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A THEOREM OF KREIN REVISITED

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ABSTRACT. M. Krein proved in [**9**] that if T is a continuous operator on a normed space leaving invariant an open cone, then its adjoint T^* has an eigenvector. We present generalizations of this result as well as some applications to C^*-algebras, operators on l_1, operators with invariant sets, contractions on Banach lattices, the Invariant Subspace Problem, and von Neumann algebras.

1. Introduction. M. Krein proved in [**9**, Theorem 3.3] that if T is a continuous operator on a normed space leaving invariant a nonempty open cone, then its adjoint T^* has an eigenvector. Krein's result has an immediate application to the Invariant Subspace problem because of the following observation. If T is a bounded operator on a Banach space and not a multiple of the identity, and $T^*f = \lambda f$, then the kernel of f is a closed nontrivial subspace of codimension 1 which is invariant under T. Moreover, $\overline{\mathrm{Range}\,(\lambda I - T)}$ is a closed nontrivial subspace which is proper (it is contained in the kernel of f) and hyperinvariant for T; that is, it is invariant under every operator commuting with T.

Several proofs and modifications of Krein's theorem appear in the literature, see, e.g. [**3**, Theorems 6.3 and 7.1] and [**12**, p. 315]. We prove yet another version of Krein's theorem: if T is a positive operator on an ordered normed space in which the unit ball has a dominating point, then T^* has a positive eigenvector. We deduce the original Krein's version of the theorem from this, as well as several applications and related results.

In particular, we show that if a bounded operator T on a Banach space satisfies any of the following conditions, then T^* has an eigenvector. Moreover, if the condition holds for a commutative family of operators, then the family of the adjoint operators has a common eigenvector.

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- T leaves invariant a cone with an interior point;
- T is a positive operator on a unital C^*-algebra;
- T is an operator on l_1 such that entries of its matrix satisfy $t_{kk} \pm t_{kj} \geq \sum_{i \neq k} |t_{ik} \pm t_{ij}|$ for some k and for all $j \neq k$;
- T leaves invariant a convex set whose interior is non-void and doesn't contain zero;
- T is a contraction with a fixed point;
- T is a positive contraction on a Banach lattice and $Te > e$ for some $e > 0$.

We also show that under the last condition T has a closed invariant order ideal. Finally, we prove a noncommutative version of this result for rearrangement invariant operator spaces arising from von Neumann algebras.

Throughout the paper X denotes a real or complex normed space, X^* the dual of X, T a bounded linear operator on X, and B_X the closed unit ball of X.

Definition 1. We call a subset K of a normed space X a *cone* if K is closed under addition and nonnegative scalar multiplication, and there exists a nonzero vector $x \in K$ such that $-x \notin K$.

Our definition of a cone is a most general one. In the literature such objects are sometimes called *wedges*, while for a cone it is often assumed in addition that $x \in K$ implies $-x \notin K$ for *every* nonzero x. This additional condition ensures that the relation on X defined via "$x \leq y$ if and only if $y - x \in K$" is a linear order relation, and, vice versa, every linear order relation defines a cone satisfying this condition, namely, the cone X_+ of all nonnegative elements. We will still use the symbol "$\leq$," even though in our case $x > 0$ and $x < 0$ may happen simultaneously. However, this does not create any problems, and, naturally, everything we do is still valid for the more restrictive definitions of a cone. See [**9**] for a discussion on definitions and properties of cones.

Given a closed cone K in a normed space X, we will call X an *ordered normed space* with respect to the (semi)order relation determined by K. Notice that K coincides with the cone X_+ of all nonnegative elements of X. A linear operator is said to be *positive* if $T(X_+) \subseteq X_+$. For $f \in X^*$ we write $f \geq 0$ or $f \in X_+^*$ if $f(x) \geq 0$ whenever $x \geq 0$. Clearly, X_+^* is a w^*-closed cone in X^*. It can be easily verified that if T is a positive bounded operator on X then T^* is a positive operator on X^*, that is, $T^*(X_+^*) \subseteq X_+^*$. It is known (see, e.g. [**9**]) that if K is a closed cone, then K and $-K$ can be (nonstrictly) separated by a continuous functional, or, equivalently, there exists a nontrivial positive functional in X^*.

Lemma 2. *Suppose that X is a real normed space and $e \in X$ with $\|e\| = 1$. If $f \in X^*$ then $f(e) = \|f\|$ if and only if $f(x) \leq f(e)$ for all $x \in B_X$.*

Proof. If $f(e) = \|f\|$, then $f(x) \leq |f(x)| \leq \|f\|\|x\| = f(e)$ whenever $\|x\| = 1$. Conversely, suppose $f(x) \leq f(e)$ for all x of norm one. Since $-f(x) = f(-x) \leq f(e)$, we have $|f(x)| \leq f(e)$, so that $\|f\| \leq f(e)$. Finally, $f(e) = |f(e)| \leq \|f\|$. $\square$

Definition 3. If X is an ordered normed space and $e \in B_X$, we say that e *dominates the unit ball of* X if $x \leq e$ for all $x \in B_X$. We then write $B_X \leq e$.

In this case it follows immediately from Lemma 2 that every positive functional attains its norm on e. In the proof of the following theorem we use techniques developed in the proof of a special case of Krein's theorem in [**2, 3**].

Theorem 4. *Suppose that X is an ordered real normed space and $e \in X$ such that $\|e\| = 1$ and $B_X \leq e$. If T is a positive operator on X then T^* has a positive eigenvector. Moreover, if Γ is a commutative family of positive operators on X, then their adjoints have a common positive eigenvector.*

Proof. Let $\mathcal{S} = \{f \in X_+^* : f(e) = 1\}$. Since $\mathcal{S} = X_+^* \cap \{f \in X^* : f(e) = 1\}$ then $\mathcal{S}$ is w^*-closed. Furthermore, if $f \in \mathcal{S}$ then $\|f\| = f(e) = 1$ by Lemma 2, so that $\mathcal{S} \subseteq B_X$, hence is w^*-compact. For $T \geq 0$ and $f \in \mathcal{S}$

we define

$$F_T(f) = \frac{f + T^*f}{[f + T^*f](e)} = \frac{f + T^*f}{1 + (T^*f)(e)}. \tag{1}$$

Since $T^* \geq 0$ then $F_T(f) \geq 0$. Clearly, $\big[F_T(f)\big](e) = 1$, so that $F_T(f) \in \mathcal{S}$; hence $F_T\colon \mathcal{S} \to \mathcal{S}$.

It can be easily verified that F_T is w^*-to-w^*-continuous. Indeed, if $f_\alpha \xrightarrow{w^*} f$, then for all $x \in X$ we have

$$[F_T(f_\alpha)]\,(x) = \frac{f_\alpha(x)+(T^*f_\alpha)(x)}{1 + (T^*f_\alpha)(e)} \to \frac{f(x)+(T^*f)(x)}{1 + (T^*f)(e)} = [F_T(f)]\,(x) \tag{2}$$

because T^* is w^*-to-w^*-continuous.

By the Fixed Point theorem there exists $h \in \mathcal{S}$ such that $F_T(h) = h$, i.e., $(h + T^*h)/\big(1 + (T^*h)(e)\big) = h$ so that $T^*h = \big((T^*h)(e)\big)h$; hence, h is an eigenvector of T^*.

Let Γ be a commutative family of positive operators on X. For $T \in \Gamma$ denote A_T the set of the fixed points of F_T in $\mathcal{S}$. It can be easily verified that $f \in \mathcal{S}$ belongs to A_T if and only if f is an eigenvector of T^*. Clearly, A_T is w^*-closed, hence w^*-compact. We claim that $\{A_T\}_{T\in\Gamma}$ has the finite intersection property; this would imply that it has nonempty intersection, and, therefore, the family $\{T^*\}_{T\in\Gamma}$ has a common eigenvector in $\mathcal{S}$. We prove the claim by induction on the size of the set. Suppose that $\bigcap_{T\in\Gamma_0} A_T \neq \varnothing$ for every n-element subset $\Gamma_0 \subseteq \Gamma$. Let Γ_0 be an n-element subset of Γ and $S \in \Gamma$, show that $\bigcap_{T\in\Gamma_0\cup\{S\}} A_T \neq \varnothing$. Pick $f \in \bigcap_{T\in\Gamma_0} A_T$; then for each T in Γ_0 there exists $\lambda_T \geq 0$ such that $T^*f = \lambda_T f$. Let $C_T = \ker(\lambda_T I - T^*) \cap \mathcal{S}$, then C_T is a convex w^*-closed subset of A_T, hence w^*-compact. It follows that $C = \bigcap_{T\in\Gamma_0} C_T$ is convex and w^*-compact. Furthermore, $C \neq \varnothing$ as $f \in C$. If $T \in \Gamma_0$ and $h \in C_T$, then

$$T^*F_S(h) = \frac{T^*h + T^*S^*h}{1 + (S^*h)(e)} = \frac{\lambda_T h + S^*(\lambda_T h)}{1 + (S^*h)(e)} = \lambda_T F_S(h), \tag{3}$$

so that $F_S(h) \in C_T$. It follows that $F_S(C_T) \subseteq C_T$, so that $F_S(C) \subseteq C$. The Fixed Point theorem implies that F_S has a fixed point in C; hence $A_S \cap C \neq \varnothing$. Since $C \subseteq \bigcap_{T\in\Gamma_0} A_T$, this proves the claim. $\square$

Theorem 5. *If T is a continuous operator on a real normed space, leaving invariant a cone with an interior point, then T^* has a positive eigenvector. Moreover, a commutative collection of such operators has a common positive eigenvector.*

Proof. Let Γ be a commutative family of bounded operators on X, C a cone in X such that $T(C) \subseteq C$ for each $T \in \Gamma$, and e an interior point of C. Without loss of generality, C is closed, $\|e\| > 1$ and $e + B_X \subseteq C$. Let C_0 be the cone spanned by $e + B_X$, that is, $C_0 = \{\alpha(e + x) : \alpha \geq 0, \|x\| \leq 1\}$. Put $W = (C_0 - e) \cap (e - C_0)$.

Note that $e + B_X \subseteq C_0$ so that $B_X \subset C_0 - e$. Also, $B_X = -B_X \subseteq e - C_0$, so that $B_X \subseteq W$. Furthermore, W is bounded. Indeed, if $w \in W$ then $w = \alpha_1(e + x_1) - e = e - \alpha_2(e + x_2)$ for some $\alpha_1, \alpha_2 > 0$ and $x_1, x_2 \in B_X$. It follows that $\alpha_1 x_1 + \alpha_2 x_2 = \big(2 - (\alpha_1 + \alpha_2)\big)e$. Thus, $\big|2 - (\alpha_1 + \alpha_2)\big|\|e\| = \|\alpha_1 x_1 + \alpha_2 x_2\| \leq \alpha_1 + \alpha_2$. If $\alpha_1 + \alpha_2 > 2$ then $\big((\alpha_1 + \alpha_2) - 2\big)\|e\| - (\alpha_1 + \alpha_2) \leq 0$, so that $\alpha_1 + \alpha_2 \leq (2\|e\|)/(\|e\| - 1)$. It follows that $\alpha_1 \leq \alpha_1 + \alpha_2 \leq \max\big\{2, (2\|e\|)/(\|e\| - 1)\big\}$. Finally, since $\|w\| \leq \alpha_1(\|e\| + 1) + \|e\|$, it follows that W is bounded. Thus, W is the unit ball of a norm, which is equivalent to the original norm of X. In the new norm, e will be of norm one. Finally, e dominates W with respect to the order defined by C. Now apply Theorem 4. $\square$

Remark 6. One can easily see that Theorem 5 is equivalent to the original theorem of Krein. Indeed, if T leaves invariant a nonempty open cone, then Theorem 5 states that T^* has an eigenvector.

Conversely, suppose that T leaves invariant a cone with an interior point. Let x be an interior point of the cone; then $f(x+Tx) \geq f(x) > 0$ for every positive functional $f \neq 0$, so that $x + Tx$ is again an interior point of the cone. It follows that $I + T$ leaves invariant the interior of the cone, so that $(I + T)^*$ has an eigenvector by Krein's theorem. This yields the existence of an eigenvector for T^*.

Next, we discuss some applications of Theorems 4 and 5.

Recall that an element e in a Banach lattice E is called a *strong unit* if for every positive $x \in E$ there exists a natural number n such that $x \leq ne$. It is known (see [**4**, p. 188] for details) that a Banach lattice with a strong unit is an AM-space with unit up to an equivalent norm. But in an AM-space the unit dominates the unit ball. Therefore, Theorem 4 yields the following result.

Corollary 7. *The adjoint of a positive operator on a Banach lattice with a strong unit has a positive eigenvector.*

In particular, the adjoint of a positive operator on a $C(\Omega)$ space, where Ω is a compact Hausdorff space, has a positive eigenvector. A direct proof of this fact can also be found in [**2**].

The case of complex normed spaces can often be reduced to the real case as follows. Suppose that X_c is a complexification of a real ordered normed space X, every element of X_c can be written in the form $x+iy$ for some $x, y \in X$. If T is a positive operator on X, then its complexification $T_c\colon X_c \to X_c$ defined by $T_c(x+iy) = Tx + iTy$ will be referred to as a positive operator on X_c. Notice that T coincides with the restriction of T_c to X. Suppose that $T^*f = \lambda f$ for some $f \in X^*$ and $\lambda \in \mathbf{R}$, then we can extend f to a continuous linear functional f_c on X_c via $f_c(x+iy) = f(x) + if(y)$. Then $T_c^* f_c = \lambda f_c$. Indeed,

$$\begin{aligned}
[T_c^* f_c](x+iy) &= f_c\big(T_c(x+iy)\big) = f_c(Tx+iTy) = f(Tx)+if(Ty) \\
&= (T^*f)(x)+i(T^*f)(y) = \lambda f(x)+i\lambda f(y) = \lambda f_c(x+iy).
\end{aligned} \tag{4}$$

Thus, Theorems 4 and 5 are applicable to complex normed spaces.

For example, we can apply our technique to positive operators on C^*-algebras. A C^*-algebra $\mathcal{A}$ can be viewed as the complexification of the real Banach space $\mathcal{A}_{\mathrm{sa}}$ of its self-adjoint elements. Recall that a self-adjoint element a in $\mathcal{A}$ is positive if $\sigma(a) \subset \mathbf{R}_+$. If $\mathcal{A}$ has unit e and x is a self-adjoint element of $\mathcal{A}$ such that $\|x\| \leq 1$, then the Spectral Mapping theorem implies that $\sigma(e-x) \subseteq [0,2]$; hence $x \leq e$. It follows that e dominates the unit ball of $\mathcal{A}_{\mathrm{sa}}$. Theorem 4 immediately yields the following result.

Corollary 8. *If T is a positive operator on a unital C^*-algebra, then T^* has a positive eigenvector.*

Let $(e_j)_{j=1}^{\infty}$ denote the standard unit basis of $X = \ell_1$, while $(e_i^*)_{i=1}^{\infty}$ stands for the dual basis of X^*. Recall that every bounded operator T on ℓ_1 can be written as an infinite matrix with entries $t_{ij} = \langle e_i^*, Te_j \rangle$.

Theorem 9. *Suppose that T is a bounded operator on ℓ_1 with matrix (t_{ij}), and suppose that there exists an index k such that*

$$t_{kk} \pm t_{kj} \geq \sum_{i \neq k} |t_{ik} \pm t_{ij}| \tag{5}$$

for each $j \neq k$. Then T^* has a positive eigenvector.

Proof. Without loss of generality $k = 1$. Let K be the cone spanned by $e_1 + B_X$. It is easy to see that K is spanned by the set $\{e_1 \pm e_i\}_{i=2}^{\infty}$. We claim that $K = \{(x_i) \mid x_1 \geq \sum_{i=2}^{\infty} |x_i|\}$. Indeed, it is easy to see that the later set is closed under addition and positive scalar multiplication; hence it is a cone. Furthermore, it contains $e_1 \pm e_i$ for each $i \geq 2$, so that it contains K. Finally, if a nonzero sequence (x_i) satisfies $x_1 \geq \sum_{i=2}^{\infty} |x_i|$ then $x/x_1 - e_1 \in B_X$, so that (x_i) is contained in K. Clearly, e_1 dominates the unit ball of X with respect to the order induced by K. The condition $t_{11} \pm t_{1j} \geq \sum_{i=2}^{\infty} |t_{i1} \pm t_{ij}|$ means that

$$\begin{aligned} T(e_1 \pm e_j) = Te_1 \pm Te_j &= (\text{the 1st column of } T) \\ &\pm (\text{the } j\text{-th column of } T) \in K \end{aligned} \tag{6}$$

for every $j \geq 2$; it follows that $T(K) \subseteq K$. Theorem 4 finishes the proof. $\square$

Example 10. Let K be as in the preceding proof, and let $\mathcal{C}$ be the set of all operators on ℓ_1 preserving K. Clearly, the adjoint of every operator in $\mathcal{C}$ has an eigenvector. By construction, $\mathcal{C}$ is itself a cone and a multiplicative semi-group in $\mathcal{L}(\ell_1)$. It is easy to see that $\mathcal{C}$ is closed in the strong operator topology (and, being a convex set, it is also closed in the weak operator topology). Finally, we claim that $\mathcal{C}$ has nonempty interior with respect to the norm topology of $\mathcal{L}(\ell_1)$. For example, put $S = (s_{ij})$ such that s_{ij} equals 1 if $i = j = 1$ and 0 otherwise. We claim that S is an interior point of $\mathcal{C}$. Indeed, suppose that $R = (r_{ij})$ such that $\|R\| < 1/5$, and let $T = S + R$. Show that

$T \in \mathcal{C}$. Note that $\sum_{i=1}^{\infty} |r_{ij}| = \|Re_j\| < 1/5$ for every $j \geq 1$. It follows that $t_{11} \pm t_{1j} = 1 + r_{11} \pm r_{1j} \geq 1 - 1/5 - 1/5 = 3/5$ for every $j > 1$. On the other hand,

$$\sum_{i=2}^{\infty} |t_{i1} \pm t_{ij}| = \sum_{i=2}^{\infty} |r_{i1} \pm r_{ij}| \leq \sum_{i=2}^{\infty} |r_{i1}| + \sum_{i=2}^{\infty} |r_{ij}| < \frac{2}{5}. \tag{7}$$

Hence, T satisfies (5) and, therefore, $T \in \mathcal{C}$.

Corollary 11. *If T is an operator on a real Banach space leaving invariant a convex set whose interior is nonvoid and doesn't contain zero, then T^* has an eigenvector. Moreover, a commutative collection of such operators has a common eigenvector.*

Proof. Apply Theorem 5 to the cone generated by the invariant set. □

Krein's theorem gives a natural insight and provides a simple solution to Exercise 7.5.10 of [**7**], even though at first glance the statement seems to have no connection to order structures.

Proposition 12. [**7**, Exercise 7.5.10]. *If $\|T\| = 1$ and T has a nonzero fixed point, then T^* has an eigenvector.*

Proof. Suppose that $\|T\| = 1$ and $Te = e$ for some e of norm one. Then the set $e + B_X$ is invariant under T and so is the cone generated by this set. Clearly, this cone is proper and has a nonvoid interior. Now apply Theorem 5. □

This approach can be generalized as follows.

Definition 13. If X is an ordered normed space, we say that it has *monotone norm* if $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$.

Theorem 14. *Suppose that T is a positive operator on an ordered normed space with monotone norm such that $\|T\| = 1$ and $Te \geq e$ for some $e > 0$. Then T^* has a positive eigenvector.*

Proof. Without loss of generality we can assume $\|e\| = 1$. Since the norm is monotone, we have $(e + X_+) \cap B_X^\circ = \varnothing$, where B_X° stands for the open unit ball of X. Hence, $X_+ \cap (B_X^\circ - e) = \varnothing$, so that the two sets can be separated by a positive functional f. Then f is nonnegative on $e - B_X$. Let K be the closed cone generated by X_+ and $e - B_X$. Since f is nonnegative on K, it is, indeed, a proper cone.

If $x \in B_X$ then $e - x \in K$, so that e dominates B_X in the (semi)order induced on X by K. It is given that $T(X_+) \subseteq X_+ \subseteq K$. Furthermore, if $x \in B_X$ then $Tx \in B_X$, and we have $T(e-x) = (Te-e)+(e-Tx) \in X_+ + (e - B_X) \subseteq K$, so that $T(e - B_X) \subseteq K$. It follows that $T(K) \subseteq K$. Now apply Theorem 4 to the order induced by K. ◻

Notice that the condition $\|T\| = 1$ in Proposition 12 cannot be dropped. Indeed, for any $\alpha > 1$, let T be α times the left shift on ℓ_p, $1 \leq p < \infty$, that is, $T(x_1, x_2, x_3, \dots) = (\alpha x_2, \alpha x_3, \dots)$. Then $\|T\| = \alpha$ and $(1, \alpha^{-1}, \alpha^{-2}, \dots)$ is a fixed point of T. Nevertheless, T^* clearly has no eigenvectors.

It follows immediately that under the hypothesis of Theorem 14 the operator T has an invariant subspace of codimension one. In fact, we will show that if $Te > e$ then there is a closed face of the positive cone of X which is invariant under T. Recall that $E \subset X_+$ (X_+ is the positive cone of X) is called a face of X_+ if E is itself a closed cone, and, for $x_1, x_2 \in X_+$, $x_1 + x_2 \in E$ implies $x_1, x_2 \in E$. One can easily see that a closed cone $E \subset X_+$ is a face of X_+ if and only if it is hereditary, that is, $x \in E$ whenever $0 \leq x \leq y$ and $y \in E$.

Theorem 15. *Suppose that X is an ordered normed space with monotone norm and T is a positive operator on X such that $\|T\| = 1$ and $Te > e$ for some $e > 0$. Then there exists a nontrivial closed face E of the positive cone of X which is invariant under T. Moreover, if X is a Banach lattice, then $E - E$ is closed nontrivial ideal in X, invariant under T.*

Proof. Without loss of generality, $\|e\| = 1$. Let

$$E = \{x \geq 0 : \lim_{\alpha\to 0^+} (\|e + \alpha x\| - 1)/\alpha = 0\}. \tag{8}$$

Note that if $x \in E$ and $0 \leq y \leq x$ then $y \in E$. Note also that E is nontrivial as the positive vector $Te - e \in E$ because, for $\alpha \in (0, 1)$,

$$\begin{aligned} 1 = \|e\| &\leq \|e + \alpha(Te-e)\| \leq \|(1-\alpha)e + \alpha Te\| \\ &\leq (1-\alpha)\|e\| + \alpha\|Te\| = 1. \end{aligned} \tag{9}$$

Furthermore, E is T-invariant. Indeed, suppose $\alpha > 0$ and $x \in E$. Then

$$\|e + \alpha Tx\| \leq \|Te + \alpha Tx\| \leq \|e + \alpha x\|. \tag{10}$$

Therefore,

$$\lim_{\alpha\to 0^+} (\|e + \alpha Tx\| - 1)/\alpha \leq \lim_{\alpha\to 0^+} (\|e + \alpha x\| - 1)/\alpha = 0. \tag{11}$$

It is easy to see that E is a cone. Indeed, if $x, y \in E$, then $cx \in E$ for $c > 0$, and

$$\|e + \alpha(x + y)/2\| - 1 \leq \frac{1}{2}\big((\|e + \alpha x\| - 1) + (\|e + \alpha y\| - 1)\big) = o(\alpha) \tag{12}$$

as α approaches 0. Thus, $x + y \in E$.

To show that E is closed, suppose x_i is a sequence of positive elements in E, converging to x in norm. We shall show that $x \in E$. Fix $\varepsilon > 0$. It suffices to prove that, whenever $\alpha > 0$ is sufficiently small, the inequality $\|e + \alpha x\| \leq 1 + \varepsilon\alpha$ is satisfied. Find i for which $\|x - x_i\| < \varepsilon/2$. There exists α_0 such that $\|e + \alpha x_i\| \leq 1 + \varepsilon\alpha/2$ whenever $0 < \alpha < \alpha_0$. Thus, for $\alpha \in (0, \alpha_0)$,

$$\|e + \alpha x\| \leq \|e + \alpha x_i\| + \alpha\|x - x_i\| \leq (1 + \varepsilon\alpha/2) + \varepsilon\alpha/2 = 1 + \varepsilon\alpha. \tag{13}$$

Finally, $e \notin E$; hence E is a nontrivial face of the positive cone of X.

Next, suppose that X is a Banach lattice, and put $F = E - E$. Clearly, F is an order ideal, that is, F is a linear subspace such that $x \in F$ and $|y| \leq |x|$ imply $y \in F$. Show that F is closed. Suppose $z \in \overline{F}$, and

(x_i), (y_i) are sequences in E such that $\lim_i \|z - (x_i - y_i)\| = 0$. Then $\lim_i \|z_+ - (x_i - y_i)_+\| = 0$. Let $a_i = (x_i - y_i)_+ \wedge z_+$. By the above, $\lim_i \|a_i - z_+\| = 0$. Note that

$$0 \le a_i \le (x_i - y_i)_+ \le |x_i| + |y_i| \in E, \tag{14}$$

hence $a_i \in E$. But E is closed, thus $z_+ \in E$. Similarly, $z_- \in E$, and therefore, $z \in F$.

Finally we prove that F is nontrivial. More precisely, we show that $e \notin F$. Indeed, suppose there exist $x, y \in E$ such that $\|e - (x - y)\| \le 1/3$. Then $(x - y)_+ \le |x| + |y| \in E$, so $(x - y)_+ \in E$.

Pick $\alpha > 0$ for which $\|e + \alpha(x - y)_+\| \le 1 + \alpha/3$. Then

$$\begin{aligned} 1 + \alpha = \|e + \alpha e\| &= \big\|e + \alpha(x-y)_+ + \alpha\big(e - (x-y)_+\big)\big\| \\ &\le \big\|e + \alpha(x-y)_+\big\| + \alpha\big\|e - (x-y)_+\big\| \\ &\le 1 + \frac{\alpha}{3} + \alpha\big\|e - (x-y)\big\| = 1 + \frac{2\alpha}{3}, \end{aligned} \tag{15}$$

a contradiction. □

Similar results hold for rearrangement invariant operator spaces, arising from von Neumann algebras. For the benefit of the reader, we give a brief introduction into this natural noncommutative generalization of Banach lattices.

Suppose N is a von Neumann algebra on a Hilbert space H, equipped with a faithful normal semifinite trace τ. Following [**11**], we say that a closed, densely defined linear operator x on H is *affiliated with* N if $u^* x u = x$ for every unitary $u \in N'$ (the commutant of N). An operator x is called *τ-measurable* if, for every $\varepsilon > 0$ there exists a (self-adjoint) projection $p \in N$ such that $p(H) \subset D(x)$ and $\tau(1 - p) < \varepsilon$ (1 is the identity in N). The set of all τ-measurable operators is denoted by $\tilde{N}$.

Following [**8**], we introduce for $x \in \tilde{N}$ the *generalized eigenvalue function* $\mu(\cdot, x) : [0, \infty) \to [0, \infty)$, defined by

$$\mu(t, x) = \inf\{s \ge 0 : \tau(\chi_{(s,\infty)}(|x|)) \le t\}. \tag{16}$$

Equivalently (see [**8**]), we have

$$\mu(t, x) = \inf\{\|xp\| : p \in N \text{ a projection},\ \tau(1 - p) \le t\}. \tag{17}$$

Following [**6**], we call a linear manifold $G \subset \tilde{N}$, equipped with the norm $\|\cdot\|$, a (*normed*) *rearrangement invariant operator space*, (r.i.o.s., for short), if whenever $x \in G$, $y \in \tilde{N}$, and $\mu(t,y) \leq \mu(t,x)$ for every t, then $y \in G$ and $\|y\| \leq \|x\|$. E is called *symmetric* if, in addition, $\|y\| \leq \|x\|$ whenever

$$\int_0^a \mu(t,y)\,dt \leq \int_0^a \mu(t,x)\,dt \tag{18}$$

for every $a > 0$.

To underscore the connections between r.i.o.s. and Banach lattices, consider the commutative case of $N = L_\infty(I)$, where I is an interval $(0,a)$, $a \in (0,\infty]$. By Proposition 2.a.8 of [**10**], any r.i.o.s. G which satisfies

$$L_1(I) \cap L_\infty(I) \subset G \subset L_1(I) + L_\infty(I) \tag{$*$}$$

is symmetric. We say that G has the *Fatou property* if, whenever $f \in G$, (f_n) is a sequence of nonnegative elements of G, and $f_n(\omega) \nearrow f(\omega)$, then $\|f_n\| \to \|f\|$.

Suppose N is a von Neumann algebra with a normal faithful semifinite trace τ, and G is as in the previous paragraph, with $I = (0, \tau(1))$.

Following [**6**], we define the space $G(N) = \{x \in \tilde{N} \mid \mu(\cdot,x) \in G\}$, equipped with the norm $\|x\|_{G(N)} = \|\mu(\cdot,x)\|_G$. If G satisfies $(*)$, then $N \cap N_* \subset G(N) \subset N + N_*$. We identify $L_\infty(N)$ with N itself, and $L_1(N)$ with N_*, the predual of N. If, in addition, G has Fatou property, then $G(N)$ is norm closed (see Proposition 1.7 and Corollary 2.4 of [**6**]).

If $G \subset N + N_*$ is a r.i.o.s., we denote by G_+ the set of positive elements in G, i.e. $G \cap \tilde{N}_+$. Then every self-adjoint element in G can be represented as a difference of two positive ones (see [**5**] and [**6**]). Moreover, every element $x \in G$ can be written as $x = x_1 - x_2 + i(x_3 - x_4)$, with $x_j \in G_+$. Finally, the trace τ extends naturally to $(N + N_*)_+$ by setting $\tau(x) = \int_0^\infty \mu(t,x)\,dt$ for $x \geq 0$.

Theorem 16. *Suppose N is a von Neumann algebra with a faithful normal semifinite trace τ, G is a norm closed symmetric rearrangement invariant subspace of $\tilde{N}$ satisfying $N \cap N_* \subset G \subset N + N_*$, and $T : G \to G$ is a positive contraction such that $Te > e$ for some positive $e \in G$.*

Then T has an invariant nontrivial face E of the positive cone of G Moreover, $\overline{E-E}$ is a nontrivial closed subspace of G, invariant under T.

To prove the theorem, we need to collect some facts related to conditional expectations on von Neumann algebras. Suppose N is a von Neumann algebra equipped with a normal faithful semifinite trace τ, and M is a Neumann subalgebra of N such that the restriction of τ to M is semifinite. Then (see Proposition 5.2.36 of [**13**]), there exists a positive contractive projection Φ from N onto M such that $\Phi(abc) = a\Phi(b)c$ and $\tau(\Phi(ab)) = \tau(a\Phi(b))$ whenever $a, c \in N_*$ and $b \in N$. Moreover, it follows from the proof that, for any $x \in N \cap N_*$, $\Phi(x) \in M \cap M_*$ and $\|\Phi(x)\|_{M_*} \leq \|x\|_{N_*}$. Since $N \cap N_*$ (or $M \cap M_*$) is dense in N_*, respectively, M_*, Φ can be extended to a contraction from N_* to M_*. Thus, Φ can be thought of as an operator from $N + N_*$ to $M + M_*$ respectively, which maps N to M and N_* to M_* contractively.

Lemma 17. *Suppose N, M and τ are as above, and G is a symmetric r.i.o.s. with $N \cap N_* \subset G \subset N + N_*$. Then Φ maps G into $G \cap \tilde{M}$, and $\|\Phi(x)\|_G \leq \|x\|_G$ for any $x \in G$.*

Proof. As noted above, Φ acts contractively from N to M and from N_* to M_*. For $x \in G$ we have, by Theorem 4.7 of [**6**], $\int_0^a \mu(t, \Phi(x))\,dt \leq \int_0^a \mu(t, x)\,dt$ for any $a > 0$. Thus, $\Phi(x) \in G$, and $\|\Phi(x)\| \leq \|x\|$. □

Proof of Theorem 16. Suppose $e \in G_+$, $\|e\| = 1$, and $T : G \to G$ is a positive operator such that $Te > e$.

Let

$$E = \{x \geq 0 : \lim_{\alpha \to 0^+} (\|e + \alpha x\| - 1)/\alpha = 0\}. \tag{19}$$

As in the proof of Theorem 15, we can show that E is a closed nontrivial face of G_+ (E is nonempty, and $e \notin E$). Moreover, E is invariant under T. Therefore the closed linear span of E is invariant under T. It remains to show that e does not belong to the closed linear span of E. It suffices to show that, whenever x_1, $x_2 \in E$, we have $\|e + x_1 - x_2\| \geq 1/6$.

First suppose that either $\tau(1) < \infty$, or $\lim_{t\to\infty} \mu(t,e) = 0$. Then there exists a commutative von Neumann algebra M such that $e \in \tilde{M}$ and the restriction of τ to M is semifinite. Indeed, if $\tau(1) < \infty$, we can consider the von Neumann algebra generated by projections $\chi_{(a,\infty)}(e)$, where $a > 0$. If $\lim_{t\to\infty} \mu(t,e) = 0$, observe that $\tau(\chi_{(a,\infty)}(e)) < \infty$ for any $a > 0$, and let $p = \sup_{a>0} \chi_{(a,\infty)}(e)$. Use Zorn's lemma to find mutually orthogonal projections $(p_i) \in N$ such that $\tau(p_i) < \infty$ and $\sum_i p_i = 1 - p$. Then let M be the von Neumann algebra generated by projections $\chi_{(a,\infty)}(e)$ and p_i. Clearly M satisfies our conditions.

Let Φ be the conditional expectation from N onto M. By Lemma 17, Φ acts as a contraction from G to $G_1 = G \cap \tilde{M}$. Then G_1 can be regarded as a Banach lattice.

Let

$$E_1 = \{x \in G_1 : x \geq 0,\ \lim_{\alpha\to 0^+} (\|e + \alpha x\| - 1)/\alpha = 0\}. \tag{20}$$

As in the proof of Theorem 15, $\|e + x - y\| \geq 1/3$ whenever $x, y \in E_1$.

However, $\Phi(E) \subset E_1$, and therefore

$$\|e + x - y\| \geq \|e + \Phi(x) - \Phi(y)\| \geq \frac{1}{3} \tag{21}$$

whenever $x, y \in E$.

The case of $a = \lim_{t\to\infty} \mu(t,e) > 0$ is more complicated. Note that $\|a1\|_G \leq \|e\| = 1$, hence $\|x\|_G \leq \|x\|_N \|1\|_G \leq \|x\|_N/a$ for any $x \in N$. Let $k = \lceil 6/a \rceil$, $p_i = \chi_{[ia/k,(i+1)a/k)}(e)$ for $0 \leq i \leq k-1$, $p_k = \chi_{[(k-1)a/k,a]}(e)$, and $e_1 = \chi_{(a,\infty)}(e)e + \sum_{i=1}^k (i/k)ap_i$. Then $e \geq e_1$, $e - e_1 \in N$, and

$$\|e - e_1\|_G \leq \|e - e_1\|_N/a \leq 1/6. \tag{22}$$

By definition, $\mu(t,e) = \mu(t,e_1)$ for any t. Moreover, a projection p_i can be represented as $p_i = \sum_j q_{ij}$, where projections q_{ij} are mutually orthogonal and $\tau(q_{ij}) < \infty$. Note also that $\tau(\chi_{(b,\infty)}(e)) < \infty$ whenever $b > a$, and $\chi_{(a,\infty)}(e) = \sup_{b>a} \chi_{(b,\infty)}(e)$.

Consider the (commutative) von Neumann algebra M, generated by projections q_{ij} and $\chi_{(b,\infty)}(e)$, $b > a$. Then $e_1 \in G_1 = G \cap \tilde{M}$. Let

$$E_1 = \{x \in G_1 : x \geq 0,\ \lim_{\alpha\to 0^+} (\|e_1 + \alpha x\| - 1)/\alpha = 0\}. \tag{23}$$

As above, we show that $\|e_1 + x - y\| \geq 1/3$ if $x, y \in E_1$. However, $\Phi(e) \geq e_1$ (since Φ is positive), and therefore, $\|e_1 + \Phi(x)\| \leq \|e + x\|$ for any $x \in G$. Thus, $\Phi(E) \in E_1$ and, for any $x, y \in E$, we have

(24)

$$\|e+x-y\| \geq \|\Phi(e) + \Phi(x) - \Phi(y)\| \geq \|e_1 + \Phi(x) - \Phi(y)\| - \|e-e_1\| \geq \frac{1}{3} - \frac{1}{6}.$$

The proof is complete. $\square$

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