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# CHERN-SIMONS FORMS ASSOCIATED TO HOMOGENEOUS PSEUDO-RIEMANNIAN STRUCTURES

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ABSTRACT. Forms of Chern-Simons type associated to homogeneous pseudo-Riemannian structures are considered. The corresponding secondary classes are a measure of the lack of a homogeneous pseudo-Riemannian space to be locally symmetric. Explicit computations are done for some pseudo-Riemannian Lie groups and their compact quotients.

**1. Introduction.** The characterization by É. Cartan of Riemannian locally symmetric spaces as those Riemannian manifolds whose curvature tensor is parallel was extended by Ambrose and Singer in [**1**]. They proved that a complete simply connected Riemannian manifold is homogeneous if and only if it admits a $(1,2)$ tensor field $S$ satisfying certain equations. If $S=0$ then the manifold is Riemannian symmetric.

The purpose of the present paper is to provide forms of Chern-Simons type for each pseudo-Riemannian manifold $(M,g)$ endowed with a homogeneous pseudo-Riemannian structure $S$. This construction furnishes odd-dimensional differential forms of degree greater than 1, which are null if $S=0$. Under certain conditions, these forms are closed and define secondary classes. Each of such triples $(M,g,S)$ has thus a number of such differential forms, and roughly speaking (when the corresponding group of real cohomology of the manifold is nonzero), the more nonvanishing classes of that kind a manifold has, the less symmetric it is.

We give several examples of such forms on some Lie groups equipped with left-invariant metrics: The three-dimensional unimodular Lie groups, so having instances of Abelian, nilpotent, solvable and simple Lie groups; and the five-dimensional generalized Heisenberg group $H(1,2)$, which is nilpotent. Further, we consider the corresponding

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secondary classes of the compact quotients of the previous groups, identifying them in the real cohomology spaces of the quotient spaces. In [**6**], we also studied the oscillator group.

**2. Preliminaries.** Ambrose and Singer [**1**] proved that a connected, simply connected and complete Riemannian manifold $(M, g)$ is homogeneous if and only if there exists a $(1, 2)$ tensor field $S$ on $M$, called a homogeneous Riemannian structure, satisfying certain equations, see (2.1) below. In [**4**] we have extended that characterization to pseudo-Riemannian manifolds. Specifically, let $(M, g)$ be a connected $C^\infty$ pseudo-Riemannian manifold of dimension $n$ and signature $(k, n-k)$. Let $\nabla$ be the Levi-Civita connection of $g$ and $R$ its curvature tensor field. A *homogeneous pseudo-Riemannian structure* on $(M, g)$ is a tensor field $S$ of type $(1, 2)$ on $M$ such that the connection $\tilde{\nabla} = \nabla - S$ satisfies

$$\tilde{\nabla} g = 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0. \tag{2.1}$$

If $g$ is a Lorentzian metric ($k = 1$), we say that $S$ is a homogeneous Lorentzian structure. In [**4**] we proved that if $(M, g)$ is connected, simply connected and geodesically complete, then it admits a homogeneous pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.

Let $(P, M, G)$ be a principal fiber bundle over the $n$-dimensional $C^\infty$ manifold $M$. Let $\mathcal{I}^r(G)$ be the real vector space of $\mathrm{Ad}\,(G)$-invariant polynomials of degree $r$. Let $D$ be a connection in $P$, with connection 1-form $\omega$ and curvature form $\Omega = d\omega + \omega \wedge \omega$. Let $I \in \mathcal{I}^r(G)$ be an invariant polynomial. One can consider for each $r$ the $2r$-form $I(\Omega^r) = I(\Omega, \dots, \Omega)$, which is a $2r$-form on $P$ and projects to a (unique) $2r$-form on $M$, say again $I(\Omega^r)$. This form is closed and determines a cohomology class in $H^{2r}(M, \mathbf{R})$. Let $\widetilde{D}$ be another connection in $P$ with connection 1-form $\widetilde{\omega}$ and curvature form $\widetilde{\Omega}$. Consider the connection given, for a $t \in [0, 1]$, by $\omega_t = \tilde{\omega} + t(\omega - \tilde{\omega})$, with curvature form $\Omega_t = d\omega_t + \omega_t \wedge \omega_t$. Then we have the *transgression formula*

$$I(\Omega^r) - I(\widetilde{\Omega}^r) = dQ(\omega, \tilde{\omega}), \tag{2.2}$$

where

$$Q(\omega,\tilde{\omega}) := r\int_0^1 I(\omega-\tilde{\omega},\underbrace{\Omega_t,\dots,\Omega_t}_{r-1})\,dt. \tag{2.3}$$

The Chern-Simons $(2r-1)$-form $Q(\omega,\tilde{\omega})$ on $M$ defines, if $I(\Omega^r) = I(\widetilde{\Omega}^r)$, a secondary class.

**3. Chern-Simons forms associated to a homogeneous pseudo-Riemannian structure.** We consider the bundle of pseudo orthonormal frames $p\colon \mathcal{O}_{k,n-k}(M) \to M$ over the pseudo-Riemannian $n$-manifold $(M,g)$, where $g$ is a metric of signature $(k,n-k)$. We define $\mathrm{Ad}\,(O(k,n-k))$-invariant polynomial functions $f_1,\dots,f_n$ on the Lie algebra $\mathfrak{o}(k,n-k)$ by

$$f(t,X) = \det\,(tI+X) = \sum_{r=0}^{n} f_r(X)\,t^{n-r}, \qquad X\in\mathfrak{o}(k,n-k).$$

Let $\Omega$ be the curvature form of a connection $\omega$ in $\mathcal{O}_{k,n-k}(M)$. Then, for each $f_r$, $r=1,\dots,n$, there exists a unique closed $2r$-form $\upsilon_r$ on $M$ such that $p^*(\upsilon_r) = f_r(\Omega)$. One has $\det\,(I+\Omega) = p^*(1+\upsilon_1+\cdots+\upsilon_n)$, so having characteristic forms $\upsilon_r$ of degree $2r$, and a total form $\Upsilon(\mathcal{O}_{k,n-k}(M),\omega) = 1+\sum_{r=1}^{n}\upsilon_r$. The forms $f_r(\Omega)$ are the elementary symmetric functions $s_r(\Omega)$, $r=1,\dots,n$, of the eigenvalues of $\Omega$, so that $\det\,(I+\Omega) = 1+s_1(\Omega)+s_2(\Omega)+\cdots+s_n(\Omega)$. By using Newton's recursive formulas, one can further compute the functions $s_r(\Omega)$ in terms of the traces of the powers of $\Omega$ from the expressions

$$\begin{aligned}&\mathrm{tr}\,(\Omega^r) - s_1(\Omega)\,\mathrm{tr}\,(\Omega^{r-1}) + s_2(\Omega)\,\mathrm{tr}\,(\Omega^{r-2}) - \cdots\\ &\quad + (-1)^{r-1}s_{r-1}(\Omega)\,\mathrm{tr}\,(\Omega) + (-1)^r r\,s_r(\Omega) = 0, \qquad r=1,\dots,n,\end{aligned}$$

and since $\mathrm{tr}\,\Omega = 0$, we have after computation that

$$\det\,(I+\Omega) = 1-\frac{1}{2}\,\mathrm{tr}\,(\Omega^2)+\frac{1}{3}\,\mathrm{tr}\,(\Omega^3)+\frac{1}{4}\Big(\frac{1}{2}(\mathrm{tr}\,(\Omega)^2)^2-\mathrm{tr}\,(\Omega^4)\Big)+\cdots.$$

Now, we consider here the Levi-Civita connection $\nabla$ and the linear connection $\widetilde{\nabla} = \nabla - S$, with connection form $\tilde{\omega}$ and curvature form $\widetilde{\Omega}$, as in the previous section, where $S$ is a homogeneous pseudo-Riemannian structure on $(M,g)$, so that the general equation (2.2) can be written in this case as

$$s_r(\Omega) - s_r(\widetilde{\Omega}) = dQ(\omega,\tilde{\omega}). \tag{3.1}$$

If $s_r(\Omega) = s_r(\widetilde{\Omega})$, then $Q(\omega,\tilde{\omega})$ is closed, so determining a secondary class. In particular, if $r = 2, 3$, then this happens if $\operatorname{tr}(\Omega^r) = \operatorname{tr}(\widetilde{\Omega}^r)$. We shall denote by $Q^S_{2r-1}(M,g)$, or simply by $Q^S_{2r-1}$, the form $Q(\omega,\tilde{\omega})$ in (3.1).

**Definition 3.1.** Let $(M,g)$ be a pseudo-Riemannian manifold and let $S$ be a homogeneous pseudo-Riemannian structure on $M$. We shall call the forms $Q^S_{2r-1}(M,g)$, for each $3 \leq 2r-1 \leq \dim M$, *Chern-Simons forms of pseudo-Riemannian homogeneity*, or simply *forms of homogeneity*, on $(M,g,S)$. We shall call the corresponding real cohomology classes $[Q^S_{2r-1}](M,g)$ *secondary classes of pseudo-Riemannian homogeneity*, or simply *secondary classes of homogeneity*.

The case $r = 1$ in (3.1) is trivial, as the forms $\omega - \tilde{\omega}$, $\Omega$, and $\widetilde{\Omega}$ take values in $\mathfrak{o}(k, n-k)$. For $r = 2$, we get the formula

$$Q^S_3 = -\frac{1}{2}\operatorname{tr}\Big(2\,\sigma\wedge\widetilde{\Omega} + \sigma\wedge d\sigma + 2\,\sigma\wedge\tilde{\omega}\wedge\sigma + \frac{2}{3}\,\sigma\wedge\sigma\wedge\sigma\Big), \tag{3.2}$$

where $\sigma = \omega - \tilde{\omega}$. One can obtain similar formulas for any $r$ with $2r \leq \dim M$. We give also the formula for $r = 3$:

(3.3)

$$\begin{aligned} Q^S_5 = \frac{1}{3}\operatorname{tr}\Big\{&3\,\sigma\wedge\widetilde{\Omega}^2 + \frac{3}{2}(\sigma^2\wedge\widetilde{\Omega}\wedge\tilde{\omega} + 2\,\sigma\wedge\widetilde{\Omega}\wedge\sigma\wedge\tilde{\omega} + \sigma\wedge\widetilde{\Omega}\wedge\, d\sigma \\ &+ \sigma^2\wedge\widetilde{\omega}\wedge\widetilde{\Omega} + \sigma\wedge d\sigma\wedge\widetilde{\Omega} + 2\,\sigma^4\wedge\widetilde{\omega} + \sigma^3\wedge d\sigma) \\ &+ 2\,\sigma^3\wedge\widetilde{\Omega} + 3\,\sigma^2\wedge\tilde{\omega}\wedge\sigma\wedge\tilde{\omega} + 2\,\sigma\wedge\tilde{\omega}\wedge\sigma\wedge d\sigma + \sigma^3\wedge\tilde{\omega}^2 \\ &+ \sigma^2\wedge\tilde{\omega}\wedge d\sigma + \sigma^2\wedge d\sigma\wedge\tilde{\omega} + \sigma\wedge(d\sigma)^2 + \frac{3}{5}\,\sigma^5\Big\}, \end{aligned}$$

where $A^j = A\wedge\overset{(j}{\cdots}\wedge A$ for any matrix $A$. We now give some general results for the forms $Q^S_{2r-1}$.

**Proposition 3.2.** *If $S = 0$ then $Q^S_{2r-1} = 0$, for each $r$.*

*Proof.* Immediate from (2.3). □

Let $S_1$ and $S_2$ be homogeneous pseudo-Riemannian structures on $(M_1, g_1)$ and $(M_2, g_2)$, respectively. We recall [**11**, pp. 33–34] that an isomorphism between $S_1$ and $S_2$ is an isometry $\varphi\colon (M_1, g_1) \to (M_2, g_2)$ which is also an affine transformation with respect to the connections $\widetilde{\nabla}_1 = \nabla_1 - S_1$ and $\widetilde{\nabla}_2 = \nabla_2 - S_2$. Then we have the following proposition.

**Proposition 3.3.** *If $\varphi\colon (M_1, g_1) \to (M_2, g_2)$ is an isomorphism between $S_1$ on $(M_1, g_1)$ and $S_2$ on $(M_2, g_2)$, then $\varphi^*\big(Q^{S_2}_{2r-1}\big) = Q^{S_1}_{2r-1}$, for each $r$.*

*Proof.* According to the previous definition, we have that $\varphi^*\omega_2 = \omega_1$ and $\varphi^*\tilde{\omega}_2 = \tilde{\omega}_1$. Thus we have that $\varphi^*((\omega_t)_2) = \varphi^*(\tilde{\omega}_2 + t(\omega_2 - \tilde{\omega}_2)) = (\omega_t)_1$, and so $\varphi^*((\Omega_t)_2) = \varphi^*(d(\omega_t)_2 + (\omega_t)_2 \wedge (\omega_t)_2) = (\Omega_t)_1$. Hence for any invariant polynomial $I$ we have that

$$\varphi^*\{I(\sigma_2, (\Omega_t)_2, \dots, (\Omega_t)_2)\} = I(\sigma_1, (\Omega_t)_1, \dots, (\Omega_t)_1).$$

As $I$ is multi-linear, we conclude. □

**Proposition 3.4.**

$$\mathrm{tr}\,(\Omega^r) - \mathrm{tr}\,(\widetilde{\Omega}^r) = \mathrm{tr}\left\{\sum_{l=0}^{r-1}\binom{r}{l}\Omega^l \wedge (3[S,S] - \mathcal{A}S_S)^{r-l}\right\},$$

*where $\mathcal{A}S_S$ is defined by $(\mathcal{A}S_S)(X,Y) = S_{S(X,Y)-S(Y,X)}$.*

*Proof.* First we recall that $d^\nabla S$ is defined [**7**, p. 22] by

$$(d^\nabla S)(X,Y) = \nabla_X S_Y - \nabla_Y S_X - S_{[X,Y]}, \tag{3.4}$$

and we put

$$[S,S](X,Y) = S_X S_Y - S_Y S_X = [S_X, S_Y]. \tag{3.5}$$

On the other hand, Ambrose-Singer's third equation (2.1) can be written as

$$(\nabla_X S)(Y,Z) = [S_X, S_Y](Z) - S_{S(X,Y)}Z. \tag{3.6}$$

Since $\nabla$ is torsionless, by (3.4) we can write $(d^\nabla S)(X,Y)(Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)$, and thus from (3.6) one has that

$$(d^\nabla S)(X,Y)(Z) = \{2[S_X, S_Y] - S_{S(X,Y)-S(Y,X)}\}(Z). \tag{3.7}$$

Hence, on account of (3.5) we have that $(d^\nabla S)(X,Y) = (2[S,S] - \mathcal{A}S_S)(X,Y)$. Substituting now (3.7) in Koszul's formula $\widetilde{\Omega} = \Omega + [S,S] + d^\nabla S$, see [**7**, p. 22], we obtain that $\widetilde{\Omega} = \Omega + 3[S,S] - \mathcal{A}S_S$. Finally, calculation of $\operatorname{tr}(\widetilde{\Omega}^r) = \operatorname{tr}(\Omega + 3[S,S] - \mathcal{A}S_S)^r$ gives us, on account of the property $\operatorname{tr}(\Phi \wedge \Psi) = \operatorname{tr}(\Psi \wedge \Phi)$ for any two $\operatorname{End}(TM)$-valued 2-forms $\Phi$, $\Psi$, the expression in the statement. □

In particular, if $3[S,S] = \mathcal{A}S_S$, then $\operatorname{tr}(\Omega^r) - \operatorname{tr}(\widetilde{\Omega}^r) = 0$, and $Q^S_{2r-1}$ defines, for $r = 2,3$, a secondary class $[Q^S_{2r-1}]$.

## 4. Examples of forms $Q^S_{2r-1}$ associated to homogeneous pseudo-Riemannian structures.

**4.1 The 3-dimensional unimodular Lie groups.** Let $G$ be a connected unimodular Lie group, with Lie algebra $\mathfrak{g}$, endowed with a left-invariant Riemannian metric $g$. We consider the homogeneous Riemannian structure $S$ on $(G,g)$ defined by [**11**, p. 83]

(4.1)

$$2g(S_X Y, Z) = g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y), \quad X,Y,Z \in \mathfrak{g}.$$

If $\dim G = 3$ there exists [**8**] an orthonormal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{g}$ such that

$$[E_1,E_2] = \lambda_3 E_3, \qquad [E_2,E_3] = \lambda_1 E_1, \qquad [E_3,E_1] = \lambda_2 E_2. \tag{4.2}$$

If $\nabla$ is the Levi-Civita connection of $G$ then $\nabla_{E_i}E_i = S_{E_i}E_i = 0$ and the remaining components of $\nabla$ and $S$ are given by

$$\begin{aligned}
\nabla_{E_1}E_2 &= S_{E_1}E_2 = \tfrac{1}{2}(-\lambda_1+\lambda_2+\lambda_3)E_3,\\
\nabla_{E_1}E_3 &= S_{E_1}E_3 = \tfrac{1}{2}(\lambda_1-\lambda_2-\lambda_3)E_2,\\
\nabla_{E_2}E_1 &= S_{E_2}E_1 = \tfrac{1}{2}(-\lambda_1+\lambda_2-\lambda_3)E_3,\\
\nabla_{E_2}E_3 &= S_{E_2}E_3 = \tfrac{1}{2}(\lambda_1-\lambda_2+\lambda_3)E_1,\\
\nabla_{E_3}E_1 &= S_{E_3}E_1 = \tfrac{1}{2}(\lambda_1+\lambda_2-\lambda_3)E_2,\\
\nabla_{E_3}E_2 &= S_{E_3}E_2 = \tfrac{1}{2}(-\lambda_1-\lambda_2+\lambda_3)E_1.
\end{aligned}$$

Let $\{\theta^1,\theta^2,\theta^3\}$ be the basis dual to $\{E_1,E_2,E_3\}$. We obtain for $\omega$, $\tilde{\omega}$ and $\widetilde{\Omega}$ defined as in Section 3, that $\tilde{\omega}=0$, $\widetilde{\Omega}=0$,

$$\omega = \frac{1}{2}\begin{pmatrix} 0 & (-\lambda_1-\lambda_2+\lambda_3)\theta^3 & (\lambda_1-\lambda_2+\lambda_3)\theta^2 \\ (\lambda_1+\lambda_2-\lambda_3)\theta^3 & 0 & (\lambda_1-\lambda_2-\lambda_3)\theta^1 \\ (-\lambda_1+\lambda_2-\lambda_3)\theta^2 & (-\lambda_1+\lambda_2+\lambda_3)\theta^1 & 0 \end{pmatrix}$$

and then from (3.2), after some calculations, the next proposition.

**Proposition 4.1.** *The Chern-Simons form associated to the homogeneous Riemannian structure $S$ on $G$, for arbitrarily fixed $\lambda_1,\lambda_2,\lambda_3$ as in* (4.2)*, is given by*

$$Q_3^S(G_{\lambda_1,\lambda_2,\lambda_3},g) = -\frac{1}{2}\Big(\sum\lambda_i^3 - \sum_{i\neq j}\lambda_i\lambda_j^2 + 4\lambda_1\lambda_2\lambda_3\Big)\,\theta^1\wedge\theta^2\wedge\theta^3. \tag{4.3}$$

If $S=0$ then $\lambda_i=0$, $1\le i\le 3$, and the group $G$ is commutative; in this case $Q_3^S(G_{0,0,0},g)=0$. Since $S_{E_1}E_1 = S_{E_2}E_2 = S_{E_3}E_3 = 0$, one has $c_{12}(S)=0$, and hence $S$ is of type $\mathcal{S}_2\oplus\mathcal{S}_3$, see [**11**, p. 84], [**5**]. In particular, $S$ is of type $\mathcal{S}_2$, that is, $\mathfrak{S}_{XYZ}S_{XYZ}=0$ for every $X,Y,Z\in\mathfrak{g}$, if and only if $\lambda_1+\lambda_2+\lambda_3=0$; and $S$ is of type $\mathcal{S}_3$, that is $S_XY+S_YX=0$ for $X,Y\in\mathfrak{g}$, if and only if $\lambda_1=\lambda_2=\lambda_3$. By [**8**], see also [**11**, p. 84], if $S\neq 0$ is of type $\mathcal{S}_2$ then the Lie algebra $\mathfrak{g}$ of $G$ is either the Lie algebra $\mathfrak{e}(1,1)$ of the Lie group of rigid motions of the Minkowski plane or $\mathfrak{sl}(2,\mathbf{R})$, and we have that

$$Q_3^S(G_{\lambda_1,\lambda_2,\lambda_3},g) = -\frac{1}{2}\,(\lambda_1^3-\lambda_2^3-\lambda_3^3+4\lambda_1\lambda_2\lambda_3)\,\theta^1\wedge\theta^2\wedge\theta^3,$$

with $\sum \lambda_i = 0$. If $S \neq 0$ is of type $\mathcal{S}_3$ we may suppose $\lambda_i = 1$, $i = 1, 2, 3$; then $\mathfrak{g} = \mathfrak{su}(2)$, and we have that

$$Q_3^S(SU(2), g) = -\frac{1}{2}\,\theta^1 \wedge \theta^2 \wedge \theta^3.$$

As a consequence of Milnor's classification [**8**] of three-dimensional unimodular Lie algebras, if $S$ is neither of type $\mathcal{S}_2$ nor $\mathcal{S}_3$ then $\mathfrak{g}$ is either the Heisenberg Lie algebra $\mathfrak{h}_3$ or the Lie algebra $\mathfrak{e}_2$ of the Lie group of rigid motions of the Euclidean space. If $\mathfrak{g}$ is the Lie algebra of the Heisenberg group $H_3$ we may suppose that $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$; in this case,

$$Q_3^S(H_3, g) = -\frac{1}{2}\,\theta^1 \wedge \theta^2 \wedge \theta^3.$$

If $\mathfrak{g} = \mathfrak{e}_2$, then one of the constants, suppose $\lambda_3$, is null; in this case,

$$Q_3^S(E(2)_{\lambda_1,\lambda_2}, g) = -\frac{1}{2}\,(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2\,\theta^1 \wedge \theta^2 \wedge \theta^3.$$

**4.2 The Heisenberg group.** Consider again the Heisenberg group $H_3$, that is, the simply connected Lie group corresponding to the Lie algebra $\mathfrak{h}_3 = \langle a, x, y\rangle$ with nonzero bracket $[x, y] = a$. We now endow $H_3$ with the left-invariant pseudo-Riemannian metric defined at $\mathfrak{h}_3$ by the diagonal matrix $g = \mathrm{diag}\,(\varepsilon, 1, 1)$ with respect to the given basis, where $\varepsilon = \pm 1$. Let $\{\tau, \alpha, \beta\}$ be the basis dual to $\{a, x, y\}$. Then, integrating Ambrose-Singer's equations (2.1), we obtain [**5, 11**] the 1-parameter family of homogeneous pseudo-Riemannian structures

$$S_\lambda = \lambda\,\tau \otimes (\alpha \wedge \beta) + \frac{1}{2}\,\varepsilon\,\beta \otimes (\tau \wedge \alpha) - \frac{1}{2}\,\varepsilon\,\alpha \otimes (\tau \wedge \beta), \qquad \lambda \in \mathbf{R}. \tag{4.4}$$

From this we have that

$$\omega = \frac{1}{2}\begin{pmatrix} 0 & -\beta & \alpha \\ \varepsilon\beta & 0 & \varepsilon\tau \\ -\varepsilon\alpha & -\varepsilon\tau & 0 \end{pmatrix}, \text{ and letting } \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}:$$

$$\widetilde{\omega} = \left(\frac{\varepsilon}{2} + \lambda\right)\tau\,A, \qquad \widetilde{\Omega} = \left(\frac{\varepsilon}{2} + \lambda\right)\alpha \wedge \beta\,A,$$

and then, after some computations from (3.2), we obtain the following proposition.

**Proposition 4.2.** *The form of homogeneity on* $(H_3, g_\varepsilon)$ *corresponding to the homogeneous pseudo-Riemannian structure* $S_\lambda$ *is given by*

$$Q_3^{S_\lambda}(H_3, g_\varepsilon) = -\frac{1}{2}\left(\frac{1}{2} - 2\lambda(\lambda+\varepsilon)\right)\tau \wedge \alpha \wedge \beta.$$

Notice that in the Riemannian case, that is, when $\varepsilon = 1$, and if $\lambda = -1/2$, then $S_\lambda$ is the homogeneous Riemannian structure on $H_3$ obtained in Section 4.1, where the Heisenberg group was considered as a particular case of three-dimensional unimodular Lie group.

**4.3 The generalized Heisenberg group** $H(1,2)$**.** Consider [**3**] a 2-nilpotent Lie group $N$ with the left-invariant metric induced by a, not necessarily positive definite, inner product in their Lie algebra $\mathfrak{n}$. If $\mathfrak{n}$ is a Lie algebra with inner product $\langle\ ,\ \rangle$ and $\mathfrak{z}$ is the center of $\mathfrak{n}$, one considers a decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z} = \mathfrak{U} \oplus \mathfrak{Z}$, $\mathfrak{v} = \mathfrak{V} \oplus \mathfrak{E}$, $\mathfrak{U}$ stands for the null subspace of $\mathfrak{z}$, and $\mathfrak{V} \subset \mathfrak{v}$ for a complementary null subspace. An example of the construction in [**3**] is the generalized Heisenberg group $H(1,2)$ of dimension 5, whose Lie algebra is $\mathfrak{n} = \mathfrak{U} \oplus \mathfrak{Z} \oplus \mathfrak{V} \oplus \mathfrak{E} = \langle\{u, z, v, e_1, e_2\}\rangle$, where $\mathfrak{U} = \langle\{u\}\rangle$, $\mathfrak{Z} = \langle\{z\}\rangle$, $\mathfrak{V} = \langle\{v\}\rangle$ and $\mathfrak{E} = \langle\{e_1, e_2\}\rangle$, with nonvanishing brackets $[e_1, e_2] = z$, $[v, e_2] = u$, and nontrivial inner products

$$\langle u, v\rangle = 1, \quad \langle z, z\rangle = \varepsilon, \quad \langle e_1, e_1\rangle = \bar{\varepsilon}_1, \quad \langle e_2, e_2\rangle = \bar{\varepsilon}_2,$$

where each $\varepsilon$-symbol is $\pm 1$ independently, so that the pseudo-Riemannian metric on $H(1,2)$ defined by $\langle\ ,\ \rangle$ has signature $(k, 5-k)$, $1 \leq k \leq 4$. Let $\{\eta, \theta, \tau, \alpha^1, \alpha^2\}$ denote the dual basis to $\{u, z, v, e_1, e_2\}$. Then integration of Ambrose-Singer's equations (2.1) gives us [**5**] the only homogeneous pseudo-Riemannian structure

$$\begin{aligned} S = {} & \frac{\varepsilon}{2}\,\alpha^2 \otimes (\theta \wedge \alpha^1) - \frac{\varepsilon}{2}\,\alpha^1 \otimes (\theta \wedge \alpha^2) \\ & - \frac{\varepsilon}{2}\,\theta \otimes (\alpha^1 \wedge \alpha^2) - \tau \otimes (\tau \wedge \alpha^2). \end{aligned} \tag{4.5}$$

We obtain that

$$\omega = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 2\tau \\ 0 & 0 & 0 & -\alpha^2 & \alpha^1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon\bar{\varepsilon}_1\alpha^2 & 0 & 0 & \varepsilon\bar{\varepsilon}_1\theta \\ 0 & -\varepsilon\bar{\varepsilon}_2\alpha^1 & -2\bar{\varepsilon}_2\tau & -\varepsilon\bar{\varepsilon}_2\theta & 0 \end{pmatrix}, \qquad \tilde{\omega} = 0, \qquad \widetilde{\Omega} = 0,$$

and by means of some computations from (3.2) and (3.3), we have the following proposition.

**Proposition 4.3.** *The forms of homogeneity on* $(H(1,2),\ g_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2})$ *corresponding to the homogeneous pseudo-Riemannian structure* $S$ *are*

$$Q_3^S(H(1,2), g_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2}) = -\frac{1}{2}\,\bar{\varepsilon}_1\,\bar{\varepsilon}_2\,\theta\wedge\alpha^1\wedge\alpha^2, \quad Q_5^S(H(1,2), g_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2}) = 0.$$

**5. Secondary classes $[Q^S_{2r-1}]$ of compact quotients of Lie groups.** Now, we determine the secondary classes $[Q^S_{2r-1}]$ of the compact quotients of the spaces considered in Section 4. For this, we first note that given a left-invariant form $\alpha$ on a Lie group $G$, then it is invariant under the action of a discrete subgroup $\Gamma$ of $G$, so that there exists a form $\hat{\alpha}$ on the quotient $\Gamma\backslash G$ such that $\pi^*(\hat{\alpha}) = \alpha$, where $\pi$ denotes the natural projection $\pi\colon G \to \Gamma\backslash G$. In the sequel, we shall denote by $\widehat{\alpha}$ such a projected form of a left-invariant form $\alpha$ on $G$ onto a compact quotient $\Gamma\backslash G$. If $g$ is a left-invariant metric on $G$, then it projects to a metric $\hat{g}$ on $\Gamma\backslash G$ such that the map $\pi\colon (G,g) \to (\Gamma\backslash G, \hat{g})$ is a local pseudo-Riemannian isometry. Moreover, the Levi-Civita connection $\nabla$ projects to the Levi-Civita connection $\widehat{\nabla}$ on $\Gamma\backslash G$ and each homogeneous pseudo-Riemannian structure $S$ projects to a homogeneous pseudo-Riemannian structure $\widehat{S}$ on $\Gamma\backslash G$, where $\Gamma$ is a uniform discrete subgroup of $G$.

**5.1 The three-dimensional unimodular groups.** We first recall that for a compact orientable three-dimensional manifold $M$ one has $H^3(M,\mathbf{R}) \approx \mathbf{R}$. On the other hand, the compact quotients of the three-dimensional unimodular Lie groups $G$ were classified in [**10**] and such manifolds are orientable. Thus, $H^3(\Gamma\backslash G,\mathbf{R}) \approx \mathbf{R}$ in all the cases, which we now recall.

The Abelian group $\mathbf{R}^3$ has vanishing Chern-Simons form, so its only compact quotient, the 3-torus $T^3$, has no nontrivial corresponding secondary class.

The compact quotients of the Heisenberg group are the $S^1$-bundles over the torus $T^2$ with Euler class $m \in H^2(T^2, \mathbf{Z})$. One has such a bundle for each $m \in \mathbf{Z}$.

Let $\widetilde{E^0}(2)$ be the universal covering of the identity component $E^0(2) = SO(2) \ltimes \mathbf{R}^2$ of the Euclidean group $E(2)$. The compact quotients of $\widetilde{E^0}(2)$ are the 2-torus bundles over $S^1$, which are flat manifolds with cyclic holonomy equal to either $\mathbf{Z}_2$ or $\mathbf{Z}_3$ or $\mathbf{Z}_4$ or $\mathbf{Z}_6$ or 1.

The compact quotients of the group $E(1,1)$ of rigid motions of the Minkowski plane are torus bundles over $S^1$ satisfying a supplementary condition.

The group $SU(2) \approx S^3$ is compact. Their quotients as above are either lens spaces when $\Gamma$ is a cyclic group, one for each $m \in \mathbf{Z}$, $m > 1$, or the quotient spaces by $\Gamma$, where $\Gamma$ is either the binary dihedral group, or the binary tetrahedral group, or the binary octahedral group, or the binary icosahedral group.

The compact quotients of the universal covering $\widetilde{SL}(2, \mathbf{R})$ of the Lie group $SL(2, \mathbf{R})$ are defined by a Fuchsian group $\Gamma$ of the first kind satisfying certain conditions. We have the following proposition.

**Proposition 5.1.** *For any three-dimensional unimodular Lie group $G$, the Chern-Simons form $Q_3^S(G_{\lambda_1,\lambda_2,\lambda_3}, g)$ in* (4.3) *defines the secondary class*

$$-\frac{1}{2}\left( \sum \lambda_i^3 - \sum_{i \neq j} \lambda_i \lambda_j^2 + 4\lambda_1\lambda_2\lambda_3 \right) [\widehat{\theta}^1 \wedge \widehat{\theta}^2 \wedge \widehat{\theta}^3],$$

*associated to the homogeneous pseudo-Riemannian structure $\widehat{S}$ induced on any of the compact quotients $(\Gamma \backslash G, \widehat{g})$ by the homogeneous pseudo-Riemannian structure $S$ in* (4.1)*. If $G = H_3$, $SU(2)$, the secondary class is given by*

$$-\frac{1}{2}\, [\widehat{\theta}^1 \wedge \widehat{\theta}^2 \wedge \widehat{\theta}^3].$$

*For* $G = E(1,1), \widetilde{SL}(2,\mathbf{R})$*, we have the class*

$$-\frac{1}{2}\,(\lambda_1^3 - \lambda_2^3 - \lambda_3^3 + 4\lambda_1\lambda_2\lambda_3)\,[\widehat{\theta}^1 \wedge \widehat{\theta}^2 \wedge \widehat{\theta}^3], \qquad \textstyle\sum \lambda_i = 0.$$

*If* $G = \widetilde{E^0}(2)$*, one has the class*

$$-\frac{1}{2}\,(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2\;[\hat{\theta}^1 \wedge \hat{\theta}^2 \wedge \hat{\theta}^3].$$

**5.2 The Heisenberg group.** The cohomology of the compact quotient of the Heisenberg group $H_3$ by a discrete subgroup $\Gamma$ is known to be [**2**], as a consequence of Nomizu's theorem [**9**], equal to

$$H^0(\Gamma\backslash H_3, \mathbf{R}) = \{[1]\}, \qquad H^1(\Gamma\backslash H_3, \mathbf{R}) = \{[\widehat{\alpha}], [\widehat{\beta}]\},$$
$$H^2(\Gamma\backslash H_3, \mathbf{R}) = \{[\widehat{\tau} \wedge \widehat{\alpha}], [\widehat{\tau} \wedge \widehat{\beta}]\}, \qquad H^3(\Gamma\backslash H_3, \mathbf{R}) = \{[\widehat{\tau} \wedge \widehat{\alpha} \wedge \widehat{\beta}]\}.$$

Then we have the following proposition.

**Proposition 5.2.** *The Chern-Simons form* $Q_3^{S_\lambda}(H_3, g_\varepsilon)$ *in Proposition* 4.2 *determines the secondary class* $-1/2(1/2 - 2\lambda(\lambda+\varepsilon))[\widehat{\tau} \wedge \widehat{\alpha} \wedge \widehat{\beta}]$ *associated to the homogeneous pseudo-Riemannian structure* $\widehat{S}_\lambda$ *induced on the compact quotient* $(\Gamma\backslash H_3, \widehat{g}_\varepsilon)$ *by the homogeneous pseudo-Riemannian structure* $S_\lambda$ *in* (4.4).

**5.3 The generalized Heisenberg group** $H(1,2)$**.** We can compute, again as a consequence of Nomizu's theorem, the cohomology of the compact quotient $\Gamma\backslash H(1,2)$ of the generalized Heisenberg group $H(1,2)$ by a discrete subgroup $\Gamma$, obtaining

$$H^0(\Gamma\backslash H(1,2), \mathbf{R}) = \langle 1 \rangle, \qquad H^1(\Gamma\backslash H(1,2), \mathbf{R}) = \langle [\widehat{\tau}], [\widehat{\alpha}^1], [\widehat{\alpha}^2] \rangle,$$
$$\begin{aligned} H^2(\Gamma\backslash H(1,2), \mathbf{R}) &= \langle [\widehat{\eta} \wedge \widehat{\tau}], [\widehat{\eta} \wedge \widehat{\alpha}^1 + \widehat{\theta} \wedge \widehat{\tau}], [\widehat{\eta} \wedge \widehat{\alpha}^2], \\ &\qquad [\widehat{\theta} \wedge \widehat{\alpha}^1], [\widehat{\theta} \wedge \widehat{\alpha}^2], [\widehat{\tau} \wedge \widehat{\alpha}^1] \rangle, \\ H^3(\Gamma\backslash H(1,2), \mathbf{R}) &= \langle [\widehat{\eta} \wedge \widehat{\theta} \wedge \widehat{\alpha}^2], [\widehat{\eta} \wedge \widehat{\tau} \wedge \widehat{\alpha}^1], [\widehat{\eta} \wedge \widehat{\tau} \wedge \widehat{\alpha}^2], \\ &\qquad [\widehat{\eta} \wedge \widehat{\alpha}^1 \wedge \widehat{\alpha}^2], [\widehat{\theta} \wedge \widehat{\tau} \wedge \widehat{\alpha}^1], [\widehat{\theta} \wedge \widehat{\alpha}^1 \wedge \widehat{\alpha}^2] \rangle, \\ H^4(\Gamma\backslash H(1,2), \mathbf{R}) &= \langle [\widehat{\eta} \wedge \widehat{\theta} \wedge \widehat{\tau} \wedge \widehat{\alpha}^1], [\widehat{\eta} \wedge \widehat{\theta} \wedge \widehat{\tau} \wedge \widehat{\alpha}^2], \\ &\qquad [\widehat{\eta} \wedge \widehat{\theta} \wedge \widehat{\alpha}^1 \wedge \widehat{\alpha}^2] \rangle, \\ H^5(\Gamma\backslash H(1,2), \mathbf{R}) &= \langle [\widehat{\eta} \wedge \widehat{\theta} \wedge \widehat{\tau} \wedge \widehat{\alpha}^1 \wedge \widehat{\alpha}^2] \rangle. \end{aligned}$$

Then we have the following proposition.

**Proposition 5.3.** *The Chern-Simons form* $Q_3^S(H(1,2), g_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2})$ *in Proposition* 4.3 *determines the secondary class* $-1/2\,\bar{\varepsilon}_1\,\bar{\varepsilon}_2\,[\widehat{\theta}\wedge\widehat{\alpha}^1\wedge\widehat{\alpha}^2]$ *associated to the homogeneous pseudo-Riemannian structure* $\widehat{S}$ *induced on the compact quotient* $(\Gamma\backslash H(1,2),\ \hat{g}_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2})$ *by the homogeneous pseudo-Riemannian structure* $S$ *in* (4.5).

**6. Final remarks.** For the class of pseudo-Riemannian homogeneity in Proposition 5.2, we have

$$[Q_3^{S_\lambda}](\Gamma\backslash H_3, \hat{g}_\varepsilon) = 0, \quad \text{for}\ \ \varepsilon = 1,\ \lambda = \frac{\pm\sqrt{2}-1}{2}, \quad \text{or}\ \varepsilon = -1,\ \lambda = -\frac{1}{2},$$

so that in these cases the pseudo-Riemannian compact quotient of the Heisenberg group, endowed with that homogeneous pseudo-Riemannian structure, is "more symmetric" (although they are never symmetric in the usual sense) than the spaces corresponding to the rest of values of $\lambda$. Consider a compact quotient $\Gamma\backslash H(1,2)$ of the generalized Heisenberg group. By Proposition 5.3, we have that

$$[Q_3^S](\Gamma\backslash H(1,2), \hat{g}_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2}) \neq 0, \qquad [Q_5^S](\Gamma\backslash H(1,2), \hat{g}_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2}) = 0.$$

Hence this compact quotient, endowed with that homogeneous pseudo-Riemannian structure, is "more symmetric" than other pseudo-Riemannian manifolds of the same dimension whose classes of pseudo-Riemannian homogeneity are nonnull.

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