s-1}(F(a_1), \dots, F(a_{s-1}), F(z)).
\end{aligned}$$

From (3.2) and (3.3) we obtain (3.1), for $2 \leq s \leq m-1$.

The result follows also easily for $s = m$. For $s = 1$ we get in (3.1) equalities by using Definition 1. In this case $G_{1,s-1}(F(a_1), \dots, F(a_{s-1}), F(G_{s,m-1}(a_s, \dots, a_m)))$ becomes $G_{1,m-1}(a_1, \dots, a_m)$ and $G_{1,s-1}(E(a_1), \dots, E(a_{s-1}), E(G_{s,m-1}(a_s, \dots, a_m)))$ becomes $G_{1,m-1}(a_1, \dots, a_m)$ too.

To prove the second statement, denote

$$z = G_{s+1,m-1}(a_{s+1}, \dots, a_m).$$

f_s is a convex function; therefore, applying Jensen's inequality we get

$$D(G_{s,m-1}(a_s, \dots, a_m)) = D\left(a_s f_s\left(\frac{z}{a_s}\right)\right) \geq D(a_s) f_s\left(\frac{D(z)}{D(a_s)}\right).$$

Each of f_i, $i = 1, \dots, m-2$ is increasing; therefore, we have the following

$$\begin{aligned}
(3.4)\quad &G_{1,s-1}(D(a_1), \dots, D(a_{s-1}), D(G_{s,m-1}(a_s, \dots, a_m)))\\
&= D(a_1) f_1\left(\frac{D(a_2)}{D(a_1)} \cdots f_{s-1}\left(\frac{D(G_{s,m-1}(a_s, \dots, a_m))}{D(a_{s-1})}\right) \cdots\right)\\
&\geq D(a_1) f_1\left(\frac{D(a_2)}{D(a_1)} \cdots f_{s-1}\left(\frac{D(a_s)}{D(a_{s-1})} f_s\left(\frac{D(z)}{D(a_s)}\right)\right) \cdots\right)\\
&= G_{1,s}(D(a_1), \dots, D(a_s), D(z)).
\end{aligned}$$

The proof of the statement for the concave case is completely similar and thus omitted. ◻

Theorem 6. *Let P and Q be positive isotonic linear functionals. If $f_i \in S$, $i = 1, \dots, m-1$ and a_i, $i = 1, \dots, m$ satisfy the assumptions of Theorem* 5*, then*

$$\begin{aligned} G_{1,s-1}(R(a_1), \dots, R(a_{s-1}), R(G_{s,m-1}(a_s, \dots, a_m))) \\ \le P(G_{1,m-1}(a_1, \dots, a_m)) \\ + G_{1,s-1}(Q(a_1), \dots, Q(a_{s-1}), Q(G_{s,m-1}(a_s, \dots, a_m))) \\ \le R(G_{1,m-1}(a_1, \dots, a_m)), \end{aligned} \tag{3.5}$$

where $R = P + Q$. If $f_i \in T$, $i = 1, \dots, m-1$, then a reverse inequality of (3.5) *holds.*

Proof. Let $\underline{0}$ be a null-functional. Inserting in Theorem 5 $D = R - P$, $F = Q$ and $E = \underline{0}$, we have the second inequality in (3.5). Applying Theorem 5 with $D = P$, $F = R$ and $E = Q$ we get the first inequality of (3.5).

When $s = m$ and $f_i \in S$, $i = 1, \dots, m-1$, inequality (3.5) becomes

$$\begin{aligned} G_{1,m-1}(R(a_1), \dots, R(a_m)) &\le P(G_{1,m-1}(a_1, \dots, a_m)) \\ &\quad + G_{1,m-1}(Q(a_1), \dots, Q(a_m)) \\ &\le R(G_{1,m-1}(a_1, \dots, a_m)), \end{aligned} \tag{3.6}$$

which is a sharpening of inequality (2.1). If $f_i \in T$, $i = 1, \dots, m-1$ we get the reverse signs of inequalities in (3.6). ◻

In the next section we show how applications of Theorems 5 and 6 give some known and some new results.

4. Sharpening of Hölder's and Jensen's and its reverse inequalities. The next theorem is a result of an application of Theorem 6 to some special functions.

Theorem 7. *Let P and Q be positive isotonic linear functionals and $R = P + Q$. If $p_i > 0$, $i = 1, \dots, m$ are such that $\sum_{i=1}^m 1/p_i = 1$, $a_i \in L$, $i = 1, \dots, m$ are positive functions such that $\prod_{i=1}^m a_i^{1/p_i} \in L$, then*

$$R\left(\prod_{i=1}^m a_i^{1/p_i}\right) \le P\left(\prod_{i=1}^m a_i^{1/p_i}\right) + \prod_{i=1}^m Q(a_i)^{1/p_i} \le \prod_{i=1}^m R(a_i)^{1/p_i}. \tag{4.1}$$

If $p_1 > 0$ and $p_i < 0$, $i = 2, \dots, m$ are such that $\sum_{i=1}^m 1/p_i = 1$, then the reverse of (4.1) *holds.*

Proof. If we define the functions f_i as $f_i(x) = x^{1/q_i}$, $i = 1, \dots, m-1$, where q_i, $i = 1, \dots, m-1$ are defined as in (2.3), then applying (3.6) we get (4.1). $\square$

Corollary 1. *If $p_i > 0$, $i = 1, \dots, m$ are such that $\sum_{i=1}^m 1/p_i = 1$, and if a_i, $i = 1, \dots, m$ and $\prod_{i=1}^m a_i^{1/p_i}$ are positive integrable functions, μ is a positive measure on $[a, b]$, $P(g) = \int_a^c g\, d\mu$, $Q(g) = \int_c^b g\, d\mu$, $R(g) = \int_a^b g\, d\mu$, $a < c < b$, then*

$$\begin{aligned}\int_a^b \prod_{i=1}^m a_i^{1/p_i}\, d\mu &\le \int_a^c \prod_{i=1}^m a_i^{1/p_i}\, d\mu + \prod_{i=1}^m \left(\int_c^b a_i\, d\mu\right)^{1/p_i} \\ &\le \prod_{i=1}^m \left(\int_a^b a_i\, d\mu\right)^{1/p_i}.\end{aligned} \tag{4.2}$$

If $p_1 > 0$ and $p_i < 0$, $i = 2, \dots m$ are such that $\sum_{i=1}^m 1/p_i = 1$, then a reverse of (4.2) *holds.*

Inequality (4.2) was obtained in [**1**], and it is a sharpening of Hölder's inequality. Here, we get also a similar result for reversed Hölder's inequality.

Corollary 2. *Let (Ω, Σ, μ) be a space with positive finite measure. Let $L = L_1(\Omega, \Sigma, \mu)$. If $\Omega_1, \Omega_2 \subset \Omega$, $\mu(\Omega_1), \mu(\Omega_2) \in (0, \infty)$, then for $f \in L$ define*

$$P(f) = \frac{1}{\mu(\Omega_1)} \int_{\Omega_1} f(x)\, d\mu(x), \quad Q(f) = \frac{1}{\mu(\Omega_2)} \int_{\Omega_2} f(x)\, d\mu(x)$$

and $R(f) = P(f) + Q(f)$.

From the second part of inequality (4.1), the following new inequality is obtained:

$$\frac{1}{\mu(\Omega_1)} \int_{\Omega_1} \prod_{i=1}^{m} a_i(x)^{1/p_i}\, d\mu(x) + \frac{1}{\mu(\Omega_2)} \prod_{i=1}^{m} \Big(\int_{\Omega_2} a_i(x)\, d\mu(x) \Big)^{1/p_i}$$
$$\leq \prod_{i=1}^{m} \Big(\frac{1}{\mu(\Omega_1)} \int_{\Omega_1} a_i(x)\, d\mu(x) + \frac{1}{\mu(\Omega_2)} \int_{\Omega_2} a_i(x)\, d\mu(x) \Big)^{1/p_i}$$

Corollary 3. *Let* $E = \{1, 2\}$ *and functionals* P *and* Q *be defined by*

$$P(f) = f(1), \quad Q(f) = f(2).$$

If $p_i > 0$, $i = 1, \dots m$ *are such that* $\sum 1/p_i = 1$ *and* $a_i : E \to R$ *are positive functions, then inequality* (4.1) *gives*

$$\prod_{i=1}^{m} x_i^{1/p_i} + \prod_{i=1}^{m} y_i^{1/p_i} \leq \prod_{i=1}^{m} (x_i + y_i)^{1/p_i} \tag{4.3}$$

where $x_i = a_i(1)$, $y_i = a_i(2)$.

Inequality (4.3) is a generalization of the following inequality given in [**7**].

Let $a_k > 0, k = 1, \dots, n$. Then

$$\prod_{i=1}^{n} (1 + a_k) \geq \Big(1 + \sqrt[n]{\prod_{i=1}^{n} a_k} \Big)^n.$$

On the other hand, this inequality is a special case of a result given in [**4**, pp. 31–32], (see also [**8**, p. 109]). Namely, replacing in (4.3)

$$a^{(1)} = (x_1, \dots, x_m), \quad a^{(2)} = (y_1, \dots, y_m),$$
$$a_{(i)} = x_i + y_i, \quad i = 1, \dots m,$$
$$p = (p_1, \dots, p_m), \quad w = \Big(\frac{1}{2}, \frac{1}{2} \Big)$$

we obtain

$$M_2^{[1]}(M_m^{[0]}(a^{(j)};p);w) \leq M_m^{[0]}(M_2^{[1]}(a_{(i)};w);p)$$

where $M_n^{[r]}(x;p)$ is the r-ordered weighted mean of n-tuple $x = (x_1, \dots, x_n)$ with weights $p = (p_1, \dots, p_n)$. The general case, which is given in [**4**], deals with means of order r and s.

The following result is a sharpening of Jensen's inequality, and it is a simple consequence of Theorem 6.

Theorem 8. *Let P and Q be positive isotonic linear functionals and $R = P + Q$. Let $k \in L$ with $k \geq 0$, $Q(k) > 0$, $R(k) > 0$, $f \in S$. For any function $g : E \to (0, \infty)$ such that $kg \in L$, $kf(g) \in L$ we get*

$$R(k) f\left(\frac{R(kg)}{R(k)}\right) \leq P(kf(g)) + Q(k) f\left(\frac{Q(kg)}{Q(k)}\right) \leq R(kf(g)). \tag{4.4}$$

If $f \in T$, then a reverse inequality of (4.4) *holds.*

Proof. Setting $m = 2$, $a_1 = k$, $a_2 = kg$, $f_1 = f$ in inequality (3.6) we obtain (4.4). □

Remark 7. If we specify that $P(f) = \int_a^c f\, d\mu$ and $Q(f) = \int_c^b f\, d\mu$, $c \in (a, b)$, we get a sharpening of the integral version of Jensen's inequality [**2**] when μ is a positive measure on $[a, b]$.

In the next theorem we show a result related to Theorems 5 and 6 when the functionals D, E and F are integrals.

Let us define $A(d, s)$ as

$$\begin{aligned} A(d,s) = &\int_a^d G_{1,m-1}(a_1, \dots, a_m)\, d\mu \\ &+ G_{1,s-1}\left(\int_d^b a_1 d\mu, \dots, \int_d^b a_{s-1} d\mu, \int_d^b G_{s,m-1}(a_s, \dots, a_m) d\mu\right), \end{aligned}$$

where $d \in [a,b]$ and $s = 1, 2, \dots, m$.

When $s = 1$ the second addend is equal to $\int_d^b G_{1,m-1}(a_1, \dots, a_m)\, d\mu$.

If $d = b$ we define $A(b,s) = \int_a^b G_{1,m-1}(a_1, \dots, a_m)\, d\mu$.

Here we suppose that all the integrals are well defined.

Theorem 9. *If $f_i \in S$, $i = 1, \dots, m-1$, $f_1, \dots, f_{m-2}$ are increasing functions, $a_i > 0$, $i = 1, \dots, m$, and if $a \le c \le d \le b$, then*

$$A(a,s) \le A(c,s) \le A(d,s) \le A(b,s) \tag{4.5}$$

for all $s = 1, 2, \dots, m$.

For any $d \in [a,b]$ and $s = 1, 2, \dots, m-1$, the following

$$A(d,s) \ge A(d, s+1). \tag{4.6}$$

holds.

If $f_i \in S$, $i = 1, \dots, m-1$, are replaced by $f_i \in T$, $i = 1, \dots, m-1$, then the reverse inequalities of (4.5) *and* (4.6) *hold.*

Proof. Denoting $D(f) = \int_c^d f\, d\mu = \int_a^d f\, d\mu - \int_a^c f\, d\mu$, $E(f) = \int_d^b f\, d\mu$ and $F(f) = \int_c^b f\, d\mu$ and, applying Theorem 5, we get (4.5) and (4.6). $\square$

5. Sharpening of Popoviciu's and related inequalities.

Theorem 10. *Let E and F be functionals satisfying* A1 *and D be a positive isotonic linear functional on L such that $F - E = D$.*

a) *If c_i, a_i, $i = 1, \dots, m$, satisfy the assumptions of Theorem* 3, *and if $f_i \in S$, $i = 1, \dots, m-1$, and $f_1, \dots, f_{m-2}$ are increasing functions, then*

$$\begin{aligned} &-D(G_{1,m-1}(a_1, \dots, a_m)) \\ &\quad \le G_{1,s-1}(c_1 - F(a_1), \dots, c_{s-1} - F(a_{s-1}), \\ &\qquad\qquad G_{s,m-1}(c_s, \dots, c_m) - F(G_{s,m-1}(a_s, \dots, a_m))) \\ &\qquad - G_{1,s-1}(c_1 - E(a_1), \dots, c_{s-1} - E(a_{s-1}), \\ &\qquad\qquad G_{s,m-1}(c_s, \dots, c_m) - E(G_{s,m-1}(a_s, \dots, a_m))). \end{aligned} \tag{5.1}$$

b) *The function* $s \mapsto G_{1,m-1}(c_1, \dots, c_m) - G_{1,s-1}(D(a_1), \dots, D(a_{s-1}), D(G_{s,m-1}(a_s, \dots, a_m)))$ *is increasing.* (*We assume that all the above-mentioned terms are well-defined.*)

c) *If* P *and* Q *are isotonic positive functional with* $R = P + Q$*, then*

$$
\begin{aligned}
G_{1,m-1}(c_1, \dots, c_m) &- R(G_{1,m-1}(a_1, \dots, a_m)) \\
&\le G_{1,m-1}(c_1 - Q(a_1), \dots, c_m - Q(a_m)) \\
&\quad - P(G_{1,m-1}(a_1, \dots, a_m)) \\
&\le G_{1,m-1}(c_1 - R(a_1), \dots, c_m - R(a_m)).
\end{aligned} \tag{5.2}
$$

If $f_i \in T$*,* $i = 1, \dots, m-1$*, then the reverse inequalities of* (5.1) *and* (5.2) *hold and the above-defined function is decreasing.*

The proof is similar to the proof of Theorem 5 and is based on the application of Theorems B, 1, 3 and 6.

If in (5.2) we specify: $f_i(x) = x^{1/q_i}$ where q_i is defined as in (2.3), then for $p_i > 0$, $i = 1, \dots, m$, $\sum_{i=1}^m 1/p_i = 1$ we have

$$
\begin{aligned}
\prod_{i=1}^m c_i^{1/p_i} - R\biggl(\prod_{i=1}^m a_i^{1/p_i}\biggr) &\ge \prod_{i=1}^m (c_i - Q(a_i))^{1/p_i} - P\biggl(\prod_{i=1}^m a_i^{1/p_i}\biggr) \\
&\ge \prod_{i=1}^m (c_i - R(a_i))^{1/p_i},
\end{aligned} \tag{5.3}
$$

which is a sharpening of the functional version of Popoviciu's inequality. If $p_1 > 0$, $p_i < 0, i = 2, \dots, m$, $\sum_{i=1}^m 1/p_i = 1$, then we get a reverse of (5.3).

If we choose the functionals P and Q as follow:

$$
P(f) = \int_a^c f \, d\mu, \quad Q(f) = \int_c^b f \, d\mu
$$

where μ is a positive measure, $c \in (a, b)$ then we have the following theorem.

Theorem 11. *Let* $c_i > 0$ *and* a_i*,* $i = 1, 2, \dots, m$*, be positive* μ*-integrable functions such that*

$$
c_i - \int_a^b a_i \, d\mu > 0, \quad i = 1, \dots, m.
$$

If $p_i > 0$, $i = 1, \dots, m$, $\sum_{i=1}^m 1/p_i = 1$, *then*

$$\begin{aligned}\prod_{i=1}^m c_i^{1/p_i} - \int_a^b \prod_{i=1}^m a_i^{1/p_i}\, d\mu &\geq \prod_{i=1}^m \left(c_i - \int_c^b a_i\, d\mu\right)^{1/p_i} - \int_a^c \prod_{i=1}^m a_i^{1/p_i}\, d\mu \\ &\geq \prod_{i=1}^m \left(c_i - \int_a^b a_i\, d\mu\right)^{1/p_i}.\end{aligned}$$

If $p_1 > 0$, *and* $p_i < 0$, $i = 2, \dots, m$, $\sum_{i=1}^m 1/p_i = 1$, *then the reverse inequality holds.*

This is a sharpening of well-known integral Popoviciu's inequality and its reverse inequality with only one positive weight.

If we specify the functionals D, E and F as in the proof of Theorem 9, we have the following results.

Theorem 12. *Let* c_i, a_i, $i = 1, \dots, m$, *satisfy the assumptions of Theorem* 3, *let* $f_i \in S$, $i = 1, \dots, m-1$, $f_1, \dots, f_{m-2}$ *be increasing functions. Denote*

$$\begin{aligned}B(d,s) = &-\int_a^d G_{1,m-1}(a_1, \dots, a_m)\, d\mu \\ &+ G_{1,s-1}\biggl(c_1 - \int_d^b a_1 d\mu, \dots, c_{s-1} - \int_d^b a_{s-1} d\mu, \\ &\qquad G_{s,m-1}(c_s, \dots, c_m) - \int_d^b G_{s,m-1}(a_s, \dots, a_m)\, d\mu\biggr),\end{aligned}$$

when $a \leq c \leq d \leq b$. *Then*

$$B(a,s) \geq B(c,s) \geq B(d,s) \geq B(b,s), \tag{5.4}$$

and, for any $d \in [a,b]$ *and* $s = 1, 2, \dots, m-1$, *the following holds*

$$B(d,s) \leq B(d,s+1). \tag{5.5}$$

If $f_i \in T$, $i = 1, \dots, m-1$, *then the reverse inequalities in* (5.4) *and* (5.5) *hold.*

And, finally, let us state a result which is a sharpening of inequality (1.2) given in Theorem B.

Theorem 13. *If* P, Q, k, a, g *satisfy the assumptions of Theorem* 8 *and* $u \in R$, $u - Q(k) > 0$, $u - R(k) > 0$, $(ua - Q(kg))/(u - Q(k))$, $(ua - R(kg))/(u - R(k)) \in (0, \infty)$, *then for* $f \in S$ *the following holds*

$$\begin{aligned} uf(a) - R(kf(kg)) &\leq (u - Q(k)) f\left(\frac{ua - Q(kg)}{u - Q(k)}\right) - P(kf(g)) \\ &\leq (u - R(k)) f\left(\frac{ua - R(kg)}{u - R(k)}\right). \end{aligned}$$

If $f \in T$, *then a reverse inequality holds.*

Proof. Setting $m = 2$, $a_1 = k$, $a_2 = kg$, $f_1 = f$ $c_1 = u$, $c_2 = ua$ in inequality (5.2), we obtain the result. □

Remark 8. If μ is a counting measure, then we have discrete results. This remark is applicable for each result in the paper.

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REFERENCES

1. S. Abramovich, B. Mond and J.Pečarić, *Sharpening Hölder's inequality*, J. Math. Anal. Appl. **196** (1995), 1131–1134.

2. ———, *Sharpening Jensen's inequality and majorization theorem*, J. Math. Anal. Appl. **214** (1997), 721–728.

3. S. Abramovich and J. Pečarić, *Convex and concave functions and generalized Hölder inequality*, Soochow J. **14** (1998), 261–272.

4. G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, London, 1952.

5. L. Losonczi and Z. Páles, *Inequalities for indefinite forms*, J. Math. Anal. Appl. **205** (1997), 148–156.

6. E.J. McShane, *Jensen's inequality*, Bull. Amer. Math Soc. **43** (1937), 521–527.

7. D.S. Mitrinović, *Analytic inequalities*, Springer Verlag, New York, 1970.

8. D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and new inequalities in analysis*, Kluwer Acad. Publ., Dordrecht, 1993.

9. J. Pečarić, *On Jessen's inequality for convex functions*, III, J. Math. Anal. Appl. **156** (1991), 231–239.

10. J. Pečarić, F. Proschan and Y.L. Tong, *Convex functions, partial orderings and statistical applications*, Academic Press, New York, 1992.

11. T. Popoviciu, *On an inequality*, Gaz. Mat. (Bucharest) **51** (1946), 81–85.

12. X-H. Sun, *On the generalized Hölder inequalities*, Soochow J. **23** (1997), 241–252.

13. P.M. Vasić and J. Pečarić, *On Jensen inequality for monotone functions*, An. Univ. Timişoara Ser. Ştiinţ. Mat. **17** (1979), 95–104.

14. ———, *On Hölder and some related inequalities*, Matematica Rev. D'Anal. Num. Th. L'Approx. **25** (1982), 95–103.

15. J. von Neumann, *Almost periodic function in a group*, Trans Amer. Math. Soc. **36** (1934), 445–492.

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