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INEQUALITIES OF OSTROWSKI TYPE IN TWO DIMENSIONS

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ABSTRACT. A weighted version of Ostrowski type inequality in two dimensions is established. An ordinary generalization of Ostrowski's inequality in two dimensions and a corresponding Ostrowski-Grüss inequality are also derived.

1. Introduction. In 1938 A. Ostrowski proved the following integral inequality, [**15**] or [**14**, p. 468].

Theorem 1. *Let $f : I \to R$, where $I \subset R$ is an interval, be a mapping differentiable in the interior $Int\ I$ of I, and let $a, b \in Int\ I$, $a < b$. If $|f'(t)| \le M$, for all $t \in [a, b]$ then we have*

$$(1) \qquad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{(x-(a+b/2))^2}{(b-a)^2} \right] (b-a)M,$$

for $x \in [a, b]$.

The first (direct) generalization of Ostrowski's inequality was given by Milovanović and Pečarić in [**12**]. In recent years a number of authors have written about generalizations of Ostrowski's inequality. For example, this topic is considered in [**1, 3, 5, 7**] and [**12**]. In this way some new types of inequalities are formed, such as inequalities of Ostrowski-Grüss type, inequalities of Ostrowski-Chebyshev type, etc. The first inequality of Ostrowski-Grüss type was given by Dragomir and Wang in [**5**]. It was generalized and improved in [**7**]. Cheng gave a sharp version of the mentioned inequality in [**3**]. The first multivariate version of Ostrowski's inequality was given by Milovanović in [**10**], see also [**11**] and [**14**, p. 468]. Multivariate versions of Ostrowski's

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inequality were also considered in [**2, 6**] and [**9**]. In this paper we give a weighted two-dimensional generalization of Ostrowski's inequality. For that purpose, we introduce specially defined functions, which can be considered as "harmonic functions," since they are generalizations of harmonic or Appell-like polynomials in two dimensions. In Section 3 we use the mentioned generalization to obtain an ordinary two-dimensional Ostrowski type inequality. Finally, in Section 4 we give a corresponding Ostrowski-Grüss inequality.

2. A weighted Ostrowski type inequality. Let $\Omega = [a,b] \times [a,b] \subset R^2$ and let $w : \Omega \to R$ be an integrable function such that $w(x,y) \geq 0$, for all $(x,y) \in \Omega$. We define

$$P_{k+1}(t,s) = \frac{1}{(k!)^2} \int_a^t \int_a^s (t-x)^k (s-y)^k w(x,y)\, dx\, dy, \qquad k = 0,1,2,\dots \tag{2}$$

Specially, we set

$$P_0(t,s) = w(t,s).$$

Lemma 2. *Let $P_k(t,s)$ be defined by* (2)*. Then we have*

$$\frac{\partial^2 P_{k+1}(t,s)}{\partial t \partial s} = P_k(t,s), \quad k = 0,1,2,\dots .$$

Proof. We have

$$\begin{aligned}\frac{\partial P_{k+1}(t,s)}{\partial t} &= \frac{1}{(k!)^2} \frac{\partial}{\partial t}\left[\int_a^t \int_a^s (t-x)^k (s-y)^k w(x,y)\, dx\, dy\right] \\ &= \frac{k}{(k!)^2} \int_a^t \int_a^s (t-x)^{k-1} (s-y)^k w(x,y)\, dx\, dy.\end{aligned}$$

From the above relation we get

$$\begin{aligned}\frac{\partial^2 P_{k+1}(t,s)}{\partial t \partial s} &= \frac{\partial}{\partial s}\left(\frac{\partial P_{k+1}(t,s)}{\partial t}\right)\\ &= \frac{1}{(k-1)!^2}\int_a^t\int_a^s (t-x)^{k-1}(s-y)^{k-1}w(x,y)\,dx\,dy\\ &= P_k(t,s).\end{aligned}$$

Specially, we have

$$\begin{aligned}\frac{\partial^2 P_1(t,s)}{\partial t \partial s} &= \frac{\partial^2}{\partial t \partial s}\left(\int_a^t\int_a^s w(x,y)\,dx\,dy\right)\\ &= \frac{\partial}{\partial s}\left(\int_a^s w(t,y)\,dx\right) = w(t,s) = P_0(t,s). \qquad \square\end{aligned}$$

Let $f:\Omega \to R$ be a given function. Here we always suppose that $f \in C^{2n+2}(\Omega)$. We now define

$$J_{k+1} = \int_a^b P_{k+1}(b,s)\frac{\partial^{2k+1} f(b,s)}{\partial t^k \partial s^{k+1}}\,ds, \quad k = 0,1,\dots,n, \tag{3}$$

$$w_k(y) = \frac{1}{k!}\int_a^b (b-x)^k w(x,y)\,dx \geq 0, \quad k = 0,1,\dots,n \tag{4}$$

and

$$Q_{j+1}(w_k,s) = \frac{1}{j!}\int_a^s (s-y)^j w_k(y)\,dy, \quad j = 0,1,\dots,n, \tag{5}$$

$$Q_0(w_k,s) = w_k(s). \tag{6}$$

Note that

$$Q_{k+1}(w_k, s) = P_{k+1}(b, s). \tag{7}$$

We also define

$$u_k(s) = \frac{\partial^k f(b, s)}{\partial t^k}, \quad k = 0, 1, \dots, n \tag{8}$$

such that

$$u_k^{(k+1)}(s) = \frac{\partial^{2k+1} f(b, s)}{\partial t^k \partial s^{k+1}}. \tag{9}$$

Lemma 3. *Let J_{k+1}, w_k, Q_{j+1}, u_k be defined by* (3), (4), (5) *and* (8), *respectively. Then we have*

$$J_{k+1} = \sum_{j=0}^{k} (-1)^{k-j+1} Q_{j+1}(w_k, b) u_k^{(j)}(b) + (-1)^{k+1} U_0(w_k), \tag{10}$$

where

$$U_0(w_k) = \int_a^b w_k(s) u_k(s)\, ds.$$

Proof. From (3), (7) and (9) it follows

$$J_{k+1} = \int_a^b Q_{k+1}(w_k, s) u_k^{(k+1)}(s)\, ds, \quad k = 0, 1, \dots, n. \tag{11}$$

We have

(12)

$$Q'_{j+1}(w_k, s) = \frac{1}{(j-1)!} \int_a^s (s-y)^{j-1} w_k(y)\, dy = Q_j(w_k, s), \quad j = 1, \dots, n,$$

$$Q'_1(w_k, s) = w_k(s) = Q_0(w_k, s).$$

We now set $J_{k+1} = U_{k+1}(w_k)$. Then from (11) and (12) we get

$$\begin{aligned}
&(-1)^{k+1} U_{k+1}(w_k) \\
&\quad = (-1)^{k+1} \left[Q_{k+1}(w_k, s) u_k^{(k)}(s) \,\Big|_{s=a}^{s=b} - \int_a^b Q_k(w_k, s) u_k^{(k)}(s)\, ds \right] \\
&\quad = (-1)^{k+1} Q_{k+1}(w_k, b) u_k^{(k)}(b) + (-1)^k \int_a^b Q_k(w_k, s) u_k^{(k)}(s)\, ds,
\end{aligned}$$

since $Q_{k+1}(w_k, a) = 0$. The above relation can be rewritten in the form

$$(-1)^{k+1} U_{k+1}(w_k) = (-1)^{k+1} Q_{k+1}(w_k, b) u_k^{(k)}(b) + (-1)^k U_k(w_k).$$

In a similar way we get

$$(-1)^k U_k(w_k) = (-1)^k Q_k(w_k, b) u_k^{(k-1)}(b) + (-1)^{k-1} U_{k-1}(w_k).$$

If we continue this procedure then we obtain

$$\begin{aligned}
(-1)^{k+1} J_{k+1} &= (-1)^{k+1} U_{k+1}(w_k) \\
&= \sum_{j=0}^{k} (-1)^j Q_{j+1}(w_k, b) u_k^{(j)}(b) + U_0(w_k).
\end{aligned}$$

From the above relation we easily get (10). □

We now define

$$K_{k+1} = \int_a^b \frac{\partial P_{k+1}(t,b)}{\partial t} \frac{\partial^{2k} f(t,b)}{\partial t^k \partial s^k}\, dt, \quad k = 0, 1, \ldots, n, \tag{13}$$

$$z_k(x) = \frac{1}{k!} \int_a^b (b-y)^k w(x,y)\, dy \geq 0, \quad k = 0, 1, \ldots, n \tag{14}$$

and

$$R_j(z_k, t) = \frac{1}{(j-1)!} \int_a^t (t-x)^{j-1} z_k(x)\, dx, \quad j = 1, 2, \ldots, n, \tag{15}$$

$$R_0(z_k,t)=z_k(t). \tag{16}$$

Note that

$$\frac{\partial P_{k+1}(t,b)}{\partial t}=R_k(z_k,t). \tag{17}$$

We also define

$$v_k(t)=\frac{\partial^k f(t,b)}{\partial s^k},\quad k=0,1,\dots,n \tag{18}$$

such that

$$v_k^{(k)}(t)=\frac{\partial^{2k} f(t,b)}{\partial t^k\partial s^k}. \tag{19}$$

Lemma 4. *Let K_{k+1}, z_k, R_j, v_k be defined by* (13), (14), (15) *and* (18)*, respectively. Then we have*

$$K_{k+1}=\sum_{j=1}^{k}(-1)^{k-j}R_j(z_k,b)v_k^{(j-1)}(b)+(-1)^kV_1(z_k), \tag{20}$$

where

$$V_1(z_k)=\int_a^b z_k(t)v_k(t)\,dt.$$

Proof. From (13), (17) and (19) it follows

$$K_{k+1}=\int_a^b R_k(z_k,t)v_k^{(k)}(t)\,dt. \tag{21}$$

We have

(22)

$$R_j'(z_k,t)=\frac{1}{(j-2)!}\int_a^t (t-x)^{j-2}z_k(x)\,dx=R_{j-1}(z_k,t),\quad j=2,\dots,n,$$

$$R_1'(z_k,t)=z_k(t)=R_0(z_k,t).$$

We now set $K_{k+1} = V_{k+1}(z_k)$. Then from (21) and (22) we get

$$
\begin{aligned}
&(-1)^k V_{k+1}(z_k) \\
&\quad = (-1)^k R_k(z_k, b) v_k^{(k-1)}(b) + (-1)^{k-1} \int_a^b R_{k-1}(z_k, t) v_k^{(k-1)}(t)\, dt,
\end{aligned}
$$

since $R_k(z_k, a) = 0$. We can rewrite the above relation in the form

$$
(-1)^k V_{k+1}(z_k) = (-1)^k R_k(z_k, b) v_k^{(k-1)}(b) + (-1)^{k-1} V_k(z_k).
$$

In a similar way we obtain

$$
(-1)^{k-1} V_k(z_k) = (-1)^{k-1} R_{k-1}(z_k, b) v_k^{(k-2)}(b) + (-1)^{k-2} V_{k-1}(z_k).
$$

If we continue this procedure then we get

$$
\begin{aligned}
(-1)^k K_{k+1} &= (-1)^k V_{k+1}(z_k) \\
&= \sum_{j=1}^{k} (-1)^j R_j(z_k, b) v_k^{(j-1)}(b) + V_1(z_k).
\end{aligned}
$$

From the above relation we easily get (20). $\square$

Theorem 5. *Let $\Omega = [a,b] \times [a,b] \subset R^2$, and let $w : \Omega \to R$ be an integrable function, $w(x,y) \geq 0$. If $f \in C^{2n+2}(\Omega)$ and*

$$
M_{2n+2} = \max_{(t,s)\in\Omega} \left| \frac{\partial^{2n+2} f(t,s)}{\partial t^{n+1} \partial s^{n+1}} \right|, \quad M_w = \max_{(t,s)\in\Omega} w(t,s) \tag{23}
$$

then we have the identity

$$
\int_a^b \int_a^b w(t,s) f(t,s)\, dt\, ds = \sum_{i=0}^{n} K_{i+1} - \sum_{i=0}^{n} J_{i+1} + I_{n+1}, \tag{24}
$$

where

$$
I_{n+1} = \int_a^b \int_a^b P_{n+1}(t,s) \frac{\partial^{2n+2} f(t,s)}{\partial t^{n+1} \partial s^{n+1}}\, dt\, ds
$$

and the inequality

$$
\left| \int_a^b \int_a^b w(t,s) f(t,s)\, dt\, ds - \sum_{i=0}^{n} K_{i+1} + \sum_{i=0}^{n} J_{i+1} \right| \le \frac{M_{2n+2} M_w}{(n+2)!^2} (b-a)^{2n+4}, \tag{25}
$$

where J_{i+1}, K_{i+1} *are given by Lemmas* 3 *and* 4.

Proof. Integrating by parts, we obtain

$$
\begin{aligned}
I_{n+1} &= \int_a^b \int_a^b P_{n+1}(t,s) \frac{\partial^{2n+2} f(t,s)}{\partial t^{n+1} \partial s^{n+1}}\, dt\, ds \\
&= \int_a^b ds \int_a^b P_{n+1}(t,s) \frac{\partial}{\partial t} \left(\frac{\partial^{2n+1} f(t,s)}{\partial t^n \partial s^{n+1}} \right) dt \\
&= \int_a^b ds \left[P_{n+1}(t,s) \frac{\partial^{2n+1} f(t,s)}{\partial t^n \partial s^{n+1}} \Big|_{t=a}^{t=b} \right] \\
&\quad - \int_a^b \int_a^b \frac{\partial P_{n+1}(t,s)}{\partial t} \frac{\partial^{2n+1} f(t,s)}{\partial t^n \partial s^{n+1}}\, dt\, ds \\
&= \int_a^b P_{n+1}(b,s) \frac{\partial^{2n+1} f(b,s)}{\partial t^n \partial s^{n+1}}\, ds \\
&\quad - \int_a^b \int_a^b \frac{\partial P_{n+1}(t,s)}{\partial t} \frac{\partial^{2n+1} f(t,s)}{\partial t^n \partial s^{n+1}}\, dt\, ds,
\end{aligned} \tag{26}
$$

since $P_{n+1}(a,s) = 0$.

If we introduce the notation

$$
L_{n+1} = \int_a^b \int_a^b \frac{\partial P_{n+1}(t,s)}{\partial t} \frac{\partial^{2n+1} f(t,s)}{\partial t^n \partial s^{n+1}}\, dt\, ds \tag{27}
$$

then we can write

$$I_{n+1} = J_{n+1} - L_{n+1}.$$

Integrating by parts, we get

$$\begin{aligned}
L_{n+1} &= \int_a^b \int_a^b \frac{\partial P_{n+1}(t,s)}{\partial t} \frac{\partial}{\partial s} \left(\frac{\partial^{2n} f(t,s)}{\partial t^n \partial s^n} \right) dt\, ds \\
&= \int_a^b dt \left[\frac{\partial P_{n+1}(t,s)}{\partial t} \frac{\partial^{2n} f(t,s)}{\partial t^n \partial s^n} \Big|_{s=a}^{s=b} \right] \\
&\quad - \int_a^b \int_a^b \frac{\partial^2 P_{n+1}(t,s)}{\partial t \partial s} \frac{\partial^{2n} f(t,s)}{\partial t^n \partial s^n} dt\, ds \\
&= \int_a^b \frac{\partial P_{n+1}(t,b)}{\partial t} \frac{\partial^{2n} f(t,b)}{\partial t^n \partial s^n} dt - \int_a^b \int_a^b P_n(t,s) \frac{\partial^{2n} f(t,s)}{\partial t^n \partial s^n} dt\, ds,
\end{aligned}$$

since $(\partial P_{n+1}(t,a)/\partial t) = 0$ and Lemma 2 holds. Thus,

$$L_{n+1} = K_{n+1} - I_n.$$

Hence, we have

$$I_{n+1} = J_{n+1} - K_{n+1} + I_n. \tag{28}$$

The above described procedure is the first step of the whole procedure. In a similar way we get

$$I_n = J_n - K_n + I_{n-1}.$$

If we now continue the above described procedure, then we get

$$I_{n+1} = \sum_{i=1}^{n} J_{i+1} - \sum_{i=1}^{n} K_{i+1} + I_1. \tag{29}$$

We have

$$I_1 = \int_a^b \int_a^b P_1(t,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} dt\, ds. \tag{30}$$

Using the previously given notations we have

$$u_0'(s) = \frac{\partial f(b,s)}{\partial s}, \quad w_0(y) = \int_a^b w(x,y)\,dx$$

and

$$P_1(b,s) = \int_a^b \int_a^s w(x,y)\,dx\,dy = Q_1(w_0,s).$$

From the above relations it follows

$$J_1 = \int_a^b Q_1(w_0,s)u_0'(s)\,ds \tag{31}$$

and

$$L_1 = \int_a^b \int_a^s \frac{\partial P_1(t,s)}{\partial t} \frac{\partial f(t,s)}{\partial s}\,dt\,ds.$$

such that

$$I_1 = J_1 - L_1. \tag{32}$$

We also have

$$v_0(t) = f(t,b), \quad z_0(x) = \int_a^b w(x,y)\,dy \quad \text{and} \quad \frac{\partial P_1(t,b)}{\partial t} = R_0(z_0,t).$$

From the above relations we get

$$K_1 = \int_a^b R_0(z_0,t)v_0(t)\,dt \tag{33}$$

such that

$$L_1 = K_1 - I_0, \tag{34}$$

where

$$I_0 = \int_a^b \int_a^b f(t,s) w(t,s)\, dt\, ds.$$

From (29), (32) and (34) it follows

$$I_{n+1} = \sum_{i=0}^{n} J_{i+1} - \sum_{i=0}^{n} K_{i+1} + I_0. \tag{35}$$

Hence, (24) holds.

We now estimate I_{n+1},

$$\begin{aligned}
&\left| \int_a^b \int_a^b P_{n+1}(t,s) \frac{\partial^{2n+2} f(t,s)}{\partial t^{n+1} \partial s^{n+1}}\, dt\, ds \right| \\
&\quad = \left| \int_a^b \int_a^b \frac{1}{n!^2} \left[\int_a^t \int_a^s (s-y)^n (t-x)^n w(x,y)\, dx\, dy \right] \right. \\
&\qquad\qquad \left. \times \frac{\partial^{2n+2} f(t,s)}{\partial t^{n+1} \partial s^{n+1}}\, dt\, ds \right| \\
&\quad \le \frac{M_{2n+2} M_w}{n!^2} \left| \int_a^b \int_a^b \frac{(s-a)^{n+1}}{n+1} \frac{(t-a)^{n+1}}{n+1}\, dt\, ds \right| \\
&\quad = \frac{M_{2n+2} M_w}{(n+1)!^2} \left(\frac{(b-a)^{n+2}}{n+2} \right)^2 = \frac{M_{2n+2} M_w}{(n+2)!^2} (b-a)^{2n+4}.
\end{aligned}$$

This completes the proof. □

3. An inequality of Ostrowski type. Here we use the notations introduced in Section 2. We now choose $w(x,y) = 1$. If we substitute this in (4) then we have

$$w_n(y) = \frac{1}{n!} \int_a^b (b-x)^n\, dx = \frac{(b-a)^{n+1}}{(n+1)!}$$

such that

$$Q_{j+1}(w_n, b) = \frac{1}{j!}\int_a^b (b-y)^j w_n(y)\,dy = \frac{(b-a)^{n+1}}{(n+1)!}\,\frac{(b-a)^{j+1}}{(j+1)!}.$$

We also have

$$U_0(w_n) = \int_a^b w_n(s)u_n(s)\,ds = \frac{(b-a)^{n+1}}{(n+1)!}\int_a^b u_n(s)\,ds.$$

Thus, from Lemma 3 we get

(37)

$$\begin{aligned} J_{n+1} &= \hat{J}_{n+1} \\ &= \frac{(b-a)^{n+1}}{(n+1)!}\left[\sum_{j=0}^{n} \frac{(-1)^{n-j}(b-a)^{j+1}}{(j+1)!} u_n^{(j)}(b) + (-1)^{n+1}\int_a^b u_n(s)\,ds\right]. \end{aligned}$$

If we substitute $w(x, y) = 1$ in (14) then we have

$$z_n(x) = \frac{1}{n!}\int_a^b (b-y)^n\,dy = \frac{(b-a)^{n+1}}{(n+1)!}$$

and

$$R_j(z_n, b) = \frac{1}{(j-1)!}\int_a^b (b-x)^{j-1} z_n(x)\,dx = \frac{(b-a)^{n+1}}{(n+1)!}\,\frac{(b-a)^j}{j!}.$$

We also have

$$V_1(z_n) = \int_a^b z_n(t)v_n(t)\,dt = \frac{(b-a)^{n+1}}{(n+1)!}\int_a^b v_n(t)\,dt.$$

Thus, from Lemma 4 we get

(37)

$$\begin{aligned} K_{n+1} &= \hat{K}_{n+1} \\ &= \frac{(b-a)^{n+1}}{(n+1)!}\left[\sum_{j=1}^{n} \frac{(-1)^{n-j}(b-a)^{j}}{j!} v_n^{(j-1)}(b) + (-1)^{n}\int_a^b v_n(t)\,dt\right]. \end{aligned}$$

We now introduce the notation

$$\hat{I}_{n+1} = \int_a^b \int_a^b \hat{P}_{n+1}(t,s) \frac{\partial^{2n+2} f(t,s)}{\partial t^{n+1} \partial s^{n+1}} \, dt \, ds, \tag{38}$$

where

$$\hat{P}_{n+1}(t,s) = \frac{1}{(n+1)!^2} (s-a)^{n+1} (t-a)^{n+1}. \tag{39}$$

Theorem 6. *Under the assumptions of Theorem* 5 *and the notations* (36)–(39) *we have*

$$\left| \int_a^b \int_a^b f(t,s) \, dt \, ds - \sum_{i=0}^{n} \hat{J}_{i+1} + \sum_{i=0}^{n} \hat{K}_{i+1} \right| \leq \frac{M_{2n+2}}{(n+2)!^2} (b-a)^{2n+4}.$$

Proof. The proof follows immediately from the above considerations and Theorem 5. □

4. An inequality of Ostrowski-Grüss type. Let $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space and $e \in X$, $\|e\| = 1$. Let $\gamma, \varphi, \Gamma, \Phi$ be real numbers and $x, y \in X$ such that the conditions

$$\langle \Phi e - x, x - \varphi e \rangle \geq 0 \quad \text{and} \quad \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \tag{40}$$

hold. In [**4**] we can find the inequality

$$\left| \langle x, y \rangle - \langle x, e \rangle \langle y, e \rangle \right| \leq \frac{1}{4} \left| \Phi - \varphi \right| \left| \Gamma - \gamma \right|. \tag{41}$$

We also have

$$\left| \langle x, y \rangle - \langle x, e \rangle \langle y, e \rangle \right| \leq \left(\|x\|^2 - \langle x, e \rangle^2 \right)^{1/2} \left(\|y\|^2 - \langle e, y \rangle^2 \right)^{1/2}. \tag{42}$$

Let $X = L_2(\Omega)$ and $e = 1/(b-a)$. If we define

$$T(f,g) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f(t,s)g(t,s)\,dt\,ds - \frac{1}{(b-a)^4}\int_a^b\int_a^b f(t,s)\,dt\,ds\int_a^b\int_a^b g(t,s)\,dt\,ds \tag{43}$$

then from (40) and (41) we get the Grüss inequality in $L_2(\Omega)$,

$$|T(f,g)| \le \frac{1}{4}(\Gamma-\gamma)(\Phi-\varphi), \tag{44}$$

if

$$\gamma \le f(x,y) \le \Gamma,\ \ \varphi \le g(x,y) \le \Phi, \quad (x,y) \in \Omega.$$

From (42) we get the pre-Grüss inequality

$$T(f,g)^2 \le T(f,f)T(g,g). \tag{45}$$

Theorem 7. *Under the assumptions of Theorem* 5 *we have*

$$\begin{aligned} \Bigg| \int_a^b\int_a^b f(t,s)\,dt\,ds + \sum_{i=0}^{n}\hat{J}_{i+1} - \sum_{i=0}^{n}\hat{K}_{i+1} \\ - \frac{(b-a)^{2n+4}}{(n+2)!^2}\left[v(b,b)-v(b,a)-v(a,b)+v(a,a)\right]\Bigg| \\ \le \frac{M_{2n+2}-m_{2n+2}}{2(n+1)!^2}(b-a)^{2n+4}\left[\frac{1}{(2n+3)^2}-\frac{1}{(n+2)^4}\right]^{1/2}, \end{aligned} \tag{46}$$

where $v(x,y) = \partial^{2n}f(x,y)/\partial x^n\partial y^n$ *and* $\hat{J}_{i+1}$, $\hat{K}_{i+1}$ *are defined in Theorem* 6*, while*

$$m_{2n+2} = \min_{(x,y)\in\Omega}\frac{\partial^{2n+2}f(x,y)}{\partial t^{n+1}\partial s^{n+1}}, \quad M_{2n+2} = \max_{(x,y)\in\Omega}\frac{\partial^{2n+2}f(x,y)}{\partial t^{n+1}\partial s^{n+1}}.$$

Proof. We have, see Theorems 5 and 6,

$$\int_a^b \int_a^b f(t,s)\,dt\,ds = -\sum_{i=0}^{n} \hat{J}_{i+1} + \sum_{i=0}^{n} \hat{K}_{i+1} + \hat{I}_{n+1}.$$

We add the terms

$$\begin{aligned}\hat{L}_{n+1} &= \pm \frac{1}{(b-a)^2} \int_a^b \int_a^b \hat{P}_{n+1}(t,s)\,dt\,ds \int_a^b \int_a^b \frac{\partial^{2n+2} f(t,s)}{\partial t^{n+1} \partial s^{n+1}}\,dt\,ds \\ &= \pm \frac{(b-a)^{2n+2}}{(n+2)!^2} \left[v(b,b) - v(b,a) - v(a,b) + v(a,a)\right]\end{aligned}$$

to the above relation. Then we get

$$\int_a^b \int_a^b f(t,s)\,dt\,ds + \sum_{i=0}^{n} \hat{J}_{i+1} - \sum_{i=0}^{n} \hat{K}_{i+1} - \hat{L}_{n+1} = \hat{I}_{n+1} - \hat{L}_{n+1}.$$

Hence, we have

$$\hat{I}_{n+1} - \hat{L}_{n+1} = (b-a)^2 T\left(\hat{P}_{n+1}(t,s), \frac{\partial^{2n+2} f(t,s)}{\partial t^{n+1} \partial s^{n+1}}\right)$$

where $T(\cdot,\cdot)$ is defined by (43) and

$$\begin{aligned}\left|\hat{I}_{n+1} - \hat{L}_{n+1}\right| \le (b-a)^2 T &\left(\frac{\partial^{2n+2} f(t,s)}{\partial t^{n+1} \partial s^{n+1}}, \frac{\partial^{2n+2} f(t,s)}{\partial t^{n+1} \partial s^{n+1}}\right)^{1/2} \\ &\times T\left(\hat{P}_{n+1}(t,s), \hat{P}_{n+1}(t,s)\right)^{1/2},\end{aligned}$$

since (45) holds.

We also have

$$T\left(\frac{\partial^{2n+2} f(t,s)}{\partial t^{n+1} \partial s^{n+1}}, \frac{\partial^{2n+2} f(t,s)}{\partial t^{n+1} \partial s^{n+1}}\right)^{1/2} \le \frac{1}{2}(M_{2n+2} - m_{2n+2})$$

by the Grüss inequality and

$$T\left(\hat{P}_{n+1}(t,s),\hat{P}_{n+1}(t,s)\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b \hat{P}_{n+1}(t,s)^2\,dt\,ds - \frac{1}{(b-a)^4}\left(\int_a^b\int_a^b \hat{P}_{n+1}(t,s)\,dt\,ds\right)^2 = \frac{(b-a)^{4n+4}}{(n+1)!^4}\left[\frac{1}{(2n+3)^2}-\frac{1}{(n+2)^4}\right].$$

From the above relations we see that (46) holds. □

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