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# ON CONVEX UNIVALENT FUNCTIONS WITH CONVEX UNIVALENT DERIVATIVES

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ABSTRACT. We study the functions

$$(*) \qquad \sum_{k=0}^{\infty} a_k \frac{(1+z)^k}{k!}, \quad a_0 \geq a_1 \geq \cdots \geq 0,$$

and show that they are either constant or convex univalent in the unit disk $\mathbf{D}$. Note that this set of functions is invariant under differentiation. Our result generalizes a previous one of Suffridge, and we verify a number of general conjectures of Suffridge and Shah & Trimble concerning functions with convex univalent derivatives for our particular cases. We also pose and discuss the conjecture that the functions $(*)$, under the further restriction that $a_1 = a_2$, actually belong to the much smaller class $DCP$, whose members preserve direction-convexity of univalent functions in $\mathbf{D}$, under the Hadamard product.

**1. Introduction.** This work was inspired by T.J. Suffridge's paper [**12**] where he studies the partial sums

$$(1.1) \qquad Q_n(z) = \sum_{k=0}^{n} \frac{(1+z)^k}{k!}, \quad z \in \mathbf{C}, \quad n \in \mathbf{N},$$

of the series $e^{1+z} = \sum_{k=0}^{\infty} (1+z)^k/k!$. His main result was that the $Q_n$ are convex univalent in the unit disk $\mathbf{D}$. Note that $Q_n' = Q_{n-1}$ so that all derivatives of $Q_n$ are as well convex univalent or constants. He conjectured that the normalized functions

$$(1.2) \quad C_n(z) := \frac{Q_n(z) - Q_n(0)}{Q_n'(0)} = \sum_{k=1}^{n} \left( \frac{\sum_{l=0}^{n-k} 1/l!}{\sum_{l=0}^{n-1} 1/l!} \right) \frac{z^k}{k!}, \quad n \in \mathbf{N},$$

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are in some sense extremal within the set of functions $\mathcal{F}_n$ with the corresponding property.

To be precise, let $\mathcal{A}$ denote the set of analytic functions in the unit disc $\mathbf{D}$, and let $\mathcal{A}_0$ stand for the set of functions $f \in \mathcal{A}$ satisfying $f(0) = 0$, $f'(0) = 1$. Furthermore, let

$$\tilde{\mathcal{K}} := \{f \in \mathcal{A} \; : \; f \equiv \text{ const. or } f \text{ convex univalent in } \mathbf{D}\},$$
$$\mathcal{K}_n := \left\{f \in \mathcal{A}_0 \; : \; f^{(k)} \in \tilde{\mathcal{K}}, \; k = 0, 1, \dots, n\right\}, \quad n = 0, 1, \dots ,$$

and $\mathcal{F}_n := \mathcal{K}_n \cap \mathcal{P}_n$, where $\mathcal{P}_n$ are the complex polynomials of degree $\le n$. We shall also use

$$\mathcal{K}_\infty = \bigcap_{n=1}^{\infty} \mathcal{K}_n.$$

The study of univalent functions with univalent derivatives has a long tradition, see for instance [**4**–**12**].

**Conjecture 1** (Suffridge [**12**]). *For $f \in \mathcal{F}_n$ we have*

$$|f^{(k)}(0)| \le C_n^{(k)}(0), \quad k = 1, \dots, n,$$

*with equality only for $f = C_n$, or one of its 'rotations.'*

For $n = 2, 3, 4$ this has been established in [**12**].

For $\mathcal{K}_\infty$ there exists an older conjecture by Shah and Trimble (note that $\lim_{n\to\infty} C_n(z) = e^z - 1 \in \mathcal{K}_\infty$):

**Conjecture 2.** *For $f \in \mathcal{K}_\infty$ we have*

$$\left|f^{(k)}(0)\right| \le 1, \quad k \in \mathbf{N}, \tag{1.3}$$

*and*

$$1 - e^{-|z|} \le |f(z)| \le e^{|z|} - 1, \quad z \in \mathbf{D}. \tag{1.4}$$

We set $\mathcal{W}_n := \overline{\text{co}}\,\{C_1, \dots, C_n\}$, $n \in \mathbf{N}$, and $\mathcal{W}_\infty := \overline{\text{co}}\,\{C_n \; : \; n \in \mathbf{N}\}$, where $\overline{\text{co}}$ stands for the closed convex hull of set (in the topology of compact convergence in $\mathbf{D}$). Our first aim is to establish the following result.

**Theorem 1.** *$\mathcal{W}_\infty \subset \mathcal{K}_\infty$ and $\mathcal{W}_n \subset \mathcal{K}_n$ for $n \in \mathbf{N}$. Moreover, Conjectures* 1 *and* 2 *hold for $\mathcal{W}_n$ and $\mathcal{W}_\infty$, respectively. In addition, for $f \in \mathcal{W}_\infty$, we find*

$$e^{-|z|} \le |f'(z)| \le e^{|z|}, \quad z \in \mathbf{D}. \tag{1.5}$$

Note that this theorem implies in particular that all functions of the form

$$\sum_{k=0}^{\infty} \frac{a_k}{k!}(1+z)^k, \quad a_0 \ge a_1 \ge \cdots \ge 0,$$

are convex univalent in $\mathbf{D}$. In fact, we believe that most of these functions enjoy a much stronger property. We briefly recall the definition of the class $DCP$, compare [**2**]. A domain $\Omega \subset \mathbf{C}$ is said to be *convex in the direction* $e^{i\phi}$, $\phi \in \mathbf{R}$, if and only if for every $a \in \mathbf{C}$ the set

$$\Omega \cap \left\{a + te^{i\phi} \,:\, t \in \mathbf{R}\right\}$$

is either connected or empty. Accordingly, the classes $\mathcal{K}(\phi)$, $\phi \in \mathbf{R}$, of functions *convex in the direction* $e^{i\phi}$ are defined as

$$\left\{f \in \mathcal{A} \,:\, f \text{ is univalent and } f(D) \text{ is convex in the direction } e^{i\phi}\right\}.$$

We say that a function $g \in \mathcal{A}$ is *Direction-Convexity-Preserving*, $g \in DCP$, if and only if $g * f \in K(\phi)$ for all $f \in K(\phi)$ and all $\phi \in \mathbf{R}$. Here $*$ stands for the Hadamard product of analytic functions. Functions in $DCP$ have many other intriguing properties in the context of convex harmonic functions and Jordan curves in the plane with convex interior domain. We refer to [**2, 3**] for more details.

**Conjecture 3.** *For $0 \ne a_1 = a_2 \ge a_3 \ge \cdots \ge 0$ we have*

$$\sum_{k=1}^{\infty} \frac{a_k}{k!}(1+z)^k \in DCP. \tag{1.6}$$

It is well known that each $f \in DCP$ is convex univalent in $\mathbf{D}$. Therefore Conjecture 3, where it applies, is stronger than the first part of Theorem 1, which just says that the functions (1.6) are convex univalent in $\mathbf{D}$.

The restriction $a_1 = a_2$ in the statement of Conjecture 3 cannot just be dropped, as it is readily verified that, for instance,

$$2\frac{1+z}{1!} + \frac{(1+z)^2}{2!} + \frac{(1+z)^3}{3!} \notin DCP. \tag{1.7}$$

Of course, there may exist weaker restrictions which do the same.

We can prove Conjecture 3 for polynomials of the form (1.6) of degree $\leq 10$ using computer algebra and numerical polynomial solvers on real algebraic polynomials of degree $< 20$. It is likely that one can go up to much higher degrees with more computational efforts. This does not seem to be a useful project if there is no very specific reason. Instead, we show that the conjecture can be reduced to a set of real inequalities which should be established theoretically. We shall discuss this in Section 3.

A weaker form of Conjecture 3 was already mentioned in [**1**], and the special case of (1.6) where $a_k = 1$, $k \in \mathbf{N}$, has first been established by Kurth (unpublished):

**Theorem 2.** $e^{1+z} \in DCP$.

We give an independent short proof of Theorem 2 in Section 3.

**2. Proof of Theorem 1.** It is easily established that $f \in \mathcal{W}_\infty$ if and only if

$$f(z) = \lambda\,(e^z - 1) + \sum_{k=1}^{\infty} \lambda_k C_k(z)$$

$$\Big(\lambda, \lambda_k \geq 0, \sum_{k=1}^{\infty} \lambda_k = 1 - \lambda\Big),$$

and thus

$$f(z) = \sum_{k=1}^{\infty} a_k \frac{(1+z)^k - 1}{k!}, \tag{2.1}$$

where

$$a_k = \frac{\lambda}{e} + \sum_{j=k}^{\infty} \frac{\lambda_j}{\sum_{m=1}^{j}(1/(m-1)!)}, \quad k \in \mathbf{N}. \tag{2.2}$$

This shows that indeed $f \in \mathcal{W}_\infty$ if and only if there is a sequence

$$a_1 \geq a_2 \geq \cdots \geq 0, \quad \text{with} \quad f'(0) = \sum_{k=1}^{\infty} \frac{a_k}{(k-1)!} = 1, \tag{2.3}$$

such that (2.1) holds. The set $\mathcal{K} := \tilde{\mathcal{K}} \cap \mathcal{A}_0$ is known to be compact. Therefore it will be sufficient to prove the convexity of such $f \in \mathcal{W}_\infty$ for which, in the representation (2.1), only finitely many of the $a_k$ are nonvanishing. After dropping the unimportant normalization we are left with the proof of

**Lemma 1.** *For some $n \in \mathbf{N}$ let $1 = a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then the function*

$$P(z) := \sum_{k=1}^{n} a_k \frac{(1+z)^k}{k!} \tag{2.4}$$

*is convex univalent in $\mathbf{D}$.*

*Proof.* We need to show that

$$\gamma_P(z) := \operatorname{Re}\left[1 + \frac{zP''(z)}{P'(z)}\right] \geq 0, \quad z \in \mathbf{D}. \tag{2.5}$$

We have

$$P'(z) = \sum_{k=0}^{n-1} a_{k+1} \frac{(1+z)^k}{k!} = \sum_{j=0}^{n-1} \lambda_j Q_j(z), \tag{2.6}$$

where the polynomials $Q_j$ are as in (1.1), and $\lambda_j \geq 0$, $\sum_{j=0}^{n-1} \lambda_j = 1$. Since the $Q_j$ are known to be convex univalent [**12**] and have positive coefficients, it is clear that $\operatorname{Re} Q_j(z) \geq Q_j(-1) = 1$, $z \in \mathbf{D}$. Then (2.6) implies that $\operatorname{Re} P'(z) \geq 1$ in $\mathbf{D}$ as well. Furthermore, the $Q_j$

are typically real and so is $P'$. Therefore there exists $w \in \mathcal{A}$ with $|w(z)| \le 1$, $z \in \mathbf{D}$, and typically real such that

$$P'(z) = \frac{2}{1-w(z)}. \tag{2.7}$$

Using $a_{n+1} := 0$ and

$$\frac{|1+z|^k}{k!} \le \frac{|1+z|^2 2^{k-2}}{k!} \le \mathrm{Re}\,(1+z), \quad k \ge 2, \quad z \in \mathbf{D},$$

we get, for $n \ge 3$,

$$\begin{aligned}\mathrm{Re}\left[\frac{z}{P'(z)}\sum_{k=2}^{n-1}(a_{k+1}-a_{k+2})\frac{(1+z)^k}{k!}\right]& \\ &\ge -\sum_{k=2}^{n-1}(a_{k+1}-a_{k+2})\frac{|1+z|^k}{k!} \\ &\ge -a_3\mathrm{Re}\,(1+z).\end{aligned}$$

Note that this holds also for $n < 3$, with all terms $= 0$. In what follows we observe that $a_2 = a_3 = 0$ for $n = 1$ and $a_3 = 0$ for $n = 2$. Calculation gives for $n \in \mathbf{N}$:

$$\begin{aligned}\gamma_P(z) &= \mathrm{Re}\left[1+z+\frac{z}{P'(z)}\sum_{k=0}^{n-1}(a_{k+2}-a_{k+1})\frac{(1+z)^k}{k!}\right] \\ &\ge \mathrm{Re}\left[(1+z)(1-a_3)+\frac{z}{P'(z)}\{a_2-a_1+(1+z)(a_3-a_2)\}\right]\end{aligned}$$

If $a_3 = 1$, then $a_2 = 1$ as well, and we are already done: $\gamma_P(z) \ge 0$. Otherwise we write $\lambda := (a_2-a_3)/(1-a_3)$ and it remains to show that

$$\mathrm{Re}\left[1+z-z\frac{1-w(z)}{2}(1+\lambda z)\right] \ge 0, \quad \lambda \in [0,1], \quad z \in \mathbf{D}. \tag{2.8}$$

The minimum of this expression will be attained on $\partial\mathbf{D}$. Furthermore, for symmetry reasons, we need to study $z = e^{i\phi}$, $0 \le \phi \le \pi$, only. Since $w$ is typically real we have for those $\phi$: $w(z) = \rho e^{i\psi}$ with $0 \le \rho \le 1$, $0 \le \psi \le \pi$. Because of linearity we have to establish (2.8) only for the extremal cases $\lambda = 0, 1$. For $\lambda = 0$ however, (2.8) is obvious. In the other case we are left with ($\tau := \phi/2$):

$$\cos(\tau) - \frac{1}{2}\cos(3\tau) + \frac{1}{2}\,\rho\cos(3\tau+\psi) \geq 0, \tag{2.9}$$

for $0 \leq \tau \leq \pi/2,\ 0 \leq \psi \leq \pi,\ 0 \leq \rho \leq 1$.

Since $\cos(\tau) - \cos(3\tau)/2 \geq 0$ in our range of $\tau$ it is clear that (2.9) holds whenever $\cos(3\tau+\psi) \geq 0$. In the other cases, $3\tau + \psi \in (\pi/2, 3\pi/2)$, the choice $\rho = 1$ is always the worst choice, which we assume from now on. For $3\tau \geq \pi$ we have $\cos(3\tau+\psi) \geq \cos(3\tau)$ which then implies (2.9). If, however, $3\tau \leq \pi$ we have to show that $\cos(\tau)-\cos(3\tau)/2 \geq 1/2$, which is a rather simple affair. This completes the proof of Lemma 1. $\square$

We have shown that $\mathcal{W}_\infty \subset \mathcal{K}$. Since $\mathcal{W}_\infty$ is invariant under differentiation (modulo renormalization), as can be seen from (2.1), it is clearly a subset of $\mathcal{K}_\infty$ as well. The first part of Theorem 1 is therefore established (the statements concerning $\mathcal{W}_n$ follow from the one for $\mathcal{W}_\infty$).

We turn to the estimates. Conjecture 1, with $n$ fixed, holds for the polynomials $C_j$, $j \leq n$, and therefore also for the members of $\mathcal{W}_n$. The same argument takes care of the coefficient estimate in Conjecture 2. Standard arguments concerning the 'integration' of lower and upper estimates for univalent functions show that (1.5) implies the distortion part in Conjecture 2 as far as $\mathcal{W}_\infty$ is concerned. We thus turn to the proof of (1.5) and use $f \in \mathcal{W}_\infty$ in the representation (2.1). Then either $f' \equiv 1$, and nothing has to be proved, or $f'$ is convex univalent with non-negative coefficients. Hence, $0 < f'(-|z|) \leq |f'(z)| \leq f'(|z|)$ in $\mathbf{D}$. Using the normalization in (2.3) we conclude that we need to establish

$$\sum_{k=1}^{\infty} \frac{a_k}{(k-1)!}\left((1-r)^{k-1} - e^{-r}\right) \geq 0, \tag{2.10}$$

$$\sum_{k=1}^{\infty} \frac{a_k}{(k-1)!}\left((1+r)^{k-1} - e^{r}\right) \leq 0, \tag{2.11}$$

for $0 \leq r \leq 1$. Of course, we can now remove the normalization in (2.3) when dealing with (2.10) and (2.11), only keeping the nonnegativity and monotoneity of the $a_k$. It is readily seen that

$$(1-r)^{k-1} - e^{-r} \le 0, \quad k \ge 2,$$

so that the sum in (2.10) will be as small as possible, with $a_1 > 0$ fixed, if all $a_k$, $k \ge 2$, are as large as possible, i.e., $a_k = a_1$ for all $k$. Then, however, the sum on the left of (2.10) is $\equiv 0$. This proves (2.10). A similar argument establishes (2.11).

**3. Proof of Theorem 2 and comments on Conjecture 3.** The following criterion for membership in $DCP$ is [**2**, Theorem 4].

**Lemma 2.** *Let $g$ be nonconstant and analytic in $\mathbf{D}$, continuous in $\overline{\mathbf{D}}$, and such that $u(x) := \operatorname{Re} g(e^{ix})$, $x \in \mathbf{R}$, is three times continuously differentiable. Then $g \in DCP$ if and only if $g$ is convex univalent and*

$$\sigma_u(x) := u'(x)\, u'''(x) - (u''(x))^2 \le 0, \quad x \in \mathbf{R}.$$

*Proof of Theorem* 2. In view of Lemma 2 we have to study

$$u(x) := \operatorname{Re} \exp(e^{ix}) = e^{\cos(x)} \cos(\sin(x)).$$

$e^z$ is convex univalent and $u$ is obviously smooth enough for the lemma to apply. A little calculation yields

$$\sigma_u(x) = -\frac{1}{2}\, e^{2\,\cos(x)} f(x),$$

where

$$f(x) := 3 + 5\cos(x) + \cos(2x) - \cos(3x + 2\sin(x)).$$

For reasons of symmetry we need to prove $f(x) \ge 0$ for $0 \le x \le \pi$ only.

First note that $3 + 5\cos(x) + \cos(2x) = 2 + 5\cos(x) + 2\cos^2(x) \ge 1$ holds for $0 \le x \le x_0 = \arccos((-5+\sqrt{17})/4) = 1.79181\ldots$. Hence it remains to establish

$$f(\pi - x) = 3 - 5\cos(x) + \cos(2x) + \cos(3x - 2\sin(x)) \ge 0,$$

for $0 \leq x \leq \pi - x_0 < 1.35$. Since

$$x - \sin(x) \leq \frac{x^3}{6}, \qquad x + \frac{x^3}{3} \leq \pi,$$

hold for $0 \leq x \leq \pi/2$, we can reduce the problem to

$$3 - 5\cos(x) + \cos(2x) + \cos\left(x + \frac{x^3}{3}\right) \geq 0, \quad 0 \leq x \leq 1.35. \tag{3.1}$$

Using the series expansion of the cosine about the origin we find $c_1(x) \leq \cos(x) \leq c_2(x)$, $0 \leq x \leq \pi$, where

$$c_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{720}, \qquad c_2(x) = c_1(x) + \frac{x^8}{40320}.$$

It thus suffices to establish the nonnegativity of

$$h(x) := 3 - 5\,c_2(x) + c_1(2x) + c_1\left(x + \frac{x^3}{3}\right),$$

and to this end we write

$$\begin{aligned} g(x) &:= \frac{1}{x^4}\left(h(x) - \frac{1003x^8}{40320} - \frac{5x^{10}}{1296}\right) \\ &= \frac{1}{6} - \frac{x^2}{12} - \frac{x^8}{1944} - \frac{x^{10}}{3888} - \frac{x^{12}}{29160} - \frac{x^{14}}{524880}. \end{aligned}$$

But then, for $0 \leq x \leq 1.35$,

$$\frac{h(x)}{x^4} \geq g(x) \geq g(1.35) = 0.00256\ldots > 0,$$

and this settles our assertion. □

In view of Theorem 1 and Lemma 2 it will be sufficient for a proof of Conjecture 3 to show that $\sigma_u \leq 0$ for

$$u(x) := \sum_{k=2}^{\infty} \lambda_k u_k(x),$$

where

$$u_k(x) := \mathrm{Re} \sum_{j=1}^{k} \frac{(1+e^{ix})^j}{j!}, \quad k = 2, 3, \dots ,$$

$$\lambda_k \geq 0, \quad \sum_{k=2}^{\infty} \lambda_k = 1.$$

This is equivalent to

$$\sum_{j,k=2}^{\infty} \lambda_j \lambda_k u_{j,k}(x) \geq 0, \quad x \in \mathbf{R},$$

with

$$u_{j,k}(x) = u_j''(x) u_k''(x) - \frac{1}{2}(u_j'(x) u_k'''(x) + u_k'(x) u_j'''(x)).$$

We conjecture that

$$u_{j,k}(x) \geq 0, \quad j, k \geq 2, \quad x \in \mathbf{R}, \tag{3.2}$$

and this will obviously imply Conjecture 3. Using computer algebra and a numerical polynomial solver (we used the software package *Mathematica* 4.0) it is easy to verify (3.2) for small values of $j, k$. We did so for $2 \leq j$, $k \leq 10$, and found no counterexample to (3.2). Note, however, that $u_{1,3}(x)$ assumes negative values, which also leads to the counterexample (1.7) mentioned in the introduction.

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