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CONFORMALLY RECURRENT SEMI-RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper we give a complete classification of conformally recurrent semi-Riemannian manifolds with *harmonic conformal* curvature tensor and to give another generalization of conformally symmetric Riemannian manifolds. Moreover, we give a nontrivial example which is neither locally symmetric nor conformally flat.

1. Introduction. Let us denote by M an $n(\geq 4)$-dimensional semi-Riemannian manifold with semi-Riemannian metric g and Riemannian connection ∇ and let R, respectively S or r, be the Riemannian curvature tensor, respectively the Ricci tensor or the scalar curvature, on M.

It is said to be *conformally recurrent* if the conformal curvature tensor C with components C_{ijkl} so that

$$\begin{aligned} C_{ijkl} = R_{ijkl} &- \frac{1}{n-2}\left(S_{il}g_{jk} - S_{ik}g_{jl} + S_{jk}g_{il} - S_{jl}g_{ik}\right) \\ &+ \frac{r}{(n-1)(n-2)}\left(g_{il}g_{jk} - g_{ik}g_{jl}\right) \end{aligned} \tag{1.1}$$

is recurrent, i.e., there is a 1-form α such that $\nabla C = \alpha \otimes C$, where R_{ijkl}, S_{ij} and g_{ij} are components of R, S and g on M. In particular, it is said to be *conformally symmetric* if $\nabla C = 0$. As is easily seen, the class of conformally recurrent semi-Riemannian manifolds includes all the classes of conformally symmetric, conformally flat and locally symmetric semi-Riemannian manifolds. Among them such kind of Riemannian manifolds are studied by Besse [**2**], Ryan [**12**], Simon [**13**], Weyl [**15, 16**], Yano [**17**], Yano and Bochner [**18**], for example.

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Conformally symmetric semi-Riemannian manifolds are investigated by Derdziński and Roter [**6**]. In particular, in the Riemannian case, Derdziński and Roter [**6**] and Miyazawa [**9**] proved the following

Theorem A. *An $n(\geq 4)$-dimensional conformally symmetric manifold is conformally flat or locally symmetric.*

The symmetric tensor K of type $(0,2)$ with components K_{ij} is called the *Weyl tensor* if it satisfies

$$K_{ijl} - K_{ilj} = \frac{1}{2(n-1)}\,(k_l g_{ij} - k_j g_{il}), \tag{1.2}$$

where $k = \operatorname{Tr} K$ and K_{ijl}, respectively k_j, are components of the covariant derivative ∇K, respectively ∇k.

On the other hand, in Weyl [**15**] and [**16**] it can be easily seen that the Ricci tensor is a Weyl tensor when we only consider an $n(\geq 4)$-dimensional conformally flat Riemannian manifold, see Eisenhart [**7**]. In particular, Derdziński and Roter [**6**] investigated the structure of analytic conformally symmetric indefinite Riemannian manifold of index 1 which is neither conformally flat nor locally symmetric.

We denote by M an $n(\geqq 4)$-dimensional semi-Riemannian manifold with semi-Riemannian metric g and semi-Riemannian connection ∇. For a tensor field $(0, r+1)$ the codifferential δT of T is defined by

$$\delta T(X_1, \dots, X_r) = \sum_{i=1}^{r} \varepsilon_i \nabla_{E_i} T(E_i, X_1, \dots, X_r)$$

for any vector fields $X_1, \dots, X_r$, where $\{E_i\}$ is an orthonormal frame on M. If $\delta C = 0$, then M is said to have *harmonic conformal curvature tensor*, see Besse [**2**].

In this paper we want to make a generalization of such results in the direction of a certain kind of curvature-like tensor fields. In order to do this we introduce the notion of conformal recurrent curvature tensor, that is, the covariant derivative of the conformal curvature tensor C satisfies $\nabla C = \alpha \otimes C$ for a certain 1-form α. Moreover, let us say a semi-Riemannian manifold M has *harmonic conformal curvature tensor* if its conformal curvature tensor C satisfies $\delta C = 0$, that is,

$$\sum\nolimits_r \varepsilon_r C_{rjkmr} = 0.$$

If the semi-Riemannian manifold M is conformally symmetric, then it is trivial that it is conformally recurrent for $\mathcal{A} = 0$ and it has a harmonic conformal curvature tensor.

Now in this paper we want to show the following

Theorem. *Let M be an $n(\geq 4)$-dimensional Riemannian manifold. If M is conformally recurrent and it has a harmonic conformal curvature tensor and if the scalar curvature is nonzero constant, then it is conformally flat or locally symmetric.*

When the 1-form α vanishes identically in above theorem, it can be explained that a conformal Riemannian symmetric manifold M is locally symmetric or conformally flat. So our theorem is also a generalization of Theorem A. In Remark 3.3 given in Section 3 we will explain that the condition concerned with the scalar curvature is not necessary.

On the other hand, in Section 4 we will show that among the indefinite class of conformal recurrent manifolds with harmonic conformal curvature tensor there are so many kind of examples which are neither locally symmetric nor conformally flat, but its scalar curvature is vanishing. So in an indefinite version of such a theorem, the condition that nonzero constant scalar curvature is essential.

2. Preliminaries. Let M be an $n(\geqq 2)$-dimensional semi-Riemannian manifold of index s, $0 \leqq s \leqq n$, equipped with semi-Riemannian metric tensor g and let R, respectively S or r, be the Riemannian curvature tensor, respectively the Ricci tensor or the scalar curvature, on M. In particular, if $0 < s < n$, then M is said to be *indefinite*.

We can choose a local field $\{E_j\} = \{E_1, \ldots, E_n\}$ of orthonormal frames on a neighborhood of M. Here and in the sequel the indices $i, j, k, \ldots$ run from 1 to n. With respect to the indefinite Riemannian metric we have $g(E_j, E_k) = \varepsilon_j \delta_{jk}$, where

$$\varepsilon_j = -1 \quad \text{or} \quad 1, \text{ according to whether } 1 \leqq j \leqq s \text{ or } s+1 \leqq j \leqq n.$$

Let $\{\theta_i\}$, $\{\theta_{ij}\}$ and $\{\Theta_{ij}\}$ be the canonical form, the connection form and the curvature form on M, respectively, with respect to the local field $\{E_j\}$ of orthonormal frames. Then we have the structure equations

$$\begin{aligned} d\theta_i + \sum_j \varepsilon_j \theta_{ij} \wedge \theta_j = 0, \quad \theta_{ij} + \theta_{ji} = 0, \\ d\theta_{ij} + \sum_k \varepsilon_k \theta_{ik} \wedge \theta_{kj} = \Theta_{ij}, \\ \Theta_{ij} = -\frac{1}{2} \sum_{k,l} \varepsilon_{kl} R_{ijkl} \theta_k \wedge \theta_l, \end{aligned}$$

where $\varepsilon_{ij\cdots k} = \varepsilon_i \varepsilon_j \cdots \varepsilon_k$ and R_{ijkl} denotes the components of the Riemannian curvature tensor R of M.

Now, let C be the conformal curvature tensor with components C_{ijkl} on M, which is given by

$$\begin{aligned} C_{ijkl} = R_{ijkl} - \frac{1}{n-2} \{\varepsilon_i(\delta_{il} S_{jk} - \delta_{ik} S_{jl}) + \varepsilon_j(S_{il}\delta_{jk} - S_{ik}\delta_{jl})\} \\ + \frac{r}{(n-1)(n-2)} \varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}), \end{aligned} \tag{2.1}$$

where $S_{ij} = \sum_l \varepsilon_l R_{lijl}$ are the components of the Ricci tensor S with respect to the local field $\{e_j\}$ of orthonormal frames and $r = \sum_j \varepsilon_j S_{jj}$ is the scalar curvature.

Remark 2.1. If M is Einstein, then the conformal curvature tensor C satisfies

$$C_{ijkl} = R_{ijkl} - \frac{r}{n(n-1)} \varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}).$$

This yields that the conformal curvature tensors of Einstein Riemannian manifolds are the concircular curvature one. In particular, if M is of constant curvature, the conformal curvature tensor vanishes identically, Yano and Bochner [**18**].

Let D^rM be the vector bundle consisting of differentiable r-forms and $DM = \sum_{r=0}^{n} D^rM$, where D^0M is the algebra of differentiable functions on M. For any tensor field K in D^rM the components K_{ijklh} of the covariant derivative ∇K of K are defined by (for simplicity, we consider the case $r = 4$)

$$\begin{aligned}\sum_h \varepsilon_h K_{ijklh}\theta_h \\ = dK_{ijkl} - \sum_h \varepsilon_h (K_{hjkl}\theta_{hi} + K_{ihkl}\theta_{hj} + K_{ijhl}\theta_{hk} + K_{ijkh}\theta_{hl}).\end{aligned}$$

Now we denote by TM the tangent bundle of M. Let T be a quadrilinear mapping of $TM \times TM \times TM \times TM$ into $\mathbb{R}$ satisfying the curvature-like properties:

(a) $\quad T(X, Y, Z, U) = -T(Y, X, Z, U) = -T(X, Y, U, Z),$

(b) $\quad T(X, Y, Z, U) = T(Z, U, X, Y),$

(c) $\quad T(X, Y, Z, U) + T(Y, Z, X, U) + T(Z, X, Y, U) = 0.$

Then T is called the *curvature-like tensor* on M. See Kobayashi and Nomizu [**5**], for example. Let T_{ijkl} be the components of T associated with the orthonormal frame $\{E_j\}$; then the components T_{ijkl} are given by $T_{ijkl} = T(E_i, E_j, E_k, E_l)$. By the conditions (a), (b) and (c), the following properties of the components of T hold corresponding to the conditions (a), (b) and (c):

$$T_{ijkl} = -T_{jikl} = -T_{ijlk}, \tag{2.2}$$

$$T_{ijkl} = T_{klij} = T_{lkji}, \tag{2.3}$$

$$T_{ijkl} + T_{jkil} + T_{kijl} = 0. \tag{2.4}$$

If the components T_{ijkl} of a tensor T in $D^4M = \otimes^4 T^*M$ satisfy (2.2), (2.3) and (2.4), then it becomes a curvature-like tensor.

Lemma 2.1. *On a semi-Riemannian manifold, the conformal curvature tensor C is curvature-like.*

For any integer a and b such that $1 \leqq a < b \leqq s$ the metric contraction reduced by a and b is denoted by $\mathcal{C}_{ab} : T^r_sM \to T^r_{s-2}M$ with respect to the orthonormal frame $\{E_j\}$. The symmetric tensor U in D^2M is called the *Weyl tensor* if its components of the covariant derivative ∇U of U satisfy

$$U_{ijk} - \frac{1}{2(n-1)}\, u_k \varepsilon_i \delta_{ij} = U_{ikj} - \frac{1}{2(n-1)}\, u_j \varepsilon_i \delta_{ik}, \tag{2.5}$$

where $u = \mathcal{C}_{12}U$. In particular, if u is constant, then U is called the *Codazzi tensor*. We put $\nabla_X U(Y,Z) = \nabla U(Y,Z,X)$. Then it is easily seen that $\sum_k \varepsilon_k U_{kjk} = u_k/2$.

Now let C be the conformal curvature tensor with components C_{ijkl} on M. The semi-Riemannian manifold is said to be *conformally flat* if $C = 0$. For the geometric meaning of conformally flat Riemannian manifolds, see Yano and Bochner [**18**], for example. In particular, if M is a space of constant curvature, the conformal curvature tensor vanishes identically.

The Ricci-like tensor $\mathrm{Ric}\,(C)$ of C is defined by $\mathcal{C}_{14}(C) = \mathcal{C}_{23}(C)$. Then the components C_{jk} of $\mathrm{Ric}\,(C)$ are given by $C_{jk} = \sum_r \varepsilon_r C_{rjkr}$. We have then

$$C_{jk} = 0. \tag{2.6}$$

3. Conformally recurrent spaces. Let M be an $n(\geq 2)$-dimensional semi-Riemannian manifold of index $2s$, $0 \leq s \leq n$, with Riemannian connection ∇ and let R, respectively S or r, be the Riemannian curvature tensor, respectively the Ricci tensor or the scalar curvature, on M.

Now let C be the conformal curvature tensor with components C_{ijkl} with respect to the field $\{E_j\}$ of orthonormal frames given by

$$\begin{aligned} C_{ijkl} = R_{ijkl} &- \frac{1}{n-2}\,(\varepsilon_j S_{il}\delta_{jk} - \varepsilon_j S_{ik}\delta_{jl} + \varepsilon_i S_{jk}\delta_{il} - \varepsilon_i S_{jl}\delta_{ik}) \\ &+ \frac{r}{(n-1)(n-2)}\,\varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \end{aligned} \tag{3.1}$$

Differentiating C of (3.1) covariantly, we have

$$
\begin{aligned}
C_{ijklm} = R_{ijklm} - \frac{1}{n-2}\,(&\varepsilon_j S_{ilm}\delta_{jk} - \varepsilon_j S_{ikm}\delta_{jl} \\
&+ \varepsilon_i S_{jkm}\delta_{il} - \varepsilon_i S_{jlm}\delta_{ik}) \\
&+ \frac{r_m}{(n-1)(n-2)}\,\varepsilon_{ij}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}),
\end{aligned} \tag{3.2}
$$

where C_{ijklm}, respectively R_{ijklm}, S_{jkm} or r_m, are the components of the covariant derivative ∇C of C, respectively the covariant derivative ∇R of R, ∇S of S or dr.

By the second Bianchi identity

$$R_{ijklm} + R_{ijlmk} + R_{ijmkl} = 0$$

for R and putting $i = m$ in (3.2) and summing up with respect to i, we obtain

$$\sum\nolimits_r \varepsilon_r C_{rjkmr} = (n-3)\{S_{jkm} - S_{jmk} - \varepsilon_j(r_m\delta_{jk} - r_k\delta_{jm})/2(n-1)\}/(n-2).$$

If M has a harmonic conformal curvature tensor, then we have by definition

$$\sum\nolimits_r \varepsilon_r C_{rjkmr} = 0, \tag{3.3}$$

from which the following property is derived

Lemma 3.1. *Let M be an $n(\geq 4)$-dimensional semi-Riemannian manifold. If M has a harmonic conformal curvature tensor, then the Ricci tensor is a Weyl tensor.*

Lemma 3.2. *Let M be an $n(\geq 4)$-dimensional semi-Riemannian manifold. If M has a harmonic conformal curvature tensor, then it satisfies*

$$\sum\nolimits_r \varepsilon_r (R_{rikm}S_{rj} + R_{rimj}S_{rk} + R_{rijk}S_{rm}) = 0. \tag{3.4}$$

Proof. By the assumption, Lemma 3.1 gives that the Ricci tensor S is the Weyl tensor. By the definition of (2.5) we have

$$S_{ijk} - S_{ikj} = \varepsilon_i(r_k\delta_{ij} - r_j\delta_{ik})/2(n-1). \tag{3.5}$$

Differentiating covariantly, we get

$$S_{ijkm} - S_{ikjm} = (r_{km}\varepsilon_i\delta_{ij} - r_{jm}\varepsilon_i\delta_{ik})/2(n-1).$$

Interchanging the indices k and m and subtracting the resulting equation from this, we obtain

$$S_{ijkm} - S_{ijmk} + S_{imjk} - S_{ikjm} = \varepsilon_i(r_{jk}\delta_{im} - r_{jm}\varepsilon_{ik})/2(n-1),$$

where we have used the property that r_{ij} is symmetric with respect to i and j, because r is the function. Thus we have that the left side is equivalent to

$$\begin{aligned}
&= (S_{ijkm} - S_{ijmk}) + (S_{imjk} - S_{imkj}) + (S_{imkj} - S_{ikjm})\\
&= (S_{ijkm} - S_{ijmk}) + (S_{imjk} - S_{imkj})\\
&\quad + \big[\{S_{ikmj} + \varepsilon_i(r_{kj}\delta_{im} - r_{mj}\delta_{ik})/2(n-1)\} - S_{ikjm}\big]\\
&= -\sum\nolimits_r \varepsilon_r(R_{mkir}S_{rj} + R_{mkjr}S_{ir}) - \sum\nolimits_r \varepsilon_r(R_{kjir}S_{rm} + R_{kjmr}S_{ir})\\
&\quad - \sum\nolimits_r \varepsilon_r(R_{jmir}S_{rk} + R_{jmkr}S_{ir}) + \varepsilon_i(r_{kj}\delta_{im} - r_{mj}\delta_{ik})/2(n-1)\\
&= -\sum\nolimits_r \varepsilon_r(R_{mkir}S_{rj} + R_{jmir}S_{rk} + R_{kjir}S_{rm}\\
&\quad + \varepsilon_i(r_{kj}\delta_{im} - r_{mj}\delta_{ik})/2(n-1)
\end{aligned}$$

where the second equality follows from (3.2), the third equality is derived from the Ricci identity for the Ricci tensor S_{ij} and the fourth equality follows from the first Bianchi identity. It yields that we have (3.4). It completes the proof. □

Lemma 3.3. *Let M be an $(n\geq 4)$-dimensional semi-Riemannian manifold. If M is conformally recurrent and if S is the Weyl tensor, then we obtain*

$$\sum\nolimits_r \varepsilon_r(C_{rikm}S_{rj} + C_{rimj}S_{rk} + C_{rijk}S_{rm}) = 0, \tag{3.6}$$

$$\sum\nolimits_r \varepsilon_r(C_{rikm}S_{rjn} + C_{rimj}S_{rkn} + C_{rijk}S_{rmn}) = 0. \tag{3.7}$$

Proof. Substituting the components R_{ijkm} of (2.1) into the lefthand side of (3.4) and calculating directly, we get the equation (3.6).

Now differentiating (3.6) covariantly and taking account of (3.6), then the conformal recurrence implies (3.7). This completes the proof. □

Theorem 3.4. *Let M be an $n(\geq 4)$-dimensional semi-Riemannian manifold of index s, $0 \leq s \leq n$, with Riemannian connection ∇. Assume that M is conformally recurrent and has a harmonic conformal curvature tensor. If the scalar curvature is constant, then it satisfies*

$$\|\alpha\|^2 \|C\|^2 \|\nabla S - \alpha S\|^2 = 0,$$

where $\| * \|^2$ denotes the squared norm of the scalar product on M.

Proof. Let C_{ijkmnp} be the components of the covariant derivative $\nabla^2 C$ of ∇C. They are given by

$$C_{ijkmnp} = (\alpha_n \alpha_p + \alpha_{np}) C_{ijkm}. \tag{3.8}$$

Now we define by f the scalar product of C, namely we put $f = \langle C, C \rangle$. Let M' be the subset of M consisting of points x in M such that $f(x) = 0$. Then we have

$$\nabla f = 2\langle \nabla C, C \rangle = 2\alpha f$$

on the open subset $M - M'$ and hence we have $\alpha = \nabla f / 2f$, from which it follows that

$$2\alpha = \nabla \log |f|.$$

This implies that

$$\alpha_{ij} = \alpha_{ji} \quad \text{on } M - M'. \tag{3.9}$$

So, on $M - M'$, by (3.8) and (3.9) we have $C_{ijkmnp} = C_{ijkmpn}$. Accordingly, by the Ricci identity we get

(3.10)

$$\sum_r \varepsilon_r (R_{pnir} C_{rjkm} + R_{pnjr} C_{irkm} + R_{pnkr} C_{ijrm} + R_{pnmr} C_{ijkr}) = 0.$$

Differentiating the above equation covariantly and taking account of $C_{ijkmn} = \alpha_n C_{ijkm}$, we have

$$\begin{aligned}\sum_r \varepsilon_r \big\{ &(R_{pnirq} C_{rjkm} + R_{pnjrq} C_{irkm} + R_{pnkrq} C_{ijrm} + R_{pnmrq} C_{ijkr}) \\ &+ \alpha_q (R_{pnir} C_{rjkm} + R_{pnjr} C_{irkm} + R_{pnkr} C_{ijrm} + R_{pnmr} C_{ijkr}) \big\} = 0.\end{aligned}$$

Hence we have by (3.10)

(3.11)

$$\sum_r \varepsilon_r (R_{pnirq} C_{rjkm} + R_{pnjrq} C_{irkm} + R_{pnkrq} C_{ijrm} + R_{pnmrq} C_{ijkr}) = 0.$$

On the other hand, by (2.1), (3.2) and $\alpha_h C_{ijkl} = C_{ijklh}$ we have

$$\begin{aligned}
&\alpha_n [R_{ijkm} - \{\varepsilon_i (S_{jk}\delta_{im} - S_{jm}\delta_{ik}) + \varepsilon_j (\delta_{jk} S_{im} - \delta_{jm} S_{ik})\}/(n-2) \\
&\quad + r\varepsilon_i (\delta_{jk}\delta_{im} - \delta_{jm}\delta_{ik})/(n-1)(n-2)] \\
&= R_{ijkmn} - \{\varepsilon_i (S_{jkn}\delta_{im} - S_{jmn}\delta_{ik}) + \varepsilon_j (\delta_{jk} S_{imn} - \delta_{jm} S_{ikn})\}/(n-2) \\
&\quad + r_n \varepsilon_{ij} (\delta_{jk}\delta_{im} - \delta_{jm}\delta_{ik})/(n-1)(n-2)
\end{aligned}$$

and hence we get

(3.12)

$$\begin{aligned}
&R_{ijkmn} \\
&= \alpha_n R_{ijkm} + \varepsilon_i \{(S_{jkn}\delta_{im} - S_{jmn}\delta_{ik}) - \alpha_n (S_{jk}\delta_{im} - S_{jm}\delta_{ik})\}/(n-2) \\
&\quad + \varepsilon_j \{(\delta_{jk} S_{imn} - \delta_{jm} S_{ikn}) - \alpha_n (\delta_{jk} S_{im} - \delta_{jm} S_{ik})\}/(n-2) \\
&\quad + (r\alpha_n - r_n)\varepsilon_{ij} (\delta_{jk}\delta_{im} - \delta_{jm}\delta_{ik})/(n-1)(n-2).
\end{aligned}$$

From (3.11) and (3.12) it follows that

$$\begin{aligned}
&\alpha_q \sum_r \varepsilon_r (R_{pnir} C_{rjkm} + R_{pnjr} C_{irkm} + R_{pnkr} C_{ijrm} + R_{pnmr} C_{ijkr}) \\
&\quad + \sum_r \varepsilon_{rp} [\{S_{niq}\delta_{pr} - S_{nrq}\delta_{pi}) - \alpha_q (S_{ni}\delta_{pr} - S_{nr}\delta_{pi})\} C_{rjkm} \\
&\quad + \{(S_{njq}\delta_{pr} - S_{nrq}\delta_{pj}) - \alpha_q (S_{nj}\delta_{pr} - S_{nr}\delta_{pj})\} C_{irkm} \\
&\quad + \{(S_{nkq}\delta_{pr} - S_{nrq}\delta_{pk}) - \alpha_q (S_{nk}\delta_{pr} - S_{nr}\delta_{pk})\} C_{ijrm} \\
&\quad + \{(S_{nmq}\delta_{pr} - S_{nrq}\delta_{pm}) - \alpha_q (S_{nm}\delta_{pr} - S_{nr}\delta_{pm})\} C_{ijkr}]/(n-2) \\
&\quad + \sum_r \varepsilon_r n [\{(\delta_{ni} S_{prq} - \delta_{nr} S_{piq}) - \alpha_q (\delta_{ni} S_{pr} - \delta_{nr} S_{pi})\} C_{rjkm} \\
&\quad + \{(\delta_{nj} S_{prq} - \delta_{nr} S_{pjq}) - \alpha_q (\delta_{nj} S_{pr} - \delta_{nr} S_{pj})\} C_{irkm} \\
&\quad + \{(\delta_{nk} S_{prq} - \delta_{nr} S_{pkq}) - \alpha_q (\delta_{nk} S_{pr} - \delta_{nr} S_{pk})\} C_{ijrm} \\
&\quad + \{(\delta_{nm} S_{prq} - \delta_{nr} S_{pmq}) - \alpha_q (\delta_{nm} S_{pr} - \delta_{nr} S_{pm})\} C_{ijkr}]/(n-2) \\
&\quad + (r\alpha_q - r_q)\{\varepsilon_n (\delta_{ni} C_{pjkm} + \delta_{nj} C_{ipkm} + \delta_{nk} C_{ijpm} + \delta_{nm} C_{ijkp}) \\
&\quad - \varepsilon_p (\delta_{pi} C_{njkm} + \delta_{pj} C_{inkm} + \delta_{pk} C_{ijnm} + \delta_{pm} C_{ijkn})\}/(n-1)(n-2) \\
&= 0
\end{aligned}$$

which can be reformed by (3.10) as

$$\begin{aligned}
&\sum_r \varepsilon_{rp}[\{S_{niq}\delta_{pr} - S_{nrq}\delta_{pi})C_{rjkm} + (S_{njq}\delta_{pr} - S_{nrq}\delta_{pj})C_{irkm} \\
&\quad + \{(S_{nkq}\delta_{pr} - S_{nrq}\delta_{pk})C_{ijrm} - (S_{nmq}\delta_{pr} - S_{nrq}\delta_{pm})C_{ijkr}\} \\
&\quad - \alpha_q\{(S_{ni}\delta_{pr} - S_{nr}\delta_{pi})C_{rjkm} + (S_{nj}\delta_{pr} - S_{nr}\delta_{pj})C_{irkm} \\
&\quad + (S_{nk}\delta_{pr} - S_{nr}\delta_{pk})C_{ijrm} + (S_{nm}\delta_{pr} - S_{nr}\delta_{pm})C_{ijkr}]/(n-2) \\
&\quad + \sum_r \varepsilon_r n[\{\delta_{ni}S_{prq} - \delta_{nr}S_{piq}) - \alpha_q(\delta_{ni}S_{pr} - \delta_{nr}S_{pi})\}C_{rjkm} \\
&\quad + \{(\delta_{nj}S_{prq} - \delta_{nr}S_{pjq}) - \alpha_q(\delta_{nj}S_{pr} - \delta_{nr}S_{pj})\}C_{irkm} \\
&\quad + \{(\delta_{nk}S_{prq} - \delta_{nr}S_{pkq}) - \alpha_q(\delta_{nk}S_{pr} - \delta_{nr}S_{pk})\}C_{ijrm} \\
&\quad + \{(\delta_{nm}S_{prq} - \delta_{nr}S_{pmq}) - \alpha_q(\delta_{nm}S_{pr} - \delta_{nr}S_{pm})\}C_{ijkr}]/(n-2) \\
&\quad + (r\alpha_q - r_q)\{\varepsilon_n(\delta_{ni}C_{pjkm} + \delta_{nj}C_{ipkm} + \delta_{nk}C_{ijpm} + \delta_{nm}C_{ijkp}) \\
&\quad - \varepsilon_p(\delta_{pi}C_{njkm} + \delta_{pj}C_{inkm} + \delta_{pk}C_{ijnm} + \delta_{pm}C_{ijkn})\}/(n-1)(n-2) \\
&= 0,
\end{aligned}$$

and hence by multiplying both sides by $(n-2)$ we obtain

$$\begin{aligned}
&(S_{niq}C_{pjkm} + S_{njq}C_{ipkm} + S_{nkq}C_{ijpm} + S_{nmq}C_{ijkp}) \\
&\quad - (S_{piq}C_{njkm} + S_{pjq}C_{inkm} + S_{pkq}C_{ijnm} + S_{pmq}C_{ijkn}) \\
&\quad - \sum_r \varepsilon_{rp}(\delta_{pi}C_{rjkm} + \delta_{pj}C_{irkm} + \delta_{pk}C_{ijrm} + \delta_{pm}C_{ijkr})S_{nrq} \\
&\quad + \sum_r \varepsilon_{rn}(\delta_{ni}C_{rjkm} + \delta_{nj}C_{irkm} + \delta_{nk}C_{ijrm} + \delta_{nm}C_{ijkr})S_{prq} \\
&\quad - \alpha_q\big\{(S_{ni}C_{pjkm} + S_{nj}C_{ipkm} + S_{nk}C_{ijpm} + S_{nm}C_{ijkp}) \\
&\quad - (S_{pi}C_{njkm} + S_{pj}C_{inkm} + S_{pk}C_{ijnm} + S_{pm}C_{ijkn})\} \\
&\quad + \alpha_q\Big\{\sum_r \varepsilon_{rp}(\delta_{pi}C_{rjkm} + \delta_{pj}C_{irkm} + \delta_{pk}C_{ijrm} + \delta_{pm}C_{ijkr})S_{nr} \\
&\quad - \sum_r \varepsilon_{rn}(\delta_{ni}C_{pjkm} + \delta_{nj}C_{irkm} + \delta_{nk}C_{ijpm} + \delta_{nm}C_{ijkr})S_{pr}\Big\} \\
&\quad + (r\alpha_q - r_q)\{\varepsilon_n(\delta_{ni}C_{pjkm} + \delta_{nj}C_{ipkm} + \delta_{nk}C_{ijpm} + \delta_{nm}C_{ijkp}) \\
&\quad - \varepsilon_p(\delta_{pi}C_{njkm} + \delta_{pj}C_{inkm} + \delta_{pk}C_{ijnm} + \delta_{pm}C_{ijkn})\}/(n-1) \\
&= 0.
\end{aligned}$$

Accordingly, we have

(3.13)

$$\begin{aligned}
&\{(S_{niq}-\alpha_q S_{ni})C_{pjkm}+(S_{njq}-\alpha_q S_{nj})C_{ipkm}\\
&\quad+(S_{nkq}-\alpha_q S_{nk})C_{ijpm}+(S_{nmq}-\alpha_q S_{nm})C_{ijkp}\}\\
&\quad-\{(S_{piq}-\alpha_q)C_{njkm}+(S_{pjq}-\alpha_q S_{pj})C_{inkm}\\
&\quad+(S_{pkq}-\alpha_q)C_{ijnm}+(S_{pmq}-\alpha_q S_{pm})C_{ijkn})\}\\
&\quad-\sum\nolimits_r \varepsilon_{rp}(\delta_{pi}C_{rjkm}+\delta_{pj}C_{irkm}+\delta_{pk}C_{ijrm}+\delta_{pm}C_{ijkr})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot(S_{nrq}-\alpha_q S_{nr})\\
&\quad+\sum\nolimits_r \varepsilon_{rn}(\delta_{ni}C_{rjkm}+\delta_{nj}C_{irkm}+\delta_{nk}C_{ijrm}+\delta_{nm}C_{ijkr})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot(S_{prq}-\alpha_q S_{pr})\\
&\quad+(r\alpha_q-r_q)\{\varepsilon_n(\delta_{ni}C_{pjkm}+\delta_{nj}C_{ipkm}+\delta_{nk}C_{ijpm}+\delta_{nm}C_{ijkp})\\
&\quad-\varepsilon_p(\delta_{pi}C_{njkm}+\delta_{pj}C_{inkm}+\delta_{pk}C_{ijnm}+\delta_{pm}C_{ijkn})\}/(n-1)\\
&=0.
\end{aligned}$$

Now putting $i=r$ in (3.13), summing up with respect to $\sum_i$ and taking account of (2.6), Lemma 2.1 and the first Bianchi identity for C, we have

(3.14)

$$\begin{aligned}
&\sum\nolimits_r \varepsilon_r\Big[(n-2)(S_{nrq}-\alpha_q S_{nr})C_{rjkm}+(S_{jrq}-\alpha_q S_{jr})C_{rnkm}\\
&\quad+(S_{krq}-\alpha_q S_{kr})C_{rjnm}+(S_{mrq}-\alpha_q S_{mr})C_{rjkn}\\
&\quad+\sum\nolimits_s \varepsilon_s\{\varepsilon_n\delta_{nk}(S_{rsq}-\alpha_q S_{rs})C_{rjms}-\varepsilon_n\delta_{nm}(S_{rsq}-\alpha_q S_{rs})C_{rjks}\}\Big]\\
&=0.
\end{aligned}$$

Next we assume that M is conformally recurrent and M has a harmonic conformal curvature tensor, namely, it satisfies $\sum_r \varepsilon_r C_{rjkmr}=0$. Then we have

$$\sum\nolimits_r \varepsilon_r \alpha_r C_{rjkm}=0. \tag{3.15}$$

Putting $m=q$ in (3.14), summing up with respect to m and taking account of (3.15) and (3.5), we have

(3.16)

$$\begin{aligned}&\sum\nolimits_{r,s}\varepsilon_{rs}\{(n-2)S_{nrs}C_{rjks}+S_{jrs}C_{rnks}+S_{krs}C_{rjns}-\alpha_s S_{rs}C_{rjkn}\}\\&\quad+\sum\nolimits_{r}\varepsilon_r r_r C_{rjkn}/2+\sum\nolimits_{r,s,t}\varepsilon_n\varepsilon_{rst}\delta_{nk}S_{rst}C_{rjts}\\&\quad-\sum\nolimits_{rs}\varepsilon_{rs}(S_{rsn}-\alpha_n S_{rs})C_{rjks}=0.\end{aligned}$$

The third term of (3.16) vanishes identically, since it satisfies

$$\begin{aligned}\text{the third term }&=\sum\nolimits_{r,s,t}\varepsilon_n\varepsilon_{rst}\delta_{nk}(S_{rst}-S_{rts})C_{rjts}/2\\&=\sum\nolimits_{r,s,t}\varepsilon_{rst}\varepsilon_n\delta_{nk}\varepsilon_r(r_t\delta_{rs}-r_s\delta_{rt})C_{rjts}/4(n-1)=0\end{aligned}$$

where the first equality follows from (2.2), the second one is derived by (3.5) and the last one is derived from (2.6).

On the other hand, we get

$$\begin{aligned}\sum\nolimits_{r,s}\varepsilon_{rst}(S_{rns}-S_{rsn})C_{rjks}&=\sum\nolimits_{r,s}\varepsilon_{rs}\varepsilon_r(r_s\delta_{rn}-r_n\delta_{rs})C_{rjks}/2(n-1)\\&=\sum\nolimits_{s}\varepsilon_s r_s C_{njks}/2(n-1)\end{aligned}$$

where the first equality is derived by (3.4) and the second one follows from (2.6). Thus (3.16) is deformed as

(3.17)

$$\begin{aligned}&\sum\nolimits_{r,s}\varepsilon_{rs}\{(n-3)S_{nrs}C_{rjks}+S_{jrs}C_{rnks}+S_{krs}C_{rjns}-\alpha_s S_{rs}C_{rjkn}\}\\&\quad+\sum\nolimits_{r}r_r\varepsilon_r C_{rjkn}/2+\sum\nolimits_{r}r_r\varepsilon_r C_{njkr}/2(n-1)+\sum\nolimits_{r,s}\alpha_n S_{rs}C_{rjks}=0.\end{aligned}$$

Since M is conformally recurrent and M has a harmonic conformal curvature tensor, by (3.7) we have

$$\sum\nolimits_{r}\varepsilon_r(C_{rikm}S_{rjn}+C_{rimj}S_{rkn}+C_{rijk}S_{rmn})=0.$$

Putting m and n by s in (3.7) and multiplying ε_l and summing up with respect to s, we have

$$\sum\nolimits_{r,s}\varepsilon_{rs}(S_{jrs}C_{riks}-S_{krs}C_{rijs})+\frac{1}{2}\sum\nolimits_{s}\varepsilon_s r_s C_{sijk}=0. \tag{3.18}$$

By (3.17) and (3.18) we have

$$\sum_{r,s}\varepsilon_{rs}\{(n-1)S_{nrs}C_{rjks}-\alpha_s S_{rs}C_{rjkn}\}-\sum_r r_r\varepsilon_r C_{rjkn}/2 \\ +\sum_r r_r\varepsilon_r C_{njkr}/2(n-1)+\sum_{r,s}\varepsilon_{rs}\alpha_n S_{rs}C_{rjks}=0$$

Then from this by using the conformal recurrence and the assumption of constant scalar curvature we have

$$(n-1)\sum_{rs}S_{nrs}C_{rjks}+\sum_{rs}S_{rs}(C_{rjkns}-C_{rjksn})=0.$$

Here we note that the indices j and k in the first and the third terms are symmetric with each other, because S_{nrs} and S_{rs} are symmetric with respect to the indices r and s. From such a fact, if take a skew-symmetric part to the above equation, then it follows that

$$0=\sum_{r,s}\varepsilon_{rs}S_{rs}(C_{rjkns}-C_{rkjns})=\sum_{r,s}\varepsilon_{rs}S_{rs}(C_{rjkns}+C_{rknjs}) \\ =-\sum_{r,s}\varepsilon_{rs}S_{rs}C_{rnjks}.$$

Hence we are able to assert that

$$\sum_{r,s}\varepsilon_{rs}S_{nrs}C_{rjks}=\sum_{r,s}\varepsilon_{rs}S_{rsn}C_{rjks}=0. \tag{3.19}$$

Transvecting (3.14) to $\alpha_m\alpha_n\alpha_q$, summing up with respect to m,n and q, and taking account of (3.15) and (3.19), we have

$$\|\alpha\|^2\sum_{r,s}\varepsilon_{rs}S_{rs}C_{rjks}=0, \tag{3.20}$$

where $\|\alpha\|^2=\|\sum_r\varepsilon_r\alpha_r\alpha_r\|$. By (3.14), (3.19) and (3.20) we have

$$\|\alpha\|^2\sum_r\varepsilon_r[\{(n-2)(S_{nrq}-\alpha_q S_{nr})C_{rjkm}+(S_{jrq}-\alpha_q S_{jr})C_{rnkm}\} \\ +(S_{krq}-\alpha_q S_{kr})C_{rjnm}+(S_{mrq}-\alpha_q S_{mr})C_{rjkn}]=0.$$

By Lemma 3.3 we have

$$\sum_r\varepsilon_r\{(S_{krq}-\alpha_q S_{kr})C_{rjnm}+(S_{mrq}-\alpha_q S_{mr})C_{rjkn}\} \\ =\sum_r\varepsilon_r\{(S_{krq}-\alpha_q S_{kr})C_{rjnm}-(S_{krq}-\alpha_q S_{kr})C_{rjnm} \\ -(S_{nrq}-\alpha_q S_{nr})C_{rjmk}\} \\ =-\sum_r\varepsilon_r(S_{nrq}-\alpha_q S_{nr})C_{rjmk}.$$

From the above two equations we obtain

$$\|\alpha\|^2\sum_r \varepsilon_r\{(n-1)(S_{nrq}-\alpha_q S_{nr})C_{rjkm}+(S_{jrq}-\alpha_q S_{jr})C_{rnkm}\} = 0$$

which implies that

$$\|\alpha\|^2\sum_r \varepsilon_r(S_{jrq}-\alpha_q S_{jr})C_{rnkm} = -(n-1)\|\alpha\|^2\sum_r \varepsilon_r(S_{nrq}-\alpha_q S_{nr})C_{rjkm}.$$

From this it follows that

$$\|\alpha\|\sum_r \varepsilon_r(S_{jrq} - \alpha_q S_{jr})C_{rnkm} = 0. \tag{3.21}$$

Transvecting $S_{niq} - \alpha_q S_{ni}$ or C_{pjkm} to (3.13) and applying equations (3.20) and (3.21), we can obtain

$$\|\alpha\|^2\|\nabla S - \alpha\otimes S\|^2 C = 0 \quad \text{or} \quad \|\alpha\|^2\|C\|^2(\nabla S - \alpha\otimes S) = 0 \tag{3.22}$$

on $M - M'$. It completes the proof. $\square$

From this and Theorem 3.4 we want to give the following lemma which will be useful to prove our main theorem

Lemma 3.5. *Let M be an $n(\geq 4)$-dimensional Riemannian manifold with Riemannian connection ∇. If M is conformally recurrent and if M has a harmonic conformal curvature tensor and constant scalar curvature, we have*

$$C\otimes(\nabla R - \alpha\otimes R) = 0.$$

Proof. By Theorem 3.4 we have

$$\alpha\otimes C\otimes(\nabla S - \alpha\otimes S) = 0.$$

Let M_1 be the subsets consisting of points x in M at which $\alpha(x) = 0$.

First we suppose that M_1 is not empty. If $\mathrm{Int}\, M_1$ is empty, the nonvanishing 1-form α gives $C = 0$ or $\nabla S - \alpha\otimes S = 0$. Then by the assumption of conformal recurrence we know that $\nabla R - \alpha\otimes R = 0$. So in such a subcase the conclusion is given by the continuity of C and $\nabla R - \alpha\otimes R$.

Suppose that $\mathrm{Int}\, M_1$ is not empty. Then in such a subcase M is conformal symmetric. Then by Theorem A due to Derdzinski and Roter [**5**] and Miyazawa [**9**] we have $C = 0$ or $\nabla R = 0$ on $\mathrm{Int}\, M_1$. Hence it follows that we have $C = 0$ or $\nabla R = \alpha \otimes R$ on $\mathrm{Int}\, M_1$. Thus we have $C \otimes (\nabla R - \alpha \otimes R) = 0$ on M.

Now we suppose that M_1 is empty. Then nonvanishing 1-form α implies $C = 0$ or $\nabla S - \alpha \otimes S = 0$. When $\nabla S - \alpha \otimes S = 0$, we also have $\nabla R - \alpha \otimes R = 0$, because M is conformally recurrent. $\square$

By virtue of Lemma 3.5 we have the following

Theorem 3.6. *Let M be an $n(\geq 4)$-dimensional Riemannian manifold with Riemannian connection ∇. Suppose that M is conformally recurrent and has a harmonic conformal curvature tensor. If the scalar curvature is a nonzero constant, then M is conformally flat or M is locally symmetric.*

Proof. Let M'' be the subset of points x in M at which

$$(\nabla R - \alpha \otimes R)(x) = 0.$$

Then we have $(\nabla r - \alpha r)(x) = 0$ on M''. Since we have assumed that the scalar curvature is nonzero constant, we get $\alpha(x) = 0$ on M''. Then from this together with Lemma 3.5 it follows that $\alpha = 0$ or $C = 0$, that is, $\alpha \otimes C = 0$ on M.

Now let us consider the open subset M^* consisting of points x at which $C(x) = 0$. Then on such an open subset we have $\nabla C = 0$ and hence the inner product $\langle C, C \rangle$ is constant. By the continuity of $\langle C, C \rangle$, if M^* is not empty, then $\langle C, C \rangle = 0$ on M, namely $C = 0$ on M. That is, M is conformally flat. If M^* is empty, then the fact $\alpha \otimes C = 0$ implies $\alpha = 0$ on M. In such a case we know that M is conformally symmetric. From this together with Theorem A we complete the proof of our theorem. $\square$

Remark 3.1. In their paper [**8**] Goldberg and Okumura proved that in an $n(\geq 4)$-dimensional compact conformally flat Riemannian manifold, if the length of the Ricci tensor is constant and less than $r/\sqrt{n-1}$, then M is a space of constant curvature.

Remark 3.2. In the next section we will show that the assumption that the scalar curvature is a nonzero constant in Theorem 3.6 is essential when we consider an indefinite version of Theorem 3.6. That is, we will show a class of an indefinite complex hypersurfaces which is neither conformally flat nor locally symmetric, but its scalar curvature is vanishing.

Remark 3.3. But in a Riemannian version the referee suggests that the assumption concerned with the scalar curvature will not be necessary. Namely, one can verify that *any conformally recurrent Riemannian manifold of dimension* $n \geq 4$ *which has harmonic conformal curvature tensor is conformally symmetric in the sense that* $\nabla C = 0$ (and Theorem A gives conformally flat or locally symmetric). Namely, he has given us another possible argument which is much more shorter than our proof as follows.

First, let g be a Riemannian-product metric, positive-definite or not, on a product manifold of dimension $n \geq 4$, with both factor manifolds of positive dimensions. If $C(X, \cdot, \cdot, \cdot) = 0$ for all vectors X tangent to the first factor, and $\nabla C = \alpha \otimes C$ with a 1-form α such that the vector X obtained from α by index-raising ($X^j = g^{jk}\alpha_k$) is tangent to the first factor, then $\nabla C = 0$ identically on M. (Here $C(X, \cdot, \cdot, \cdot) = 0$ means that $C(X, Y, Z, U) = 0$ for all vectors Y, Z, U).

In fact, in product coordinates obtained from coordinates x^a in the first factor manifold and x^λ in the second factor, our assumptions mean that all components of C vanish except, possibly, those of the form $C_{\lambda\mu\nu\xi}$, while $\alpha_\lambda = 0$. (We let $\lambda, \mu, \nu, \xi, \rho$ vary through one index range, and a, b through the other.) Due to the definition of C (formula (1.1) in the paper), relations $C_{a\lambda b\mu} = 0$, contracted against g^{ab} or $g^{\lambda\mu}$, show that both factor metrics are Einstein, even if one or both of them happen to be two-dimensional. In particular, they both have constant scalar curvatures, which now easily implies that the only (possibly) nonzero components of ∇C are $C_{\lambda\mu\nu\xi,\rho}$. As $\nabla C = \alpha \otimes C$ and $\alpha_\rho = 0$, this gives $\nabla C = 0$.

It follows now that a *Riemannian* manifold of dimension $n \geq 4$ with $\delta C = 0$ and $\nabla C = \alpha \otimes C$ must have $\nabla C = 0$ everywhere.

In fact, suppose on the contrary that $\nabla C \neq 0$ somewhere. Thus, we can pick a nonempty connected open set M' such that $C \neq 0$ and $\nabla C \neq 0$, everywhere in M'. Since $\nabla C = \alpha \otimes C$, defining the norm $|C|$ of C by the usual formula $|C|^2 = g^{ip}g^{jq}g^{ks}g^{lt}C_{ijkl}C_{pqst}$ we obtain $\nabla T = 0$ on M', where T is the tensor field on M' given by $T = C/|C|$. (To see this, first note that, by transvecting both sides of $\nabla C = \alpha \otimes C$ with C, we obtain $\alpha = d\log|C|$ on M'). The tangent vectors X such that $T(X, \cdot, \cdot, \cdot) = 0$, i.e., $T(X, Y, Z, U) = 0$ for all Y, Z, U, form a distribution $\mathcal{D}$ on M' which is parallel (since so is T). Its dimension $\dim\mathcal{D}$ satisfies $0 < \dim\mathcal{D} < n$ since not all vectors lie in $\mathcal{D}$, as $C \neq 0$, but some nonzero vectors do (namely, the vector X obtained from α by index-raising, at any point of M', is in $\mathcal{D}$, due to the assumption that $\nabla C = \alpha \otimes C$ and $\delta C = 0$. The parallel distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are, locally in M', tangent to the factors of a Riemannian-product decomposition of the original metric which satisfies all the hypotheses of the preceding paragraph. Therefore, $\nabla C = 0$ on M', contradicting our very choice of M'.

4. Example. For any integer $p(\geq 2)$ and any complex number c such that $|c|\geq 1$ we define an indefinite complex Euclidean space C_n^{2n+1} of index $2n$ is defined as follows.

Let $\{z^j, z^{j^*}, z^{2n+1}\} = \{z^1, \cdots, z^{2n+1}\}$ be a complex coordinate of C_n^{2n+1}. Then $M = M(p, c)$ is an indefinite complete complex hypersurface of index $2n$ defined by

$$z^{2n+1} = \sum\nolimits_j h_j(z^j + cz^{j^*}), \quad h_j(z) = z^p,$$

where c is any complex number such that $|c|\geq 1$. The range of indices are given as follows:

$$i, j, \cdots = 1, \cdots, n, \quad A, B, \cdots = 1, \cdots, 2n, \quad \alpha, \beta, \cdots = 1, \cdots, 4n,$$
$$j^* = n + j, \quad A^* = 2n + A.$$

Usually in a semi-Kaehler manifold M we are able to choose a local field of orthonormal frame $\{E_1, \cdots, E_n, E_{1*}, \cdots, E_{n^*} = JE_n\}$ on a neighborhood of M. Then $U_j = 1/\sqrt{2}(E_j - iE_{j^*})$ and $U_{\bar{j}} = 1/\sqrt{2}(E_j + iE_{j^*})$ constitute a local field of unitary frames on M. Moreover, its semi-Kaehler metric is given by $g = 2\sum\varepsilon_j\omega_j\otimes\bar{\omega}_j$, where $\omega_j = \theta_j + i\theta_{j^*}$, and $\bar{\omega}_j = \theta_j - i\theta_{j^*}$.

Then the components h_{AB} of the second fundamental form, see Aiyama, Ikawa, Kwon and Nakagawa [**1**], are given by

$$h_{ij} = p(p-1)\delta_{ij}z^{p-2}, \qquad h_{i^*j} = p(p-1)c\delta_{ij}z^{p-2}, \\ h_{i^*j^*} = p(p-1)c^2\delta_{ij}z^{p-2}. \tag{4.1}$$

Let $S_{A\bar{B}}$ be the components of the extended Ricci tensor S of M with respect to the complex coordinate $\{z^j, z^{j^*}\}$. Then from the formula due to Aiyama, Nakagawa and Suh [**2**] and Choi, Kwon and Suh [**4**, **5**] we obtain that

$$\begin{aligned} S_{i\bar{j}} &= -\sum\nolimits_R h_{iR}\bar{h}_{Rj} \\ &= -\sum\nolimits_k \varepsilon_k h_{ik}\bar{h}_{kj} - \sum\nolimits_{k^*} h_{ik^*}\bar{h}_{k^*j} \\ &= \sum\nolimits_k h_{ik}\bar{h}_{kj} - \sum\nolimits_k h_{ik^*}\bar{h}_{k^*j} \\ &= (1-|c|^2)p^2(p-1)^2\delta_{ij}|z|^{2(p-2)}. \end{aligned}$$

Similarly, the other components are given by

$$\begin{aligned} S_{i\bar{j}^*} &= -\sum\nolimits_R \varepsilon_R h_{iR}\bar{h}_{Rj^*} = (1-|c|^2)p^2(p-1)^2\delta_{ij}|z|^{2(p-2)}, \\ S_{i^*\bar{j}^*} &= -\sum\nolimits_R \varepsilon_R h_{iR}\bar{h}_{Rj^*} = (1-|c|^2)p^2(p-1)^2\delta_{ij}|z|^{2(p-2)} \end{aligned}$$

which means that if $|c|^2 = 1$, then the Ricci tensor S on M is flat. Then by (3.2) and (3.13) we know that the conformal curvature tensor is harmonic, that is, coclosed $\delta C = 0$.

Next for the components h_{ABC} of the covariant derivatives of the second fundamental form we have

$$\begin{aligned} h_{ijk} &= p(p-1)(p-2)\delta_{ij}\delta_{ik}z^{p-3}, \\ h_{i^*jk} &= p(p-1)(p-2)c\delta_{ij}\delta_{ik}z^{p-3}, \\ h_{i^*j^*k} &= p(p-1)(p-2)c^2\delta_{ij}\delta_{ik}z^{p-3}, \\ h_{i^*j^*k^*} &= p(p-1)(p-2)c^3\delta_{ij}\delta_{ik}z^{p-3}. \end{aligned} \tag{4.2}$$

We should note that the expression is by the complex coordinates. Let

$$\{x^A, y^A, x^{2n+1}, y^{2n+1}\}$$

be the real coordinate of C_n^{2n+1}. Let $K_{\overline{A}BC\overline{D}}$ be the components of the extended Riemannian curvature tensor R of M with respect to the complex coordinate $\{z^j, z^{j^*}\}$ defined by

$$K_{\overline{A}BC\overline{D}} = g(R(U_{\overline{A}}, U_B)U_C, U_{\overline{D}}),$$

and let

$$R_{\alpha\beta\gamma\delta} = g(R(E_\alpha, E_\beta)E_\gamma, E_\delta)$$

be the components of the Riemannian curvature tensor R of M with respect to the real coordinates $\{x^A, y^A\}$. Then by the theory of complex hypersurfaces, see Aiyama, Nakagawa and Suh [**2**], in an indefinite Kaehler manifold we have

$$K_{\overline{A}BC\overline{D}} = -h_{BC}\bar{h}_{AD}, \qquad K_{\overline{A}BC\overline{D}E} = -h_{BCE}\bar{h}_{AD}, \tag{4.3}$$

$$K_{\overline{A}BC\overline{D}} = -\{R_{ABCD} + R_{A^*BC^*D} + i(R_{A^*BCD} - R_{ABC^*D})\}. \tag{4.4}$$

By (4.1) and (4.3) we have

$$(*) \qquad \begin{aligned} K_{\bar{i}jk\bar{m}} &= -h_{jk}\bar{h}_{im} = -p^2(p-1)^2\delta_{jk}\delta_{im}|z|^{2(p-2)}, \\ K_{\bar{i}jk\bar{m}^*} &= -h_{jk}\bar{h}_{im*} = -cp^2(p-1)^2\delta_{jk}\delta_{im}|z|^{2(p-2)}. \end{aligned}$$

Others are similarly given, from which it follows that M is not necessarily flat. Furthermore we have

$$\begin{aligned} K_{\bar{i}jk\bar{m}n} &= -h_{jkn}\bar{h}_{im} = -p^2(p-1)^2(p-2)\delta_{jm}\delta_{ik}|z|^{2(p-2)}z^{-1} \\ &= (p-2)\delta_{jn}z^{-1}K_{i\bar{j}k\bar{m}} = \alpha_j\delta_{jn}K_{\bar{i}jk\bar{m}}, \end{aligned}$$

where $\alpha_j = d\beta_j$ and the smooth function β_j is defined by

$$\beta_j = \log\frac{h_j(z)}{z^2} = \log z^{p-2}, \quad p\geq 3,$$

from which it follows that

$$K_{\bar{i}jk\bar{m}n} = \alpha_j\delta_{jn}K_{\bar{i}jk\bar{m}}.$$

Similarly, we get

$$K_{\bar{i}jk\bar{m}^*n} = \alpha_j\delta_{jn}K_{\bar{i}jk\bar{m}^*}.$$

Accordingly, if $p\geq 3$, then M is not locally symmetric and we are able to get

$$K_{\overline{A}BC\overline{D}E} = \alpha_E K_{\overline{A}BC\overline{D}}. \tag{4.5}$$

On the other hand, by (4.4) and the expression of $R_{\alpha\beta\gamma\delta}$ we have

$$K_{\bar{i}i i\bar{i}} = -R_{i^* i i^* i} = R_{i i^* i^* i} = g(R(E_i, JE_i)JE_i, E_i). \tag{4.6}$$

In general, since M is the semi-Kaehler manifold, the components $R_{\alpha\beta\gamma\delta}$ of the Riemannian curvature tensor R satisfy

$$R_{A^*BCD} = -R_{AB^*CD}, \quad R_{A^*B^*CD} = R_{ABCD}. \tag{4.7}$$

For indices i, j such that $i\neq j$, we have known $K_{\bar{i}jC\overline{D}} = 0$ from the formula $(*)$. By (4.4) we get

$$R_{ijkm} + R_{ij^*km^*} = 0, \quad R_{i^*jkm} - R_{ijk^*m} = 0. \tag{4.8}$$

Accordingly, the first equation of (4.8) is deformed as

$$\begin{aligned} R_{ijCD} + R_{ij^*CD^*} &= R_{ijCD} - R_{j^*iCD^*} = R_{ijCD} - R_{jiC^*D^*} \\ &= R_{ijCD} - R_{jiCD} = 2R_{ijCD} \\ &= 0, \end{aligned}$$

where the first equality is derived by the general property of the Riemannian curvature tensor, the second one follows from (4.8) and the general property of the Riemannian curvature tensor, and the third one is also derived from (4.7) and the general property of the Riemannian curvature tensor. Thus we have

$$R_{ijCD} = 0, \quad i\neq j. \tag{4.9}$$

On the relation between the real natural frame and the complex natural frame we have (4.4) and by the definition of the covariant derivative the components $K_{\bar{A}BC\bar{D}E}$ are given by

(4.10)

$$\begin{aligned} &K_{\overline{A}BC\overline{D}E} \\ &\quad = -\{R_{ABCDE} + R_{AB^*CD^*E} + i(R_{A^*BCDE} - R_{ABCD^*E})\}/2 \\ &\qquad + i\{R_{ABCDE^*} + R_{AB^*CD^*E^*} + i(R_{A^*BCDE^*} - R_{ABCD^*E^*})\}/2. \end{aligned}$$

On the other hand, from (4.5) we get $K_{\bar{i}ii\bar{i}i} = \alpha_i K_{\bar{i}ii\bar{i}}$. Then from this together with (4.8) and (4.10) it follows that

$$R_{i^*iii^*E} = 2\alpha_E R_{i^*iii^*}. \tag{4.11}$$

Similarly, we get

$$R_{ABCDE} = 2\alpha_E R_{ABCD}.$$

In such a case the Ricci tensor is flat if $|c| = 1$ and the complex hypersurface M of index $2n$ in a $(2n+1)$-dimensional indefinite complex Euclidean space C_n^{2n+1} of index $2n$ defined above is conformally recurrent. Of course its conformal curvature tensor is coclosed, which is neither locally symmetric nor conformally flat if $p \geq 3$. Moreover, we know that the scalar curvature is identically vanishing, because its Ricci tensor is vanishing on M.

This example shows that in an indefinite version of Theorem 3.6 the assumption that the scalar curvature is a nonzero constant is essential.

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REFERENCES

1. R. Aiyama, T. Ikawa, J-H. Kwon and H. Nakagawa, *Complex hypersurfaces in an indefinite complex space form*, Tokyo J. Math. **10** (1987), 349–361.

2. R. Aiyama, H. Nakagawa and Y.J. Suh, *Semi-Kaehler submanifolds of an indefinite complex space form*, Kodai Math. J. **11** (1988), 325–343.

3. A.L. Besse, *Einstein manifolds*, Springer-Verlag, New York, 1987.

4. Y.S. Choi, J.-H. Kwon and Y.J. Suh, *On semi-symmetric complex hypersurfaces of a semi-definite complex space form*, Rocky Mountain J. Math. **31** (2001), 417–435.

5. ———, *On semi-Ryan complex submanifolds in an indefinite complex space form*, Rocky Mountain J. Math. **31** (2001), 873–897.

6. A. Derdziński and W. Roter, *On conformally symmetric manifolds with metrics of indices* 0 *and* 1, Tensor, N.S. **31** (1978), 255–259.

7. L.P. Eisenhart, *Riemannian geometry*, Princeton University Press, Princeton, 1934.

8. S. Goldberg and M. Okumura, *Conformally flat manifolds and a pinching problem on the Ricci tensor*, Proc. Amer. Math. Soc. **58** (1976), 234–236.

9. T. Miyazawa, *Some theorems on conformally symmetric spaces*, Tensor, N.S. **32** (1978), 24–26.

10. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

11. W. Roter, *On conformally symmetric* 2 *Ricci-recurrent spaces*, Colloq. Math. **26** (1972), 115–122.

12. P.J. Ryan, *A class of complex hypersurfaces*, Colloq. Math. **26** (1972), 175–182.

13. U. Simon, *Compact conformally Riemannian spaces*, Math. Z. **132** (1973), 173–177.

14. S. Tanno, *Curvature tensors and covariant derivative*, Annal. Math. Pura Appl. **96** (1973), 233–241.

15. H. Weyl, *Reine Infinitesimalgeometrie*, Math. Z. **26** (1918), 384–411.

16. ———, *Zur Infinitesimalgeometriae*: *Einordnung der projecktiven und konformen Auffassung*, Göttigen Nachr., 1921, pp. 99–112.

17. K. Yano, *The theory of Lie derivatives and its applications*, North-Holland, Amsterdam, 1957.

18. K. Yano and S. Bochner, *Curvature and Betti numbers*, Ann. Math. Studies No. 32, Princeton University Press, 1953.

19. K. Yano and M. Kon, *Structures on manifolds*, World Scientific, Singapore, 1984.

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