ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 3, 2005

COMPOSITION FOLLOWED BY DIFFERENTIATION BETWEEN BERGMAN AND HARDY SPACES

R.A. HIBSCHWEILER AND N. PORTNOY

ABSTRACT. Let Φ be an analytic self-map of the disc, and let H^p denote the Hardy space. The operator DC_Φ is defined for functions analytic in the disc by $DC_\Phi(f) = (f \circ \Phi)'$. We show that compactness and boundedness of the map $DC_\Phi : H^p \to H^q$, $p, q \geq 1$, are equivalent to the conditions $\Phi' \in H^q$ and $||\Phi||_\infty < 1$. For $\alpha > -1$ and $p \geq 1$, A^p_α denotes the weighted Bergman space. In the case $1 \leq p \leq q$, $DC_\Phi : A^p_\alpha \to A^q_\beta$ is bounded if and only if a related measure obeys a Carleson-type condition. Compactness is characterized by the analogous little-oh condition. For $1 \leq q < p$, Khinchine's inequality is used to show that boundedness and compactness are equivalent to an integrability condition on a weighted integral.

1. The Hardy space H^p, $p \geq 1$, is the Banach space of functions analytic in $U = \{z : |z| < 1\}$ satisfying

$$\|f\|_{H^p} = \sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(\mathrm{r}e^{i\theta})|^p \, d\theta \right\}^{1/p} < \infty.$$

References for the Hardy spaces include [**2**] and [**3**].

Let Φ be a nonconstant self-map of U, and let $C_\Phi(f) = f \circ \Phi$ for functions f analytic in the disc. Many authors [**1, 6, 7, 10**] have studied boundedness and compactness of C_Φ on the Hardy spaces. It is known [**12**] that if C_Φ is compact on H^p for some $p \geq 1$, then C_Φ is compact on all the Hardy spaces. Shapiro [**11**] characterized the self-maps Φ for which C_Φ is compact on H^2.

2000 AMS *Mathematics Subject Classification.* Primary 47B38, Secondary 30H05.
Key words and phrases. Composition operators, differentiation, Hardy spaces, Bergman spaces.
Some of this research was completed as part of the second author's dissertation under the direction of E.A. Nordgren. Both authors wish to thank Nordgren for many helpful conversations.
Received by the editors on January 23, 2001, and in revised form on July 10, 2001.

The focus of this note is to characterize maps Φ for which the operator $DC_\Phi(f) = (f \circ \Phi)'$ is bounded or compact on the Hardy and Bergman spaces. In his unpublished dissertation [**8**, Theorem 3.0.9], the second author used a result of MacCluer to prove that, for $p, q \geq 1$, $DC_\Phi : H^p \to H^q$ is bounded if and only if it is compact if and only if $\Phi' \in H^q$ and $||\Phi||_\infty < 1$. We present the proof here and extend the result to the setting of the weighted Bergman spaces, where the solution is more subtle.

Note that if $DC_\Phi : H^p \to H^q$ is bounded, then $\Phi' \in H^1$. It follows that Φ extends continuously to $\overline{U}$ [**2**, Theorem 3.11]. Thus in this section we may assume that Φ is analytic in the disc and continuous on the closed disc.

The proof of Theorem 1 requires the notion of Carleson sets. For $|\zeta| = 1$ and $0 < \delta < 1$,

$$S(\zeta, \delta) = \bigl\{z \in \overline{U} : |z - \zeta| < \delta\bigr\}.$$

In the rest of this work, C will denote a positive constant, the exact value of which may differ from one appearance to the next.

Theorem 1. *Let $p \geq 1$, and let Φ be an analytic self-map of the disc with $\Phi' \in H^1$. The following are equivalent.*

(1) *$DC_\Phi : H^p \to H^1$ is bounded.*

(2) *$DC_\Phi : H^p \to H^1$ is compact.*

(3) $||\Phi||_\infty < 1$.

Proof. It is clear that (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Thus it will suffice to prove that (1) $\Rightarrow$ (3).

Suppose that $||\Phi||_\infty = 1$ and $\Phi' \in H^1$. It follows that Φ extends to a continuous function on $\overline{U}$ and Φ is absolutely continuous on ∂U [**2**, Theorem 3.11]. Let σ denote normalized Lebesgue measure on ∂U, and define a finite measure ν on Borel subsets of the circle by

$$\nu(E) = \int_E |\Phi'| \, d\sigma.$$

As part of the proof of [**6**, Theorem 2.3], MacCluer showed that for such Φ and ν,

$$\sup_{|\zeta|=1} \nu(\Phi^{-1}S(\zeta,\delta)\cap\partial U) \neq o(\delta) \quad \text{as } \delta\to 0. \tag{1}$$

We give a brief outline of her argument. Without loss of generality, suppose that $\Phi(1)=1$, and let $A_\delta = \Phi^{-1}(S(1,\delta))\cap\partial U$. The integral

$$\int_{A_\delta} |\Phi'|\,d\sigma$$

gives the arc length of the image of A_δ under Φ. Since $1\in A_\delta$, and since $\Phi(1)=1$ the continuity of Φ implies that the arc length must be at least 2δ, and thus (1) holds.

In what follows we view Φ as a function defined on ∂U, that is, $\Phi:\partial U\to\overline{U}$. Relation (1) implies that there is a sequence $(\zeta_n)\subset\partial U$, a sequence (δ_n) of positive numbers with $\delta_n\to 0$ and a positive constant β such that

$$\nu\Phi^{-1}(S(\zeta_n,\delta_n))\geq\beta\delta_n, \quad n=1,2,\dots .$$

Let $a_n=(1-\delta_n)\zeta_n$, and let $f_n(z)=(1-|a_n|^2)^{1/p}(1-\bar{a}_n z)^{-2/p}$. A calculation shows that $\|f_n\|_{H^p}=1$ for $n=1,2,\dots$. Note that

$$\begin{aligned}\|DC_\Phi(f_n)\|_{H^1} &= \int_{\partial U}|f_n'\circ\Phi|\,|\Phi'|\,d\sigma\\ &= \int_{\partial U}|f_n'\circ\Phi|\,d\nu\\ &= \int_{\overline{U}}|f_n'|\,d(\nu\Phi^{-1})\\ &\geq \int_{S(\zeta_n,\delta_n)}|f_n'|\,d(\nu\Phi^{-1}).\end{aligned}$$

By a calculation, $|f_n'(z)|\geq C\delta_n^{-(1+p)/p}$ for $z\in S(\zeta_n,\delta_n)$. It follows that

$$\|DC_\Phi(f_n)\|_{H^1}\geq\delta_n^{-(1+p)/p}\,\nu\Phi^{-1}(S(\zeta_n,\delta_n))\geq C\beta\delta_n^{-1/p}.$$

Thus $DC_\Phi:H^p\to H^1$ is unbounded if $||\Phi||_\infty=1$. This completes the proof. $\square$

Theorem 1 has the following corollary.

Corollary 1. *Let $p, q \geq 1$, and let $\Phi' \in H^q$. The following are equivalent.*

(1) $DC_\Phi : H^p \to H^q$ *is bounded.*

(2) $DC_\Phi : H^p \to H^q$ *is compact.*

(3) $||\Phi||_\infty < 1$.

Proof. It suffices to prove that (1) $\Rightarrow$ (3). Since the inclusion $I : H^q \to H^1$ is bounded, the first assertion implies that the map $DC_\Phi : H^p \to H^1$ is bounded. By Theorem 1, $||\Phi||_\infty < 1$. $\square$

Let $\alpha > -1$, $p \geq 1$, and let A denote normalized area measure on the disc. The weighted Bergman space A^p_α is the Banach space of functions analytic in U with

$$\|f\|^p_{A^p_\alpha} = \int_U |f(z)|^p (\log(1/|z|))^\alpha \, dA(z) < \infty.$$

It will be convenient to let $dA_\alpha = (\log(1/|z|))^\alpha \, dA(z)$. Note that dA_α can be replaced by the measure $(1 - |z|^2)^\alpha \, dA(z)$. This results in the same space of functions with an equivalent norm.

Smith [**13**, p. 2336] noted that the appropriate definition for A^p_{-1} is the Hardy space H^p. Theorem 1 will be extended from the setting of the Hardy spaces to the spaces A^p_α, $\alpha > -1$.

Theorem 2. *Let $p \geq 1$, and let $\alpha > -1$. Let Φ be a self-map of U with $\Phi' \in H^1$. The following are equivalent.*

(1) $DC_\Phi : A^p_\alpha \to H^1$ *is bounded.*

(2) $DC_\Phi : A^p_\alpha \to H^1$ *is compact.*

(3) $||\Phi||_\infty < 1$.

Proof. It is enough to show that (1) $\Rightarrow$ (3). Thus suppose that $\Phi' \in H^1$ and $||\Phi||_\infty = 1$. Let (ζ_n), (δ_n) and (a_n) be sequences as described in the proof of Theorem 1, and define

$$f_n(z) = (1 - |a_n|^2)^{(\alpha+2)/p} (1 - \overline{a_n} z)^{-2(\alpha+2)/p}.$$

Then $\|f_n\|_{A^p_\alpha} \approx C$ [**13**, p. 2340]. An argument as in the previous proof yields

$$\|(DC_\Phi)(f_n)\|_{H^1} \longrightarrow \infty \quad \text{as } n \to \infty.$$

Thus $DC_\Phi : A^p_\alpha \to H^1$ is not bounded if $||\Phi||_\infty = 1$. This completes the proof. □

The proof of Corollary 2 is omitted, since it is similar to the proof of Corollary 1.

Corollary 2. *Let $p, q \geq 1$, and let $\alpha > -1$. Suppose that $\Phi' \in H^q$. The following are equivalent.*

(1) *$DC_\Phi : A^p_\alpha \to H^q$ is bounded.*

(2) *$DC_\Phi : A^p_\alpha \to H^q$ is compact.*

(3) $||\Phi||_\infty < 1$.

2. In this section we will characterize Φ for which $DC_\Phi : A^p_\alpha \to A^q_\beta$, $\alpha, \beta > -1$, is bounded or compact. This will be done in terms of more general theorems that characterize measures μ for which the differentiation operator $D : A^p_\alpha \to L^q(\mu)$ is bounded.

While the Carleson sets $S(\zeta, \delta)$ were useful in Section 1, it will be more convenient here to use pseudohyperbolic discs. Recall that the pseudohyperbolic metric ρ is defined by

$$\rho\,(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right| (z, w \in U).$$

In what follows, $D(a)$ denotes the pseudohyperbolic disc $\{z : \rho(a, z) < 1/8\}$.

Luecking characterized positive measures μ with the property $\|f^{(n)}\|_{L^q(\mu)} \leq C\|f\|_{A^p_\alpha}$. Theorem 3 gives Luecking's result [**5**, Theorem 2.2] for $n = 1$ in case $1 \leq p \leq q$.

Theorem 3 (Luecking). *Let $1 \leq p \leq q$, and let $\alpha > -1$. Let $\mu \geq 0$ be a finite measure on U. The following are equivalent.*

(1) *$\|f'\|_{L^q(\mu)} \leq C\|f\|_{A^p_\alpha}$ for all $f \in A^p_\alpha$.*

(2) $\mu(D(a)) = O((1-|a|^2)^{q(\alpha+2+p)/p})$ *as* $|a| \to 1$.

For the case $1 \leq q < p$, Luecking used Khinchine's inequality and other estimates to obtain a version of Theorem 4 for $f^{(n)}$, where $f \in A_0^p$ [**4**, Theorem 1]. We are interested in the case $n = 1$ and $f \in A_\alpha^p$. Theorem 4 is a slight modification of Luecking's result, so the proof is not given here.

Theorem 4 (Luecking). *Let* $1 \leq q < p$, *and let* $\alpha > -1$. *Let* $\mu \geq 0$ *be a finite measure on* U. *Let* $L(z) = (1-|z|^2)^{-(\alpha+2+q)}\mu(D(z))$. *The following are equivalent.*

(1) $\|f'\|_{L^q(\mu)} \leq C\|f\|_{A_\alpha^p}$ *for all* $f \in A_\alpha^p$.

(2) $L \in L^{p/(p-q)}(A_\alpha)$.

Let $|\lambda| < 1$, and let $w_\lambda(z) = (\lambda - z)/(1-\bar{\lambda}z)$. Since $C_{w_\lambda} : A_\alpha^p \to A_\alpha^p$ is bounded, we may assume that $\Phi(0) = 0$ in the rest of this work.

Theorems 3 and 4 will now be applied to the operator DC_Φ. To do so, we need a relative of the Nevanlinna counting function, which will participate in the change of variable.

Definition 1. Let Φ be a self-map of U with $\Phi(0) = 0$. Let $q \geq 1$, and let $\beta > -1$. For $w \in U$, $w \neq 0$,

$$\tau_{q,\beta}(w) = \sum |\Phi'(z)|^{q-2}(\log(1/|z|))^\beta.$$

The sum extends over all solutions of $\Phi(z) = w$.

Corollary 3. *Let* $1 \leq p \leq q$, *and let* $\alpha, \beta > -1$. *Let* Φ *be an analytic self-map of the disc with* $\Phi(0) = 0$ *and* $\Phi' \in A_\beta^q$. *Let* $d\mu(w) = \tau_{q,\beta}(w)\,dA(w)$. *The following are equivalent.*

(1) $DC_\Phi : A_\alpha^p \to A_\beta^q$ *is bounded.*

(2) $\mu(D(a)) = 0((1-|a|^2)^{q(\alpha+2+p)/p})$ *as* $|a| \to 1$.

Furthermore, the operator is compact if and only if the analogous little-oh condition is satisfied.

Proof. Since $\Phi' \in A^q_\beta$, a change of variable [**1**, Theorem 2.32] implies that μ is a finite measure. Thus Theorem 3 applies. Note that

$$\begin{aligned}\|(f\circ\Phi)'\|^q_{A^q_\beta} &= \int_U |f'(\Phi(z))|^q|\Phi'(z)|^q(\log(1/|z|))^\beta\,dA(z)\\ &= \int_U |f'(w)|^q\tau_{q,\beta}(w)\,dA(w)\\ &= \int_U |f'(w)|^q\,d\mu(w) = \|f'\|^q_{L^q(\mu)}.\end{aligned}$$

If $DC_\Phi : A^p_\alpha \to A^q_\beta$ is bounded, then

$$\|f'\|_{L^q(\mu)} = \|(DC_\Phi)(f)\|_{A^q_\beta} \le C\|f\|_{A^p_\alpha} \quad \text{for all } f\in A^p_\alpha.$$

Theorem 3 implies that

$$\mu(D(a)) = O((1-|a|^2))^{q(\alpha+2+p)/p} \quad \text{as } |a|\to 1. \tag{2}$$

For the converse, suppose that assertion (2) holds. Theorem 3 implies that

$$\|(DC_\Phi)(f)\|_{A^q_\beta} = \|f'\|_{L^q(\mu)} \le C\|f\|_{A^p_\alpha},$$

and thus $DC_\Phi : A^p_\alpha \to A^q_\beta$ is bounded.

Next suppose that $DC_\Phi : A^p_\alpha \to A^q_\beta$ is compact. Let $a\in U$, and let

$$f_a(z) = (1-|a|^2)^{(\alpha+2)/p}(1-\bar a z)^{-2(\alpha+2)/p}.$$

Then $\|f_a\|_{A^p_\alpha} \approx C$ [**13**, p. 2340] and $f_a \to 0$ uniformly on compact sets as $|a|\to 1$. The standard compactness criterion implies that, given $\varepsilon > 0$, there exists $0<r<1$ such that $\|(DC_\Phi)(f_a)\|^q_{A^q_\beta} < \varepsilon$ for $|a| > r$. Thus

$$\varepsilon > \int_U |f_a'(w)|^q\,d\mu(w) \ge \int_{D(a)} |f_a'|^q\,d\mu \quad \text{for } |a|>r.$$

An estimate on $|f_a'|$ for $z\in D(a)$ yields

$$\mu(D(a)) < \varepsilon(1-|a|^2)^{q(\alpha+2+p)/p} \quad \text{for all } a \quad \text{with } |a|>r. \tag{3}$$

Finally assume that relation (3) holds, and let (f_n) be a bounded sequence in A^p_α with $f_n \to 0$ uniformly on compact sets. To show that $DC_\Phi : A^p_\alpha \to A^q_\beta$ is compact, it will suffice to show that

$$I_n = \|(DC_\Phi)(f_n)\|^q_{A^q_\beta} = \|f'_n\|^q_{L^q(\mu)} \longrightarrow 0 \quad \text{as } n \to \infty.$$

By a standard estimate [**4**, p. 338],

$$I_n \leq C \int_U \frac{1}{(1-|z|^2)^{2+q}} \int_{D(z)} |f_n(w)|^q \, dA(w)\, d\mu(z).$$

Note that $\chi_{D(z)}(w) = \chi_{D(w)}(z)$ and $1-|w|^2 \approx 1-|z|^2$ for $z \in D(w)$. Fubini's theorem now yields

$$I_n \leq C \int_U |f_n(w)|^q \frac{\mu(D(w))}{(1-|w|^2)^{2+q}} \, dA(w).$$

Smith [**13**, Lemma 2.5] showed that, for $f \in A^p_\alpha$ and $w \in U$,

$$|f(w)| \leq C\|f\|_{A^p_\alpha}(1-|w|^2)^{-(\alpha+2)/p}.$$

Let $\Gamma = (\alpha q + 2q + qp - \alpha p)/p$. Since $\|f_n\|_{A^p_\alpha} \cong C$, Smith's estimate yields

$$I_n \leq C \int_U |f_n(w)|^p \frac{\mu(D(w))}{(1-|w|^2)^\Gamma} \, dA(w).$$

Relation (3) implies that, for a given $\varepsilon > 0$, there exists $0 < r < 1$ such that

$$\int_{|w|>r} |f_n(w)|^p \frac{\mu(D(w))}{(1-|w|^2)^\Gamma} \, dA(w) < \varepsilon \|f_n\|^p_{A^p_\alpha} \leq C\varepsilon. \tag{4}$$

Since $f_n \to 0$ uniformly on compact subsets,

$$\begin{aligned}\int_{|w|\leq r} |f_n(w)|^p \frac{\mu(D(w))}{(1-|w|^2)^\Gamma} \, dA(w) &\leq C\varepsilon \int_U \mu(U)\, dA(w) \\ &= C\varepsilon \quad \text{for large } n.\end{aligned} \tag{5}$$

Relations (4) and (5) yield $I_n \to 0$, as required. The proof is complete. $\square$

Corollary 4. *Let $1 \le q < p$, and let $\alpha, \beta > -1$. Let $d\mu(w) = \tau_{q,\beta}(w)\, dA(w)$, and let $\Phi' \in A^q_\beta$. Let $L(z) = (1-|z|^2)^{-(\alpha+q+2)}\mu(D(z))$. The following are equivalent.*

(1) $DC_\Phi : A^p_\alpha \to A^q_\beta$ *is bounded.*

(2) $DC_\Phi : A^p_\alpha \to A^q_\beta$ *is compact.*

(3) $L \in L^{p/(p-q)}(A_\alpha)$.

Proof. After a change of variable as in Corollary 3, Theorem 4 gives the equivalence of (1) and (3).

It is clear that (2) implies (1).

It remains to verify that (3) implies (2). Assume that $\|f_n\|_{A^p_\alpha} \le C$ and $f_n \to 0$ uniformly on compact sets. It will suffice to show that

$$I_n = \|DC_\Phi(f_n)\|^q_{A^q_\beta} = \int_U |f_n'(w)|^q \, d\mu(w) \longrightarrow 0 \quad \text{as } n \to \infty. \tag{6}$$

By estimates as in the previous proof,

$$\begin{aligned} I_n &\le \int_U \frac{1}{(1-|w|^2)^{2+q}} \int_{D(w)} |f_n(z)|^q \, dA(z)\, d\mu(w) \\ &\le C \int_U |f_n(z)|^q L(z)\, dA_\alpha(z). \end{aligned}$$

Let $\varepsilon > 0$. The hypothesis on L implies that there exists r, $0 < r < 1$, with the property

$$\int_{|z|>r} L(z)^{p/(p-q)}\, dA_\alpha(z) < \varepsilon^{p/(p-q)}.$$

It follows by Hölder's inequality that

$$\begin{aligned} &\int_{|z|>r} |f_n(z)|^q L(z)\, dA_\alpha(z) \\ &\qquad \le \left(\int_U |f_n|^p \, dA_\alpha\right)^{q/p} \left(\int_{|z|>r} L^{p/(p-q)}\, dA_\alpha\right)^{(p-q)/p} \\ &\qquad \le \varepsilon \|f_n\|^q_{A^p_\alpha} \le C\varepsilon. \end{aligned} \tag{7}$$

Since $f_n \to 0$ uniformly on compact subsets,

$$\int_{|z|\le r} |f_n(z)|^q L(z)\, dA_\alpha(z) \le \varepsilon \int_{|z|\le r} L(z)\, dA_\alpha(z) \quad \text{for large } n.$$

Since $\Phi' \in A^q_\beta$, $\mu(U) < \infty$ and thus

$$\int_{|z|\le r} L(z)\, dA_\alpha(z) \le C \int_U \mu(U)\, dA_\alpha(z) = C.$$

Thus

$$\int_{|z|\le r} |f_n(z)|^q L(z)\, dA_\alpha(z) \le C\, \varepsilon \quad \text{for large } n. \tag{8}$$

The two inequalities (7) and (8) imply that the expression at (6) tends to 0 as $n \to \infty$. This completes the proof that (3) $\Rightarrow$ (2) and completes the proof of the corollary. $\square$

We close this section with a question and with examples to show that the conditions in Corollaries 3 and 4 do not require $||\Phi||_\infty < 1$. In the examples it will be shown that $DC_\Phi : A^p_\alpha \to A^2_1$ is bounded, for certain polygonal maps Φ with $||\Phi||_\infty = 1$.

Note that in the case $q = 2$, $\beta = 1$, the function $\tau_{q,\beta}$ simplifies to the Nevanlinna counting function $N_1(w)$. Smith [**13**, p. 2347] obtained estimates on $N_1(w)$ for polygonal maps.

Let $P \subset \overline{U}$ be a polygon with $P \cap \partial U = \{1\}$ and with angular aperture π/η at $w = 1$ ($\eta > 1$). Let Φ be a Riemann map of U onto the interior of P. Smith showed that, for such a polygonal map,

$$N_1(w) = O((1-|w|)^\eta) \quad \text{as } |w| \to 1.$$

Since $1 - |w| \approx 1 - |a|$ for $w \in D(a)$, it follows that

$$\int_{D(a)} N_1(w)\, dA(w) \le C(1-|a|)^{\eta+2} \quad \text{as } |a| \to 1. \tag{9}$$

By an easy calculation, $\Phi' \in A^2_1$. Thus Corollaries 3 and 4 apply. First suppose that $1 \le p \le 2$ and $\alpha > -1$. Let $\eta = (2\alpha + 4)/p$. Then $\eta > 1$. A calculation using (9) yields

$$\int_{D(a)} N_1(w)\, dA(w) = O((1-|a|^2)^{2(\alpha+2+p)/p}) \quad \text{as } |a| \to 1.$$

By Corollary 3, $DC_\Phi : A^p_\alpha \to A^2_1$ is bounded.

Next let $p > 2$ and $\alpha > -1$. Let $\eta > (p + 2\alpha + 2)/p$. Then $\eta > 1$ and (9) implies that

$$L(a) = \frac{\int_{D(a)} N_1(w)\, dA(w)}{(1 - |a|)^{\alpha+4}} \leq C(1 - |a|)^{\eta-\alpha-2} \quad \text{as } |a| \to 1.$$

A calculation shows that $L \in L^{p/(p-2)}(A_\alpha)$, and thus $DC_\Phi : A^p_\alpha \to A^2_1$ is bounded and compact for p and α as described.

Finally recall from Section 1 that if $DC_\Phi : A^p_\alpha \to A^p_{-1}$ is bounded, then $||\Phi||_\infty < 1$. Is this the case when $-1 < \beta < 0$ and $DC_\Phi : A^p_\alpha \to A^p_\beta$ is bounded?

3. The methods in Section 2 can be used to study the operator $C_\Phi D$. A brief discussion is given here.

Theorem 5. *Let $1 \leq p \leq q$, and let $\alpha, \beta > -1$. Let Φ be an analytic self-map of the disc with $\Phi \in A^q_\beta$. The following are equivalent.*

(1) *$C_\Phi D : A^p_\alpha \to A^q_\beta$ is bounded.*

(2) *$A_\beta \Phi^{-1}(D(a)) = O((1 - |a|^2)^{q(\alpha+2+p)/p})$ as $|a| \to 1$.*

Furthermore, $C_\Phi D : A^p_\alpha \to A^q_\beta$ is compact if and only if the analogous little-oh condition holds.

Proof. First suppose that $C_\Phi D : A^p_\alpha \to A^q_\beta$ is bounded. An argument using the test functions in Corollary 3 yields

$$C \geq \int_{D(a)} |f'_a(w)|^q\, d(A_\beta \Phi^{-1})(w).$$

An estimate on $|f'_a|$ gives the required result.

Next let $f \in A^p_\alpha$ and apply estimates as in Corollary 3 to show that if the measure $A_\beta \Phi^{-1}$ obeys the given big-oh condition, then $\|f' \circ \Phi\|_{A^q_\beta} < \infty$ for every $f \in A^p_\alpha$.

The proof of the statement about compactness is omitted. $\square$

The last theorem proceeds along the lines of Corollary 4. The proof is omitted.

Theorem 6. *Let* $1 \leq q < p$, *and let* $\alpha, \beta > -1$. *Let* $\Phi \in A^q_\beta$ *and let*

$$M(z) = (1 - |z|^2)^{-(\alpha+2+q)} A_\beta \Phi^{-1}(D(z)).$$

The following are equivalent.

(1) $C_\Phi D : A^p_\alpha \to A^q_\beta$ *is bounded.*

(2) $C_\Phi D : A^p_\alpha \to A^q_\beta$ *is compact.*

(3) $M \in L^{p/(p-q)}(A_\alpha)$.

REFERENCES

1. C.C. Cowen and B.D. MacCluer, *Composition operators on spaces of analytic functions*, New York, 1994.

2. P.L. Duren, *Theory of* H^p *spaces*, Academic Press, New York, 1970.

3. P. Koosis, *Introduction to* H^p *spaces*, Cambridge Univ. Press, Cambridge, 1980.

4. D.H. Luecking, *Embedding theorems for spaces of analytic functions via Khinchine's inequality*, Mich. Math. J. **40** (1993), 333–358.

5. ———, *Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives*, Amer. J. Math. **107** (1985), 85–111.

6. B.D. MacCluer, *Composition operators on* S^p, Houston J. Math. **13** (1987), 245–254.

7. B.D. MacCluer and J.H. Shapiro, *Angular derivatives and compact composition operators on the Hardy and Bergman spaces*, Canad. J. Math. **38** (1986), 878–906.

8. N. Portnoy, *Differentiation and composition on the Hardy and Bergman spaces*, Dissertation, University of New Hampshire, 1998.

9. R. Rochberg, *Decomposition theorems for Bergman spaces and their applications*, in *Operators and function theory* (S.C. Power, ed.), D. Reidel, Dordrecht, 1985, pp. 225–277.

10. J.H. Shapiro, *Composition operators and classical function theory*, Springer-Verlag, New York, 1993.

11. ———, *The essential norm of a composition operator*, Ann. of Math. **125** (1987), 375–404.

12. J.H. Shapiro and P.D. Taylor, *Compact, nuclear and Hilbert-Schmidt composition operators on* H^2, Indiana Univ. Math J. **23** (1973), 471–496.

13. W. Smith, *Composition operators between Bergman and Hardy spaces*, Trans. Amer. Math. Soc. **348** (1996), 2331–2348.

14. W. Smith and L. Yang, *Composition operators that improve integrability on weighted Bergman spaces*, Proc. Amer. Math. Soc. **126** (1998), 411–420.

University of New Hampshire, Durham, NH 03824
E-mail address: rah2@cisunix.unh.edu

State University of New York at Stony Brook, Stony Brook, NY 11794-3651
E-mail address: nportnoy@math.sunysb.edu