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REPRESENTATIONS AND INTERPOLATIONS OF WEIGHTED HARMONIC BERGMAN FUNCTIONS

KYESOOK NAM

ABSTRACT. On the setting of the upper half-space of the Euclidean n-space, we study representation theorems and interpolation theorems for weighted harmonic Bergman functions. Also, we consider the harmonic (little) Bloch spaces as limiting spaces.

1. Introduction. Let $\mathbf{H}$ denote the upper half space $\mathbf{R}^{n-1} \times \mathbf{R}_+$ where $\mathbf{R}_+$ denotes the set of all positive real numbers. We will write points $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' \in \mathbf{R}^{n-1}$ and $z_n > 0$.

For $\alpha > -1$ and $1 \leq p < \infty$, let $b^p_\alpha = b^p_\alpha(\mathbf{H})$ denote the *weighted harmonic Bergman space* consisting of all real-valued harmonic functions u on $\mathbf{H}$ such that

$$\|u\|_{L^p_\alpha} := \left(\int_{\mathbf{H}} |u(z)|^p \, dV_\alpha(z) \right)^{1/p} < \infty$$

where $dV_\alpha(z) = z_n^\alpha dz$ and dz is the Lebesque measure on $\mathbf{R}^n$. Then we can see easily that the space b^p_α is a Banach space. In particular, b^2_α is a Hilbert space. Hence, there is a unique Hilbert space orthogonal projection Π_α of L^2_α onto b^2_α which is called the weighted harmonic Bergman projection. It is known that this weighted harmonic Bergman projection can be realized as an integral operator against the weighted harmonic Bergman kernel $R_\alpha(z, w)$. See Section 2.

In [**6**], many fundamental weighted harmonic Bergman space properties have been studied. In this paper, we study the representation property of b^p_α-functions and the interpolation by b^p_α-functions. Our methods are taken from those in [**4**] and based on estimates of the

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weighted harmonic Bergman kernel in [**6**]. Related results for spaces of harmonic functions were given in [**7**] and [**8**].

The following theorems are special cases of the representation results and the interpolation results, respectively.

Theorem 1.1. *Let $\alpha > -1$, and let $1 < p < \infty$. There exists a sequence $\{z_m\}$ of points in* $\mathbf{H}$ *and a constant C with the following properties. For $(\lambda_m) \in l^p$, define u by*

$$u(z) = \sum \lambda_m z_{mn}^{(n+\alpha)(1-1/p)} R_\alpha(z, z_m). \tag{1.1}$$

Then $u \in b^p_\alpha$ with

$$\int_{\mathbf{H}} |u|^p \, dV_\alpha \leq C \sum |\lambda_m|^p.$$

Conversely, given $u \in b^p_\alpha$, there exists a sequence $(\lambda_m) \in l^p$ such that (1.1) *holds and*

$$\sum |\lambda_m|^p \leq C \int_{\mathbf{H}} |u|^p \, dV_\alpha.$$

The corresponding theorem for $p = 1$ is also available with a certain restriction.

Theorem 1.2. *Let $\alpha > -1$, and let $1 \leq p < \infty$. There exists a sequence $\{z_m\}$ of points in* $\mathbf{H}$ *and a constant C with the following properties. For $u \in b^p_\alpha$, we have*

$$\sum z_{mn}^{(n+\alpha)} |u(z_m)|^p \leq C \int_{\mathbf{H}} |u|^p \, dV_\alpha.$$

Conversely, given $(\lambda_m) \in l^p$, there exists a function $u \in b^p_\alpha$ such that $z_{mn}^{(n+\alpha)/p} u(z_m) = \lambda_m$ for all m and

$$\int_{\mathbf{H}} |u|^p \, dV_\alpha \leq C \sum |\lambda_m|^p.$$

These two properties of holomorphic Bergman spaces were studied in [**5**] and [**9**]. In [**5**], the representation properties of harmonic Bergman functions, as well as harmonic Bloch functions, were also proved on the unit ball in $\mathbf{R}^n$. See [**2**] for the interpolation properties of holomorphic (little) Bloch functions. On the setting of the half-space of $\mathbf{R}^n$, Choe and Yi [**4**] have studied these two properties of harmonic Bergman spaces. In [**4**], the harmonic (little) Bloch spaces are also considered as limiting spaces of b^p.

In Section 2 we give some basic properties related to the space b^p_α, the harmonic Bloch space $\widetilde{\mathcal{B}}$ and the little harmonic Bloch space $\widetilde{\mathcal{B}}_0$. In Section 3 we collect some technical lemmas which will be used in later sections. In Section 4 and Section 5 we study the representation theorems for b^p_α, $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}_0$. In Section 6 and Section 7 we prove the interpolation theorems for b^p_α, $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}_0$.

Constants. Throughout the paper the same letter C will denote various positive constants, unless otherwise specified, which may change at each occurrence. The constant C may often depend on the dimension n and some parameters like δ, p, α or β, but it will be always independent of particular functions, points or sequences under consideration. For nonnegative quantities A and B, we will often write $A \lesssim B$ or $B \gtrsim A$ if A is dominated by B times some positive constant. Also, we write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries. In this section we summarize preliminary results on b^p_α, as well as the harmonic Bloch space $\widetilde{\mathcal{B}}$ from [**6**]. Let $\alpha > -1$ and let $1 \le p < \infty$. First, we introduce the fractional derivative.

Let D denote the differentiation with respect to the last component, and let $u \in b^p_\alpha$. Then the mean value property, Jensen's inequality and Cauchy's estimate yield

$$|D^k u(z)| \lesssim z_n^{-(n+\alpha)/p-k} \tag{2.1}$$

for each $z \in \mathbf{H}$ and for every nonnegative integer k.

Let $\mathcal{F}_\beta$ be the collection of all functions v on $\mathbf{H}$ satisfying $|v(z)| \lesssim z_n^{-\beta}$ for $\beta > 0$, and let $\mathcal{F} = \cup_{\beta>0}\mathcal{F}_\beta$. If $v \in \mathcal{F}$, then $v \in \mathcal{F}_\beta$ for some $\beta > 0$.

In this case, we define the fractional derivative of v of order $-s$ by

$$\mathcal{D}^{-s}v(z) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}v(z', z_n + t)\,dt \tag{2.2}$$

for the range $0 < s < \beta$. (Here, Γ is the Gamma function.)

If $u \in b^p_\alpha$, then for every nonnegative integer k, $D^k u \in \mathcal{F}$ by (2.1). Thus for $s > 0$, we define the fractional derivative of u of order s by

$$\mathcal{D}^s u = \mathcal{D}^{-([s]-s)} D^{[s]} u. \tag{2.3}$$

Here, $[s]$ is the smallest integer greater than or equal to s and $\mathcal{D}^0 = D^0$ is the identity operator. If $s > 0$ is not an integer, then $-1 < [s]-s-1 < 0$ and $[s] \geq 1$. Thus we know from (2.1) that, for each $z \in \mathbf{H}$ and for every $u \in b^p_\alpha$,

$$\mathcal{D}^s u(z) = \frac{1}{\Gamma([s]-s)}\int_0^\infty t^{[s]-s-1}D^{[s]}u(z', z_n + t)\,dt$$

always makes sense.

Let $P(z, w)$ be the extended Poisson kernel on $\mathbf{H}$ and

$$P_z(w) := P(z, w) = \frac{2}{nV(B)}\frac{z_n + w_n}{|z - \overline{w}|^n}$$

where $z, w \in \mathbf{H}$ and $\overline{w} = (w', -w_n)$ and B is the open unit ball in $\mathbf{R}^n$. It is known that the weighted harmonic Bergman projection Π_α of L^2_α onto b^2_α is

$$\Pi_\alpha f(z) = \int_{\mathbf{H}} f(w) R_\alpha(z, w)\,dV_\alpha(w)$$

for all $f \in L^2_\alpha$ where $R_\alpha(z, w)$ is the weighted harmonic Bergman kernel and its explicit formula is

$$R_\alpha(z, w) = \frac{1}{C_\alpha}\,\mathcal{D}^{\alpha+1}P_z(w) \tag{2.4}$$

and $C_\alpha = (-1)^{[\alpha]+1}\Gamma(\alpha+1)/2^{\alpha+1}$. Also, it is known that

$$|\mathcal{D}^\beta_{z_n} R_\alpha(z, w)| \leq \frac{C}{|z - \overline{w}|^{n+\alpha+\beta}} \tag{2.5}$$

for all $z, w \in \mathbf{H}$. Here, $\beta > -n-\alpha$ and the constant C is dependent only on n, α and β. Using (2.5), we know $R_\alpha(z,\cdot) \in b^q_\alpha$ for all $1 < q \le \infty$. Thus, Π_α is well defined whenever $f \in L^p_\alpha$ for $1 \le p < \infty$. Also, for $1 \le p < \infty$, $u \in b^p_\alpha$, $z \in \mathbf{H}$,

$$u(z) = \int_{\mathbf{H}} u(w) R_\beta(z,w)\, dV_\beta(w) \tag{2.6}$$

whenever $\beta \ge \alpha$. Furthermore, we have a useful norm equivalence. If $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p + \gamma > 0$, then

$$\|u\|_{L^p_\alpha} \approx \|w_n^\gamma \mathcal{D}^\gamma u\|_{L^p_\alpha} \tag{2.7}$$

as u ranges over b^p_α.

Set $z_0 = (0,1)$. A harmonic function u on $\mathbf{H}$ is called a Bloch function if

$$\|u\|_{\mathcal{B}} = \sup_{w\in\mathbf{H}} w_n |\nabla u(w)| < \infty,$$

where ∇u denotes the gradient of u. We let $\mathcal{B}$ denote the set of Bloch functions on $\mathbf{H}$ and let $\widetilde{\mathcal{B}}$ denote the subspace of functions in $\mathcal{B}$ that vanish at z_0. Then the space $\widetilde{\mathcal{B}}$ is a Banach space under the Bloch norm $\|\ \|_{\mathcal{B}}$.

A function $u \in \widetilde{\mathcal{B}}$ is called a harmonic little Bloch function if it has the following vanishing condition

$$\lim_{z\to\partial^\infty\mathbf{H}} z_n |\nabla u(z)| = 0$$

where $\partial^\infty\mathbf{H}$ denotes the union of $\partial\mathbf{H}$ and $\{\infty\}$. Let $\widetilde{\mathcal{B}}_0$ denote the set of all harmonic little Bloch functions on $\mathbf{H}$. It is not hard to verify that $\widetilde{\mathcal{B}}_0$ is a closed subspace of $\widetilde{\mathcal{B}}$. Let $\mathcal{C}_0$ denote the set of all continuous functions on $\mathbf{H}$ vanishing at ∞.

Because $R_\alpha(z,\cdot)$ is not in L^1_α, $\Pi_\alpha f$ is not well defined for $f \in L^\infty$. So we need the following modified Bergman kernel. For $z, w \in \mathbf{H}$, define

$$\widetilde{R}_\alpha(z,w) = R_\alpha(z,w) - R_\alpha(z_0,w).$$

Then, there is a constant $C = C(n,\alpha)$ such that

$$|\widetilde{R}_\alpha(z,w)| \le C\left(\frac{|z-z_0|}{|z-\overline{w}|^{n+\alpha}|z_0-\overline{w}|} + \frac{|z-z_0|}{|z-\overline{w}||z_0-\overline{w}|^{n+\alpha}}\right) \tag{2.8}$$

for all $z, w \in \mathbf{H}$. Thus, (2.8) implies that $\widetilde{R}_\alpha(z,\cdot) \in L^1_\alpha$ for each fixed $z \in \mathbf{H}$ and then we can define $\widetilde{\Pi}_\alpha$ on L^∞ by

$$\widetilde{\Pi}_\alpha f(z) = \int_{\mathbf{H}} f(w)\widetilde{R}_\alpha(z,w)\, dV_\alpha(w)$$

for $f \in L^\infty$. Then, it turns out that $\widetilde{\Pi}_\alpha$ is a bounded linear map from L^∞ onto $\widetilde{\mathcal{B}}$. Also, $\widetilde{\Pi}_\alpha$ has the following property: If $\gamma > 0$ and $v \in \widetilde{\mathcal{B}}$ then

$$\widetilde{\Pi}_\alpha(w_n^\gamma \mathcal{D}^\gamma v)(z) = Cv(z) \tag{2.9}$$

where $C = C(\alpha, \gamma)$. The Bloch norm is also equivalent to the normal derivative norm: If $\gamma > 0$, then

$$\|u\|_{\mathcal{B}} \approx \|w_n^\gamma \mathcal{D}^\gamma u\|_\infty \tag{2.10}$$

as u ranges over $\widetilde{\mathcal{B}}$. (See [**6**] for details.)

3. Technical lemmas. In this section we prove technical lemmas which will be used in later sections. We first introduce a distance function on $\mathbf{H}$ which is useful for our purposes. The pseudohyperbolic distance between $z, w \in \mathbf{H}$ is defined by

$$\rho(z,w) = \frac{|z-w|}{|z-\overline{w}|}.$$

This ρ is an actual distance. (See [**4**].) Note that ρ is horizontal translation invariant and dilation invariant. In particular,

$$\rho(z,w) = \rho(\phi_a(z), \phi_a(w)) \tag{3.1}$$

for $z, w \in \mathbf{H}$ where $\phi_a (a \in \mathbf{H})$ denotes the function defined by

$$\phi_a(z) = \left(\frac{z' - a'}{a_n}, \frac{z_n}{a_n}\right)$$

for $z \in \mathbf{H}$. Note that the Jacobian of ϕ_a^{-1} is a_n^n. For $z \in \mathbf{H}$ and $0 < \delta < 1$, let $E_\delta(z)$ denote the pseudohyperbolic ball centered at z

with radius δ. Note that $\phi_z(E_\delta(z)) = E_\delta(z_0)$ by the invariance property (3.1). Also, a simple calculation shows that

$$E_\delta(z) = B\left(\left(z', \frac{1+\delta^2}{1-\delta^2}\, z_n\right), \frac{2\delta}{1-\delta^2}\, z_n\right) \tag{3.2}$$

so that $B(z, \delta z_n) \subset E_\delta(z) \subset B(z, 2\delta(1-\delta)^{-1} z_n)$ where $B(z, r)$ denotes the Euclidean ball centered at z with radius r. From (3.2), we have two lemmas which will be used many times in this paper. For proofs of the following lemmas, see [**4**].

Lemma 3.1. *For $z, w \in \mathbf{H}$, we have*

$$\frac{1-\rho(z,w)}{1+\rho(z,w)} \le \frac{z_n}{w_n} \le \frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

This lemma implies the following lemma.

Lemma 3.2. *For $z, w \in \mathbf{H}$, we have*

$$\frac{1-\rho(z,w)}{1+\rho(z,w)} \le \frac{|z-\overline{s}|}{|w-\overline{s}|} \le \frac{1+\rho(z,w)}{1-\rho(z,w)}$$

for all $s \in \mathbf{H}$.

The following lemma is used to prove the representation theorem. If α is a nonnegative integer, then it is proved in [**4**]. Therefore, to complete the proof of the following lemma, we only need to show the case that α is not an integer.

Lemma 3.3. *Let $\alpha > -1$ and β be real. Then*

$$\left| z_n^\beta R_\alpha(s,z) - w_n^\beta R_\alpha(s,w) \right| \le C\rho(z,w)\, \frac{z_n^\beta}{|z-\overline{s}|^{n+\alpha}}$$

whenever $\rho(z,w) < 1/2$ and $s \in \mathbf{H}$.

Proof. Suppose $\beta = 0$ and let $k = [\alpha]$. Then $k - \alpha > 0$. From the proof of Lemma 3.4 in [**4**], it is easily seen that

$$|R_k(s,z) - R_k(s,w)| \le \frac{C\rho(z,w)}{|z-\overline{s}|^{n+k}}.$$

Thus we get from (2.4),

$$\begin{aligned}
(3.3)\quad & |R_\alpha(s,z) - R_\alpha(s,w)| \\
&\le C\int_0^\infty \big|D^{k+1}P_s(z', z_n+t) - D^{k+1}P_s(w', w_n+t)\big| t^{k-\alpha-1}\,dt \\
&\le C\int_0^\infty \frac{\rho\big((z', z_n+t), (w', w_n+t)\big)}{|(z', z_n+t) - \overline{s}|^{n+k}}\, t^{k-\alpha-1}\,dt \\
&\le C\rho(z,w)\,\frac{1}{|z-\overline{s}|^{n+\alpha}}.
\end{aligned}$$

Now, let β be a real number. Then from (3.3) and (2.5), we have

$$\begin{aligned}
&\big|z_n^\beta R_\alpha(s,z) - w_n^\beta R_\alpha(s,w)\big| \\
&\qquad \le z_n^\beta\,|R_\alpha(s,z) - R_\alpha(s,w)| + z_n^\beta\,|R_\alpha(s,w)|\left|1 - \left(\frac{w_n}{z_n}\right)^\beta\right| \\
&\qquad \le C\rho(z,w)\,\frac{z_n^\beta}{|z-\overline{s}|^{n+\alpha}} + C\rho(z,w)\,\frac{z_n^\beta}{|w-\overline{s}|^{n+\alpha}} \\
&\qquad \le C\rho(z,w)\,\frac{z_n^\beta}{|z-\overline{s}|^{n+\alpha}}.
\end{aligned}$$

The last two inequalities of the above hold by Lemma 3.1 and Lemma 3.2. The proof is complete. □

Let $\alpha > -1$, and let $1 \le p < \infty$. Define Π_β on the weighted Lebesque space L^p_α by

$$\Pi_\beta f(z) = \int_{\mathbf{H}} f(w) R_\beta(z,w)\,dV_\beta(w)$$

for each $f \in L^p_\alpha$ and every $z \in \mathbf{H}$. Then we show in the following lemma Π_β is a bounded projection on L^p_α. For the proof of the following lemma, see Theorem 4.3 in [**6**].

Lemma 3.4. *Suppose $\alpha > -1$, $1 \le p < \infty$ and $\alpha + 1 < (\beta + 1)p$. Then Π_β is bounded projection of L^p_α onto b^p_α.*

By simple estimation, we have the next lemma which will be used frequently. For the proof of the following lemma, see Lemma 2.1 in [**6**].

Lemma 3.5. *For $b < 0, -1 < a + b$, we have*

$$\int_{\mathbf{H}} \frac{w_n^{a+b}}{|z - \overline{w}|^{n+a}}\, dw \le C z_n^b$$

for every $z, w \in \mathbf{H}$.

Lemma 3.6. *Let $\alpha > -1$, $1 \le p < \infty$, and let $(1 + \alpha)/p + \gamma > 0$. Suppose $0 < \delta < 1$. Then*

$$z_n^{n+p\gamma} |\mathcal{D}^\gamma u(z)|^p \le \frac{C}{\delta^{n+pk}} \int_{E_\delta(z)} |u(w)|^p\, dw$$

for all $z \in \mathbf{H}$ and for every u harmonic on $\mathbf{H}$ where $k = [\gamma]$ if $\gamma > -1$ and $k = 0$ if $\gamma \le -1$. The constant $C = C(n, p, \gamma)$ is independent of δ.

Proof. Since k is a nonnegative integer, we have from Lemma 3.6 of [**4**],

$$z_n^{n+pk} |D^k u(z)|^p \le \frac{C}{\delta^{n+pk}} \int_{E_\delta(z)} |u(w)|^p\, dw.$$

Suppose that γ is not a nonnegative integer. Then, we have from (2.3),

$$\begin{aligned} |\mathcal{D}^\gamma u(z)| &\le \frac{1}{\Gamma(k-\gamma)} \int_0^\infty |D^k u(z', z_n + t)|\, t^{k-\gamma-1}\, dt \\ &\le \frac{C}{\delta^{(n+pk)/p}} \int_0^\infty \frac{t^{k-\gamma-1}}{(z_n + t)^{(n+pk)/p}}\, dt \left(\int_{E_\delta(z)} |u(w)|^p\, dw \right)^{1/p} \\ &\le \frac{C}{z_n^{(n+pk)/p - (k-\gamma)} \delta^{(n+pk)/p}} \left(\int_{E_\delta(z)} |u(w)|^p\, dw \right)^{1/p}. \end{aligned}$$

The proof is complete. $\square$

If γ satisfies the condition of Lemma 3.6, we can show $\mathcal{D}^\gamma u$ is harmonic on $\mathbf{H}$. If γ is a nonnegative integer, then $\mathcal{D}^\gamma u$ is harmonic on $\mathbf{H}$, because it is a partial derivative of a harmonic function. If γ is not a nonnegative integer, we see also $\mathcal{D}^\gamma u$ is harmonic on $\mathbf{H}$ by passing the Laplacian through the integral.

The notation $|E|$ denotes the Lebesque measure of a Borel subset E of $\mathbf{H}$. Let $|E|_\alpha$ denote $V_\alpha(E)$. The following lemma is proved by using the mean value property and Cauchy's estimates.

Lemma 3.7. *Suppose u is harmonic on some proper open subset Ω of $\mathbf{R}^n$. Let $\alpha > -1$ and let $1 \le p < \infty$. Then, for a given open ball $E \subset \Omega$,*

$$\int_E |u(z) - u(a)|^p \, dV_\alpha(z) \le C \frac{|E|^{p/n} |E|_\alpha}{d(E, \partial\Omega)^{n+p}} \int_\Omega |u(w)|^p \, dw$$

for all $a \in E$. The constant C depends only on n, α and p.

4. Representation on weighted harmonic Bergman functions. In this section we prove the representation property of b^p_α-functions. Let $\{z_m\}$ be a sequence in $\mathbf{H}$, and let $0 < \delta < 1$. We say that $\{z_m\}$ is δ-separated if the balls $E_\delta(z_m)$ are pairwise disjoint or simply say that $\{z_m\}$ is separated if it is δ-separated for some δ. Also, we say that $\{z_m\}$ is a δ-lattice if it is $\delta/2$-separated and $\mathbf{H} = \cup E_\delta(z_m)$. Note that any "maximal" $\delta/2$-separated sequence is a δ-lattice.

From [**4**], we have the following three lemmas.

Lemma 4.1. *Fix a $1/2$-lattice $\{a_m\}$, and let $0 < \delta < 1/8$. If $\{z_m\}$ is a δ-lattice, then we can find a rearrangement $\{z_{ij} : i = 1, 2, \dots, j = 1, 2, \dots, N_i\}$ of $\{z_m\}$ and a pairwise disjoint covering $\{D_{ij}\}$ of $\mathbf{H}$ with the following properties*:

(a) $E_{\delta/2}(z_{ij}) \subset D_{ij} \subset E_\delta(z_{ij})$

(b) $E_{1/4}(a_i) \subset \cup_{j=1}^{N_i} D_{ij} \subset E_{5/8}(a_i)$

(c) $z_{ij} \in E_{1/2}(a_i)$ *for all $i = 1, 2, \dots,$ and $j = 1, 2, \dots, N_i$.*

Lemma 4.2. *Let $r > 0$ and let $0 < (1+r)\eta < 1$. If $\{z_m\}$ is an η-separated sequence, then there is a constant $M = M(n, r, \eta)$ such that more than M of the balls $E_{r\eta}(z_m)$ contain no point in common.*

Lemma 4.3. *Let N_i be the sequence defined in Lemma* 4.1. *Then*

$$\sup_i N_i \leq C\delta^{-n}$$

for some constant C depending only on n.

Analysis similar to that in the proof of Lemma 3.4 shows the following lemma which is used in the proof of Proposition 4.5.

Lemma 4.4. *Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. For $f \in L^p_\alpha$, define*

$$\Phi_\beta f(z) = \int_{\mathbf{H}} f(w) \frac{w_n^\beta}{|z - \overline{w}|^{n+\beta}}\, dw$$

for $z \in \mathbf{H}$. Then, $\Phi_\beta : L^p_\alpha \to L^p_\alpha$ is bounded.

Let $\{z_m\}$ be a sequence in $\mathbf{H}$. Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta+1)p$. For $(\lambda_m) \in l^p$, let $Q_\beta(\lambda_m)$ denote the series defined by

$$Q_\beta(\lambda_m)(z) = \sum \lambda_m z_{mn}^{(n+\beta)(1-1/p)+(\beta-\alpha)/p}\, R_\beta(z, z_m), \tag{4.1}$$

for $z \in \mathbf{H}$. For a sequence $\{z_m\}$ good enough, $Q_\beta(\lambda_m)$ will be harmonic on $\mathbf{H}$. We say that $\{z_m\}$ is a b^p_α-representing sequence of order β if $Q_\beta(l^p) = b^p_\alpha$. Lemma 4.4 implies the following proposition which shows $Q_\beta(l^p) \subset b^p_\alpha$ if the underlying sequence is separated.

Proposition 4.5. *Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. Suppose $\{z_m\}$ is a δ-separated sequence. Then $Q_\beta : l^p \to b^p_\alpha$ is bounded.*

Proof. For $(\lambda_m) \in l^p$, put $f = \sum |\lambda_m| z_{mn}^{(n+\beta)(1-1/p)+(\beta-\alpha)/p} |E_\delta(z_m)|_\beta^{-1} \chi_m$ where χ_m is the characteristic function of $E_\delta(z_m)$. By (2.5) and

Lemma 3.2, there exists a constant $C = C(n, \beta, \delta)$ such that

$$|R_\beta(z, z_m)| \le \frac{C}{|z - \overline{z}_m|^{n+\beta}} \le \frac{C}{|z - \overline{w}|^{n+\beta}}$$

for all $w \in E_\delta(z_m)$ and $z \in \mathbf{H}$. Thus, we get

$$\begin{aligned}|Q_\beta(\lambda_m)(z)| \le C \sum |\lambda_m| &\frac{z_{mn}^{(n+\beta)(1-1/p)+(\beta-\alpha)/p}}{|E_\delta(z_m)|_\beta} \\ &\times \int_{E_\delta(z_m)} \frac{w_n^\beta}{|z - \overline{w}|^{n+\beta}}\, dw = C\Phi_\beta f(z).\end{aligned}$$

Note from (3.2) and Lemma 3.1 that $|E_\delta(z_m)|_\alpha \approx z_{mn}^{n+\alpha}$. Thus, we obtain from Lemma 4.4 that

$$\begin{aligned}\|Q_\beta(\lambda_m)\|_{L_\alpha^p}^p &\le C \sum |\lambda_m|^p z_{mn}^{(n+\beta)(p-1)+(\beta-\alpha)} |E_\delta(z_m)|_\beta^{-p}\, |E_\delta(z_m)|_\alpha \\ &\le C \sum |\lambda_m|^p.\end{aligned}$$

This shows that $Q_\beta : l^p \to L_\alpha^p$ is bounded and the series in (4.1) converges in norm. Since every term in the series (4.1) is harmonic, the series converges uniformly on compact subsets of H. Consequently, we have $Q_\beta : l^p \to b_\alpha^p$ is bounded. This completes the proof. □

Now, we prove the main theorem in this section.

Theorem 4.6. *Let $\alpha > -1$, $1 \le p < \infty$ and $\alpha + 1 < (\beta+1)p$. Then there exists $\delta_0 > 0$ with the following property. Let $\{z_m\}$ be a δ-lattice with $\delta < \delta_0$ and let $Q_\beta : l^p \to b_\alpha^p$ be the associated linear operator as in (4.1). Then there is a bounded linear operator $\mathcal{P}_\beta : b_\alpha^p \to l^p$ such that $Q_\beta \mathcal{P}_\beta$ is the identity on b_α^p. In particular, $\{z_m\}$ is a b_α^p-representing sequence of order β.*

Proof. Let $u \in b_\alpha^p$. We may assume $\delta < 1/8$. Fix a 1/2-lattice $\{a_m\}$. Find a rearrangement $\{z_{ij}\}$ of $\{z_m\}$, as well as a pairwise disjoint covering $\{D_{ij}\}$ of $\mathbf{H}$, for which all properties of Lemma 4.1 are satisfied. Note from Lemma 3.1 and (3.2) that there exist C_1 and C_2 independent of δ such that

$$C_1^{-1} < \frac{w_n}{z_{ijn}} < C_1, \quad C_2^{-1}\delta^n z_{ijn}^{n+\alpha} < |E_\delta(z_{ij})|_\alpha < C_2 \delta^n z_{ijn}^{n+\alpha} \tag{4.2}$$

for all $w \in E_\delta(z_{ij})$ because $\delta < 1/8$. Then, we have from (a) in Lemma 4.1 and Lemma 3.6 that

$$z_{ijn}^{n+\alpha-(n+\beta)p}|D_{ij}|_\beta^p|u(z_{ij})|^p \le C\delta^{n(p-1)} \int_{D_{ij}} |u(w)|^p\, w_n^\alpha\, dw. \tag{4.3}$$

Let Tu denote the sequence $\big(z_{ijn}^{(n+\beta)(1/p-1)-(\beta-\alpha)/p}|D_{ij}|_\beta u(z_{ij})\big)$. Then we have from (4.3) that

$$\|Tu\|_{l^p}^p \le C\delta^{n(p-1)} \sum \int_{D_{ij}} |u(w)|^p\, w_n^\alpha\, dw = C\|u\|_{L_\alpha^p}^p.$$

This shows that $T : b_\alpha^p \to l^p$ is bounded and thus $Q_\beta T$ is bounded on b_α^p by Proposition 4.5.

Now, we show that $Q_\beta T$ is invertible on b_α^p for all δ sufficiently small. Let χ_{ij} denote the characteristic function of D_{ij}. Then we know from Lemma 3.4, $u = \Pi_\beta u = \Pi_\beta\left[\sum u\chi_{ij}\right]$. Since $Q_\beta Tu(z) = \sum |D_{ij}|_\beta u(z_{ij}) R_\beta(z, z_{ij})$, we have $u - Q_\beta Tu = u_1 + u_2$ where

$$u_1(z) = \Pi_\beta\left[\sum \big(u - u(z_{ij})\big)\,\chi_{ij}\right](z),$$
$$u_2(z) = \sum u(z_{ij}) \int_{D_{ij}} R_\beta(z,w) - R_\beta(z, z_{ij})\, dV_\beta(w).$$

Note from (c) in Lemma 4.1 that $D_{ij} \subset E_\delta(z_{ij}) \subset E_{1/2+\delta}(a_i) \subset E_{5/8}(a_i)$. Hence, we have from (4.2)

$$d(E_\delta(z_{ij}), \partial E_{2/3}(a_i)) \ge d(E_{5/8}(a_i), \partial E_{2/3}(a_i)) \ge C a_{in} \ge C z_{ijn}$$

for some absolute constant C. Thus, we get from Lemma 3.7 and (4.2) that

$$\begin{aligned}
&\int_{D_{ij}} |u(w) - u(z_{ij})|^p\, dV_\alpha(w) \\
&\qquad\le C\, \frac{|E_\delta(z_{ij})|^{p/n} |E_\delta(z_{ij})|_\alpha}{d(E_\delta(z_{ij}), \partial E_{2/3}(a_i))^{n+p}} \int_{E_{2/3}(a_i)} |u(w)|^p\, dw \\
&\qquad\le C\delta^{n+p} \int_{E_{2/3}(a_i)} |u(w)|^p\, w_n^\alpha\, dw
\end{aligned}$$

for all i,j. Here, the constant C is independent of i,j and δ. Thus, for each fixed i, Lemma 4.3 implies

$$\sum_{j=1}^{N_i}\int_{D_{ij}}|u(w)-u(z_{ij})|^p\,dV_\alpha(w)\le C\delta^p\int_{E_{2/3}(a_i)}|u|^p\,dV_\alpha. \tag{4.4}$$

Therefore, we get from Lemma 3.4 that

$$\begin{aligned}\|u_1\|^p_{L^p_\alpha}&\le C\Big\|\sum_{i,j}\big(u-u(z_{ij})\big)\,\chi_{ij}\Big\|^p_{L^p_\alpha}\\ &=C\sum_{i,j}\int_{D_{ij}}|u(w)-u(z_{ij})|^p\;dV_\alpha(w)\\ &\le C\delta^p\sum_i\int_{E_{2/3}(a_i)}|u|^p\;dV_\alpha\le C\delta^p\|u\|^p_{L^p_\alpha}.\end{aligned}\tag{4.5}$$

The last inequality of the above holds by Lemma 4.2. Here, the constant C is independent of δ.

Now, we show $\|u_2\|_{L^p_\alpha}\le C\delta\|u\|_{L^p_\alpha}$ for some constant C independent of δ. Note from Lemma 3.3 and Lemma 3.2 that

$$\begin{aligned}\int_{D_{ij}}|R_\beta(z,w)-R_\beta(z,z_{ij})|\;dV_\beta(w)&\le C\int_{D_{ij}}\frac{\rho(w,z_{ij})}{|z-\overline{z}_{ij}|^{n+\beta}}\,dV_\beta(w)\\ &\le C\delta\,\frac{1}{|z-\overline{a}_i|^{n+\beta}}\,|D_{ij}|_\beta.\end{aligned}$$

Then, we have from (4.3) and (4.2) that

$$\begin{aligned}|u_2(z)|&\le C\delta\sum_{i,j}\frac{1}{|z-\overline{a}_i|^{n+\beta}}\,|D_{ij}|_\beta\,|u(z_{ij})|\\ &\le C\delta\sum_{i,j}\frac{z_{ijn}^{\beta-\alpha}}{|z-\overline{a}_i|^{n+\beta}}\int_{D_{ij}}|u|\,dV_\alpha\\ &\le C\delta\sum_i\frac{a_{in}^{\beta-\alpha}}{|z-\overline{a}_i|^{n+\beta}}\int_{E_{2/3}(a_i)}|u|\,dV_\alpha.\end{aligned}\tag{4.6}$$

The last inequality of the above holds (b) in Lemma 4.1. Note from Lemma 3.2 and (4.2) that

$$\frac{a_{in}^{\beta-\alpha}}{|z-\overline{a}_i|^{n+\beta}}\le\frac{C}{|E_{2/3}(a_i)|_\alpha}\int_{E_{2/3}(a_i)}\frac{w_n^\beta}{|z-\overline{w}|^{n+\beta}}\,dw. \tag{4.7}$$

Let $\lambda_i = \left(\int_{E_{2/3}(a_i)} |u(w)|^p \, dV_\alpha(w)\right)^{1/p}$, and let χ_i be the characteristic function of $E_{2/3}(a_i)$. If $p = 1$, we have from (4.6) and (4.7)

$$|u_2(z)| \le \Phi_\beta \left[C\delta \sum_i \lambda_i |E_{2/3}(a_i)|_\alpha^{-1} \chi_i \right](z).$$

Thus, Lemma 4.4 and Lemma 4.2 yield

$$\|u_2\|_{L^1_\alpha} \le C\delta \sum_i |\lambda_i| = C\delta \sum_i \int_{E_{2/3}(a_i)} |u| \, dV_\alpha \le C\delta \|u\|_{L^1_\alpha}. \tag{4.8}$$

Here, the constant C is independent of δ. Assume that $p > 1$. Hölder's inequality and (4.7) imply that (4.6) is less than or equal to

$$\begin{aligned} C\delta \sum_i \frac{a_{in}^{\beta-\alpha}}{|z - \overline{a}_i|^{n+\beta}} |E_{2/3}(a_i)|_\alpha^{1/q} &\left(\int_{E_{2/3}(a_i)} |u|^p \, dV_\alpha \right)^{1/p} \\ &\le C\delta \sum_i \lambda_i |E_{2/3}(a_i)|_\alpha^{1/q-1} \int_{E_{2/3}(a_i)} \frac{1}{|z - \overline{w}|^{n+\beta}} \, dV_\beta(w) \\ &\le \Phi_\beta \left[C\delta \sum_i \lambda_i |E_{2/3}(a_i)|_\alpha^{-1/p} \chi_i \right](z) \end{aligned}$$

where q is the index conjugate to p. Now, Lemma 4.4 and Lemma 4.2 yield

$$\|u_2\|^p_{L^p_\alpha} \le C\delta^p \sum_i |\lambda_i|^p \le C\delta^p \|u\|^p_{L^p_\alpha}. \tag{4.9}$$

Here, the constant C is independent of δ. Let I be the identity on b^p_α. Then (4.5), (4.8) and (4.9) imply $\|Q_\beta T - I\| \le C\delta$ for some constant C independent of δ. Therefore, $Q_\beta T$ is invertible for all δ sufficiently small. For such δ, set $\mathcal{P}_\beta = T\,(Q_\beta T)^{-1}$. This completes the proof. $\square$

Since $\mathcal{D}^\gamma u$ is harmonic and we have (2.7), we can have a similar result with Proposition 4.8 of [**4**].

Proposition 4.7. *Let $\alpha > -1$, $1 \le p < \infty$, and let $(1+\alpha)/p + \gamma > 0$. If $\{z_m\}$ is a δ-lattice with δ sufficiently small, then*

$$\|u\|_{L^p_\alpha}^p \approx \sum z_{mn}^{n+\alpha+p\gamma} \, |\mathcal{D}^\gamma u(z_m)|^p$$

as u ranges over b^p_α.

5. Representation on $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}_0$. In this section we prove the representation property of $\widetilde{\mathcal{B}}$-functions and $\widetilde{\mathcal{B}}_0$-functions. Let $\{z_m\}$ be a sequence in $\mathbf{H}$, and let $\beta > -1$. For $(\lambda_m) \in l^\infty$, let

$$\widetilde{Q}_\beta(\lambda_m)(z) = \sum \lambda_m z_{mn}^{n+\beta} \, \widetilde{R}_\beta(z, z_m) \tag{5.1}$$

for $z \in \mathbf{H}$. We say that $\{z_m\}$ is a $\widetilde{\mathcal{B}}$-representing sequence of order β if $\widetilde{Q}_\beta(l^\infty) = \widetilde{\mathcal{B}}$. We also say that $\{z_m\}$ is a $\widetilde{\mathcal{B}}_0$-representing sequence of order β if $\widetilde{Q}_\beta(\mathcal{C}_0) = \widetilde{\mathcal{B}}_0$. As in the case of b^p_α-representation, we begin with a observation that a separated sequence represents a part of the whole space. The proof of the following proposition is the same with that of Proposition 4.9 in [**4**].

Proposition 5.1. *Let $\beta > -1$ and suppose $\{z_m\}$ is a δ-separated sequence. Then, $\widetilde{Q}_\beta : l^\infty \to \widetilde{\mathcal{B}}$ is bounded. In addition, $\widetilde{Q}_\beta$ maps $\mathcal{C}_0$ into $\widetilde{\mathcal{B}}_0$.*

If γ is a positive integer, then the following lemma is proved in [**4**]. Therefore to complete the proof of the lemma, we only need to show the case that γ is not an integer.

Lemma 5.2. *Let $\gamma > 0$. Then*

$$|z_n^\gamma \, \mathcal{D}^\gamma u(z) - w_n^\gamma \, \mathcal{D}^\gamma u(w)| \le C\rho(z, w)\|u\|_{\mathcal{B}}$$

for all $z, w \in \mathbf{H}$ and $u \in \widetilde{\mathcal{B}}$.

Proof. Let $u \in \widetilde{\mathcal{B}}$. Fix $z, w \in \mathbf{H}$. By (2.10), we may assume $\rho(z, w) < 1/2$. Note from (2.9) that $u(z) = C\widetilde{\Pi}_\alpha(s_n Du)(z) =$

$C\int_{\mathbf{H}} s_n Du(s)\widetilde{R}_\alpha(z,s)\,dV_\alpha(s)$. Thus, from the definition of the fractional derivative, we have

$$\begin{aligned}(5.2)\quad &|z_n^\gamma \mathcal{D}^\gamma u(z) - w_n^\gamma \mathcal{D}^\gamma u(w)| \\ &\le C\int_0^\infty \bigl|z_n^\gamma D^{[\gamma]}u(z', z_n+t) - w_n^\gamma D^{[\gamma]}u(w', w_n+t)\bigr|\, t^{[\gamma]-\gamma-1}\,dt \\ &\le C\int_0^\infty \int_{\mathbf{H}} |s_n Du(s)|\, \bigl|z_n^\gamma D_{z_n}^{[\gamma]}\widetilde{R}_\alpha\bigl((z', z_n+t), s\bigr) \\ &\qquad\qquad - w_n^\gamma D_{w_n}^{[\gamma]}\widetilde{R}_\alpha\bigl((w', w_n+t), s\bigr)\bigr|\, dV_\alpha(s)\, t^{[\gamma]-\gamma-1}\,dt.\end{aligned}$$

Note that $D_{z_n}^{[\gamma]}\widetilde{R}_\alpha\bigl((z', z_n+t), s\bigr) = D_{z_n}^{[\gamma]}R_\alpha\bigl((z', z_n+t), s\bigr) = CR_{\alpha+[\gamma]}\bigl((z', z_n+t), s\bigr)$. Thus, Lemma 3.3 and Fubini's theorem imply that (5.2) is less than or equal to

$$\begin{aligned}(5.3)\quad &C\|u\|_{\mathcal{B}}\int_0^\infty \int_{\mathbf{H}} \bigl|z_n^\gamma R_{\alpha+[\gamma]}\bigl((z', z_n+t), s\bigr) \\ &\qquad\qquad - w_n^\gamma R_{\alpha+[\gamma]}\bigl((w', w_n+t), s\bigr)\bigr|\, dV_\alpha(s)\, t^{[\gamma]-\gamma-1}\,dt \\ &\le C\rho(z,w)\|u\|_{\mathcal{B}} z_n^\gamma \int_{\mathbf{H}}\int_0^\infty \frac{t^{[\gamma]-\gamma-1}}{|(z', z_n+t)-\overline{s}|^{n+\alpha+[\gamma]}}\,dt\,dV_\alpha(s).\end{aligned}$$

Note that $|(z', z_n+t)-\overline{s}| \approx |z-\overline{s}|+t$ for $s\in\mathbf{H}$, $t>0$. Thus, (5.3) is less than or equal to

$$\begin{aligned}&C\rho(z,w)\|u\|_{\mathcal{B}} z_n^\gamma \int_{\mathbf{H}}\int_0^\infty \frac{t^{[\gamma]-\gamma-1}}{(|z-\overline{s}|+t)^{n+\alpha+[\gamma]}}\,dt\,dV_\alpha(s) \\ &\qquad\le C\rho(z,w)\|u\|_{\mathcal{B}} z_n^\gamma \int_{\mathbf{H}} \frac{s_n^\alpha}{|z-\overline{s}|^{n+\alpha+\gamma}}\,ds \le C\rho(z,w)\|u\|_{\mathcal{B}}\end{aligned}$$

after applying change of variable $t = |z-\overline{s}|t$ and Lemma 3.5. This completes the proof. □

Having Proposition 5.1 and Lemma 5.2, we can modify the proof of Theorem 4.6 to obtain a similar $\widetilde{\mathcal{B}}$-representation theorem.

Theorem 5.3. *Let $\beta > -1$. Then there exists a positive number δ_0 with the following property. Let $\{z_m\}$ be a δ-lattice with $\delta < \delta_0$, and let $\widetilde{Q}_\beta : l^\infty \to \widetilde{\mathcal{B}}$ be the associated linear operator as in* (5.1)*. Then there exists a bounded linear operator $\widetilde{\mathcal{P}}_\beta : \widetilde{\mathcal{B}} \to l^\infty$ such that $\widetilde{Q}_\beta \widetilde{\mathcal{P}}_\beta$ is the identity on $\widetilde{\mathcal{B}}$. Moreover, $\widetilde{\mathcal{P}}_\beta$ maps $\widetilde{\mathcal{B}}_0$ into $\mathcal{C}_0$. In particular, $\{z_m\}$ is both a $\widetilde{\mathcal{B}}$-representing and $\widetilde{\mathcal{B}}_0$-representing sequence of order β.*

Lemma 5.2 yields the following result for $\widetilde{\mathcal{B}}$ analogous to Proposition 4.7.

Proposition 5.4. *Let $\gamma > 0$. Let $\{z_m\}$ be a δ-lattice with δ sufficiently small. Then*

$$\|u\|_{\mathcal{B}} \approx \sup_m z_{mn}^{\gamma} \, |\mathcal{D}^\gamma u(z_m)|$$

as u ranges over $\widetilde{\mathcal{B}}$.

6. Interpolation on b^p_α. In this section we prove the interpolation theorem for the space b^p_α. Let $\{z_m\}$ be a sequence on $\mathbf{H}$. Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p + \gamma > 0$. For $u \in b^p_\alpha$, let $T_\gamma u$ denote the sequence of complex numbers defined by

$$T_\gamma u = \left(z_{mn}^{(n+\alpha)/p+\gamma} \, \mathcal{D}^\gamma u(z_m) \right). \tag{6.1}$$

If $T_\gamma(b^p_\alpha) = l^p$, we say that $\{z_m\}$ is a b^p_α-interpolating sequence of order γ.

The following two lemmas are used to prove that separation is necessary for b^p_α-interpolation.

Lemma 6.1. *Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p + \gamma > 0$. Let $\{z_m\}$ be a b^p_α-interpolating sequence of order γ. Then, $T_\gamma : b^p_\alpha \to l^p$ is bounded.*

Proof. Assume $u_j \to u$ in b^p_α and $T_\gamma u_j \to (\lambda_m)$ in l^p. By the closed graph theorem, we need to show $T_\gamma u = (\lambda_m)$. Note from Lemma 3.6, Lemma 3.1 and (2.7) that

$$\begin{aligned}\sum_{m=1}^{N} z_{mn}^{n+\alpha+p\gamma}\,|\mathcal{D}^\gamma u(z_m)-\mathcal{D}^\gamma u_j(z_m)|^p \\ &\le C\sum_{m=1}^{N}\int_{E_\delta(z_m)} |w_n^\gamma \mathcal{D}^\gamma (u-u_j)(w)|^p\, w_n^\alpha\,dw \\ &\le CN\,\|u-u_j\|_{L_\alpha^p}^p.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|T_\gamma u-(\lambda_m)\|_{l^p}^p &= \sum_{m=1}^{\infty}\left|z_{mn}^{(n+\alpha)/p+\gamma}\,\mathcal{D}^\gamma u(z_m)-\lambda_m\right|^p \\ &\le C\sum_{m=1}^{N} z_{mn}^{n+\alpha+p\gamma}\,|\mathcal{D}^\gamma u(z_m)-\mathcal{D}^\gamma u_j(z_m)|^p \\ &\quad + C\sum_{m=1}^{N}|z_{mn}^{(n+\alpha)/p+\gamma}\,\mathcal{D}^\gamma u_j(z_m)-\lambda_m|^p \\ &\quad + \sum_{m=N+1}^{\infty}\left|z_{mn}^{(n+\alpha)/p+\gamma}\,\mathcal{D}^\gamma u(z_m)-\lambda_m\right|^p \\ &\le CN\|u-u_j\|_{L_\alpha^p}^p + \|T_\gamma u_j-(\lambda_m)\|_{l^p}^p \\ &\quad + \sum_{m=N+1}^{\infty}\left|z_{mn}^{(n+\alpha)/p+\gamma}\,\mathcal{D}^\gamma u(z_m)-\lambda_m\right|^p\end{aligned}$$

for every N. Taking first the limit $j\to\infty$ and then $N\to\infty$, we have $T_\gamma u=(\lambda_m)$. This completes the proof. □

The following lemma is a b_α^p-version of Lemma 5.2 which is the result of $\widetilde{\mathcal{B}}$-functions. If γ is a nonnegative integer, then the following lemma is proved in [**4**]. Therefore to complete the proof of the lemma, we only need to show the case that γ is not a nonnegative integer.

Lemma 6.2. *Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p + \gamma > 0$. Then,*

$$\left| z_n^{(n+\alpha)/p+\gamma}\, \mathcal{D}^\gamma u(z) - w_n^{(n+\alpha)/p+\gamma}\, \mathcal{D}^\gamma u(w) \right| \le C\rho(z,w)\|u\|_{L^p_\alpha}$$

for all $z, w \in \mathbf{H}$ and $u \in b^p_\alpha$.

Proof. Let $u \in b^p_\alpha$ and fix $z, w \in \mathbf{H}$. By Lemma 3.6, we may assume $\rho(z,w) < 1/2$. Note from (2.6) that $u(z) = \int_{\mathbf{H}} u(s) R_\alpha(z,s)\, dV_\alpha(s)$. Thus, letting $k = [\gamma]$ if $\gamma > -1$ and $k = 0$ if $\gamma \le -1$, we have from Lemma 3.3 and Fubini's theorem that

$$\begin{aligned}
(6.2)\quad & \left| z_n^{(n+\alpha)/p+\gamma}\, \mathcal{D}^\gamma u(z) - w_n^{(n+\alpha)/p+\gamma}\, \mathcal{D}^\gamma u(w) \right| \\
&\le C \int_0^\infty \int_{\mathbf{H}} |u(s)| \left| z_n^{(n+\alpha)/p+\gamma} D^k_{z_n} R_\alpha\big((z', z_n + t), s\big) \right. \\
&\qquad \left. - w_n^{(n+\alpha)/p+\gamma} D^k_{w_n} R_\alpha\big((w', w_n + t), s\big) \right| dV_\alpha(s)\, t^{k-\gamma-1}\, dt \\
&\le C\rho(z,w) \int_{\mathbf{H}} |u(s)| z_n^{(n+\alpha)/p+\gamma} \\
&\qquad \times \int_0^\infty \frac{t^{k-\gamma-1}}{(|z-\overline{s}|+t)^{n+\alpha+k}}\, dt\; dV_\alpha(s) \\
&\le C\rho(z,w) \int_{\mathbf{H}} |u(s)| \frac{z_n^{(n+\alpha)/p+\gamma}}{|z-\overline{s}|^{n+\alpha+\gamma}}\, dV_\alpha(s)
\end{aligned}$$

after applying change of variable $t = |z-\overline{s}|t$. If $p = 1$, then we have from (6.2),

$$|z_n^{(n+\alpha)/p+\gamma}\, \mathcal{D}^\gamma u(z) - w_n^{(n+\alpha)/p+\gamma}\, \mathcal{D}^\gamma u(w)| \le C\rho(z,w)\|u\|_{L^1_\alpha}$$

because $n+\alpha+\gamma > 0$. Assume $1 < p < \infty$. Note that $(1+\alpha)/p+\gamma > 0$ implies $n+\alpha < (n+\alpha+\gamma)q$ where q is the index conjugate to p. Thus, Hölder's inequality and Lemma 3.5 imply that (6.2) is less than or equal to

$$C\rho(z,w)\|u\|_{L^p_\alpha} \left(\int_{\mathbf{H}} \frac{z_n^{(n+\alpha)q/p+\gamma q}}{|z-\overline{s}|^{(n+\alpha+\gamma)q}}\, dV_\alpha(s) \right)^{1/q} \le C\rho(z,w)\|u\|_{L^p_\alpha}.$$

The proof is complete. $\square$

Since we have Lemma 6.1 and Lemma 6.2, the proof of the following proposition is the same as that of Proposition 5.3 in [**4**] and thus omitted.

Proposition 6.3. *Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p + \gamma > 0$. Every b^p_α-interpolating sequence of order γ is separated.*

The following lemma is used to prove b^p_α-interpolation theorem.

Lemma 6.4. *Let $\alpha > -1$, $1 < p < \infty$ and $(1+\alpha)/p + \gamma > 0$. Let $\{z_m\}$ be a δ-separated sequence. Then, for $(\lambda_m) \in l^p$, we have*

$$\begin{aligned}\Big|\sum \lambda_m z_{mn}^{(n+\alpha)/q}\, \mathcal{D}^\gamma R_\alpha(z_m, w)\Big|^p &\le C\delta^{n(1-p)} w_n^{-(1+\alpha+p\gamma)/q} \\ &\quad \times \sum |\lambda_m|^p z_{mn}^{(1+\alpha)/q}\, |\mathcal{D}^\gamma R_\alpha(z_m, w)|\end{aligned}$$

for $w \in \mathbf{H}$ and q is the index conjugate to p. The constant C is independent of δ.

Proof. Note from Lemma 3.6, (2.5) and Lemma 3.5 that

$$\begin{aligned}&\sum z_{mn}^{(n+\alpha)-(1+\alpha)/p}\, |\mathcal{D}^\gamma R_\alpha(z_m, w)| \\ &\qquad\le C\delta^{-n} \sum z_{mn}^{\alpha-(1+\alpha)/p} \int_{E_{\delta/2}(z_m)} |\mathcal{D}^\gamma R_\alpha(s, w)|\, ds \\ &\qquad\le C\delta^{-n} \int_{\mathbf{H}} \frac{s_n^{\alpha-(1+\alpha)/p}}{|s-\overline{w}|^{n+\alpha+\gamma}}\, ds \\ &\qquad\le C\delta^{-n} w_n^{-(1+\alpha)/p-\gamma}\end{aligned}$$

because $1/3 < z_{mn}/s_n < 3$ for $s \in E_{\delta/2}(z_m)$. Here, the constant C is independent of δ. Thus, applying Hölder's inequality to the following two functions,

$$|\lambda_m| z_{mn}^{(1+\alpha)/pq} |\mathcal{D}^\gamma R_\alpha(z_m, w)|^{1/p}, \quad z_{mn}^{(n+\alpha)/q} z_{mn}^{-(1+\alpha)/pq} |\mathcal{D}^\gamma R_\alpha(z_m, w)|^{1/q},$$

we have

$$\begin{aligned}
&\Big|\sum \lambda_m z_{mn}^{(n+\alpha)/q}\, \mathcal{D}^\gamma R_\alpha(z_m,w)\Big|^p \\
&\qquad \le \bigg(\sum |\lambda_m|^p z_{mn}^{(1+\alpha)/q}\, |\mathcal{D}^\gamma R_\alpha(z_m,w)| \bigg) \\
&\qquad\quad \times \bigg(\sum z_{mn}^{(n+\alpha)-(1+\alpha)/p}\, |\mathcal{D}^\gamma R_\alpha(z_m,w)| \bigg)^{p/q} \\
&\qquad \le C\delta^{-np/q} w_n^{-(1+\alpha+p\gamma)/q} \sum |\lambda_m|^p z_{mn}^{(1+\alpha)/q}\, |\mathcal{D}^\gamma R_\alpha(z_m,w)|.
\end{aligned}$$

Here, the constant C is independent of δ. The proof is complete. □

Now, we prove the main theorem of this section.

Theorem 6.5. *Let $\alpha > -1$, $1 \le p < \infty$ and $(1+\alpha)/p + \gamma > 0$. Then there exists a positive number δ_0 with the following property. Let $\{z_m\}$ be a δ-separated sequence with $\delta > \delta_0$, and let $T_\gamma : b_\alpha^p \to l^p$ be the associated linear operator as in* (6.1). *Then there is a bounded linear operator $S_\gamma : l^p \to b_\alpha^p$ such that $T_\gamma S_\gamma$ is the identity on l^p. In particular, $\{z_m\}$ is a b_α^p-interpolating sequence of order γ.*

Proof. Fix γ. Note that $D^{k+1}P_z(w) = C(k)\sum_{m=0}^{k+2} C(m)(z_n+w_n)^m/|z-\overline{w}|^{n+k+m}$ for some nonnegative integer k. Thus, for the case that both α and γ are nonnegative integers, $w_n^{n+\alpha+\gamma}\mathcal{D}^\gamma R_\alpha(w,w)$ is constant. Assume that both α and γ are not nonnegative integers. Let $k = [\gamma]$ if $\gamma > -1$, and let $k = 0$ if $\gamma \le -1$. Then we have

$$\begin{aligned}
&w_n^{n+\alpha+\gamma}\mathcal{D}^\gamma R_\alpha(w,w) \\
&\quad = C w_n^{n+\alpha+\gamma} \int_0^\infty\!\!\int_0^\infty D^{k+[\alpha]+1}\, P\big((w', w_n+s), (w', w_n+t)\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times t^{[\alpha]-\alpha-1}\, dt\, s^{k-\gamma-1}\, ds \\
&\quad = C w_n^{n+\alpha+\gamma} \sum_{m=0}^{k+[\alpha]+2} C(m) \int_0^\infty\!\!\int_0^\infty \frac{t^{[\alpha]-\alpha-1} s^{k-\gamma-1}}{(2w_n+s+t)^{n+k+[\alpha]}}\, dt\, ds.
\end{aligned}$$

Thus, applying change of variable, we have that $w_n^{n+\alpha+\gamma}\mathcal{D}^\gamma R_\alpha(w,w)$ is constant depending only on α and γ. For the remaining case, we have the same result. Thus, we will let $d_{\alpha,\gamma}$ denote $w_n^{n+\alpha+\gamma}\mathcal{D}^\gamma R_\alpha(w,w)$.

Let $1 < p < \infty$. Fix $(\lambda_m) \in l^p$. Let $Q_\alpha(\lambda_m)$ denote the function by

$$Q_\alpha(\lambda_m)(z) = \sum \lambda_m z_{mn}^{(n+\alpha)/q} R_\alpha(z, z_m) \tag{6.3}$$

where $z \in \mathbf{H}$ and q is the index conjugate to p. By Proposition 4.5, we have $Q_\alpha : l^p \to b_\alpha^p$ is a bounded operator. Thus, $T_\gamma Q_\alpha$ is bounded on l^p by Lemma 6.1.

We show that $T_\gamma Q_\alpha$ is invertible on l^p for all δ sufficiently close to 1. Let I denote the identity on l^p, and let (α_j) denote the jth component of the sequence of $(T_\gamma Q_\alpha - d_{\alpha,\gamma} I)(\lambda_m)$. Since the series in (6.3) converges uniformly on compact subsets of H, interchanging differentiation and sum yields

$$\begin{aligned} \alpha_j &= z_{jn}^{(n+\alpha)/p+\gamma} \mathcal{D}^\gamma Q_\alpha(\lambda_m)(z_j) - d_{\alpha,\gamma}\lambda_j \\ &= z_{jn}^{(n+\alpha)/p+\gamma} \sum_{m \neq j} \lambda_m z_{mn}^{(n+\alpha)/q} \mathcal{D}^\gamma R_\alpha(z_m, z_j). \end{aligned}$$

Thus, Lemma 6.4 gives

$$|\alpha_j|^p \leq C\delta^{n(1-p)} z_{jn}^{(n+\alpha+\gamma)-(1+\alpha)/q} \sum_{m \neq j} |\lambda_m|^p \, z_{mn}^{(1+\alpha)/q} \, |\mathcal{D}^\gamma R_\alpha(z_m, z_j)|$$

so that

$$\begin{aligned} \sum |\alpha_j|^p &\leq C\delta^{n(1-p)} \sum_{m=1}^\infty |\lambda_m|^p \, z_{mn}^{(1+\alpha)/q} \\ &\qquad \times \sum_{j \neq m} z_{jn}^{(n+\alpha+\gamma)-(1+\alpha)/q} \, |\mathcal{D}^\gamma R_\alpha(z_m, z_j)| \\ &:= C\delta^{n(1-p)} \sum_{m=1}^\infty |\lambda_m|^p \, \beta_m \end{aligned} \tag{6.4}$$

where

$$\beta_m = z_{mn}^{(1+\alpha)/q} \sum_{j \neq m} z_{jn}^{(n+\alpha+\gamma)-(1+\alpha)/q} \, |\mathcal{D}^\gamma R_\alpha(z_m, z_j)|.$$

By Lemma 3.6 and Lemma 3.1, we have

$$\begin{aligned}
\beta_m &\le C\delta^{-n} z_{mn}^{(1+\alpha)/q} \sum_{j\neq m} z_{jn}^{\alpha+\gamma-(1+\alpha)/q} \int_{E_{\delta/2}(z_j)} |\mathcal{D}^\gamma R_\alpha(z_m,s)|\,ds \\
&\le C\delta^{-n} z_{mn}^{(1+\alpha)/q} \sum_{j\neq m} \int_{E_{\delta/2}(z_j)} s_n^{\alpha+\gamma-(1+\alpha)/q} |\mathcal{D}^\gamma R_\alpha(z_m,s)|\,ds \\
&\le C\delta^{-n} z_{mn}^{(1+\alpha)/q} \int_{\mathbf{H}\setminus E_\delta(z_m)} \frac{s_n^{\alpha+\gamma-(1+\alpha)/q}}{|s-\overline{z}_m|^{n+\alpha+\gamma}}\,ds \\
&= C\delta^{-n} \int_{\mathbf{H}\setminus E_\delta(z_0)} \frac{s_n^{\alpha+\gamma-(1+\alpha)/q}}{|s-\overline{z}_0|^{n+\alpha+\gamma}}\,ds
\end{aligned}$$

for all m. Here, the constant C is independent of δ. The last equality of the above holds by change of variable $s = \phi_{z_m}^{-1}(s)$. Thus, (6.4) is less than or equal to

$$C\delta^{-np} \int_{\mathbf{H}\setminus E_\delta(z_0)} \frac{s_n^{\alpha+\gamma-(1+\alpha)/q}}{|s-\overline{z}_0|^{n+\alpha+\gamma}}\,ds.$$

Consequently, we obtain

$$\|T_\gamma Q_\alpha - d_{\alpha,\gamma} I\|_{l^p} \le C\delta^{-n} \left(\int_{\mathbf{H}\setminus E_\delta(z_0)} \frac{s_n^{\alpha+\gamma-(1+\alpha)/q}}{|s-\overline{z}_0|^{n+\alpha+\gamma}}\,ds \right)^{1/p} \tag{6.5}$$

for some constant C independent of δ. Since Lemma 3.5 yields

$$\int_{\mathbf{H}} \frac{s_n^{\alpha+\gamma-(1+\alpha)/q}}{|s-\overline{z}_0|^{n+\alpha+\gamma}}\,ds < \infty,$$

the integral in (6.5) tends to 0 as $\delta \nearrow 1$. Thus $T_\gamma Q_\alpha$ is invertible on l^p for all δ sufficiently close to 1. For such δ, put $S_\gamma = Q_\alpha\,(T_\gamma Q_\alpha)^{-1}$.

Let $p = 1$. Fix $(\lambda_m) \in l^1$. Let $Q_{\alpha+1}(\lambda_m)$ denote by

$$Q_{\alpha+1}(\lambda_m)(z) = \sum \lambda_m z_{mn} R_{\alpha+1}(z, z_m)$$

for $z \in \mathbf{H}$. Then Proposition 4.5 and Lemma 6.1 yield that $Q_{\alpha+1} : l^1 \to b_\alpha^1$ is bounded and $T_\gamma Q_{\alpha+1}$ is bounded on l^1. Now, we show that $T_\gamma Q_{\alpha+1}$ is invertible on l^1 for all δ sufficiently close to 1. Let α_j denote the jth component of the sequence $(T_\gamma Q_{\alpha+1} - d_{\alpha+1,\gamma} I)(\lambda_m)$. Differentiating term by term yields

$$\begin{aligned}\alpha_j &= z_{jn}^{n+\alpha+\gamma}\mathcal{D}^\gamma Q_{\alpha+1}(\lambda_m)(z_j) - d_{\alpha+1,\gamma}\lambda_j\\ &= z_{jn}^{n+\alpha+\gamma}\sum_{m\neq j}\lambda_m z_{mn}\mathcal{D}^\gamma R_{\alpha+1}(z_j, z_m).\end{aligned}$$

Thus we have from Lemma 3.6 and Lemma 3.1 that

$$\begin{aligned}\sum|\alpha_j| &\le C\delta^{-n}\sum_m\sum_{j\neq m}|\lambda_m|\int_{E_{\delta/2}(z_j)}\frac{z_{mn}w_n^{\alpha+\gamma}}{|z_m-\overline{w}|^{n+\alpha+\gamma+1}}\,dw\\ &\le C\delta^{-n}\sum_m|\lambda_m|\int_{\mathbf{H}\setminus E_\delta(z_m)}\frac{z_{mn}w_n^{\alpha+\gamma}}{|z_m-\overline{w}|^{n+\alpha+\gamma+1}}\,dw\\ &= C\delta^{-n}\left(\sum_m|\lambda_m|\right)\int_{\mathbf{H}\setminus E_\delta(z_0)}\frac{w_n^{\alpha+\gamma}}{|z_0-\overline{w}|^{n+\alpha+\gamma+1}}\,dw\end{aligned}$$

where the constant C is independent of δ. Since $\alpha+\gamma > -1$, Lemma 3.5 yields

$$\int_{\mathbf{H}}\frac{w_n^{\alpha+\gamma}}{|z_0-\overline{w}|^{n+\alpha+\gamma+1}}\,dw < \infty.$$

Thus, $T_\gamma Q_{\alpha+1}$ is invertible on l^1 for all δ sufficiently close to 1. For such δ, put $S_\gamma = Q_{\alpha+1}(T_\gamma Q_{\alpha+1})^{-1}$. The proof is complete. □

7. Interpolation on $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}_0$. In this section we consider the interpolation theorems for $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}_0$. Let $\gamma > 0$, and let $\{z_m\}$ be a sequence in $\mathbf{H}$. For $u \in \widetilde{\mathcal{B}}$, define

$$\widetilde{T}_\gamma u = \big(z_{mn}^\gamma\mathcal{D}^\gamma u(z_m)\big). \tag{7.1}$$

Then (2.10) implies

$$\widetilde{T}_\gamma : \widetilde{\mathcal{B}} \longrightarrow l^\infty$$

is bounded. If $\widetilde{T}_\gamma(\widetilde{\mathcal{B}}) = l^\infty$, $\{z_m\}$ is called a $\widetilde{\mathcal{B}}$-interpolating sequence of order γ. Also, if $\widetilde{T}_\gamma(\widetilde{\mathcal{B}}_0) = \mathcal{C}_0$, $\{z_m\}$ is called a $\widetilde{\mathcal{B}}_0$-interpolating sequence of order γ.

The following proposition shows that separation is also necessary for $\widetilde{\mathcal{B}}_0$ interpolation. Since we have Lemma 5.2, the proof of the following proposition is the same as that of Proposition 5.6 in [**4**].

Proposition 7.1. *Let $\gamma > 0$. Every $\widetilde{\mathcal{B}}$-interpolating sequence of order γ is separated. Also, every $\widetilde{\mathcal{B}}_0$-interpolating sequence of order γ is separated.*

Having Proposition 5.1, we can modify the proof of Theorem 6.5 to the following theorem.

Theorem 7.2. *Let $\gamma > 0$. Then there exists a positive number δ_0 with the following property. Let $\{z_m\}$ be a δ-separated sequence with $\delta > \delta_0$, and let $\widetilde{T}_\gamma : \widetilde{\mathcal{B}} \to l^\infty$ be the associated linear operator as in (7.1). Then there exists a bounded linear operator $\widetilde{S}_\gamma : l^\infty \to \widetilde{\mathcal{B}}$ such that $\widetilde{T}_\gamma \widetilde{S}_\gamma$ is the identity on l^∞. Moreover, $\widetilde{S}_\gamma$ maps $\mathcal{C}_0$ into $\widetilde{\mathcal{B}}_0$. In particular, $\{z_m\}$ is both a $\widetilde{\mathcal{B}}$-interpolating and $\widetilde{\mathcal{B}}_0$-interpolating sequence of order γ.*

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Department of Mathematics, Hanshin University, Gyeonggi-do 447-791, Korea
E-mail address: ksnam@hanshin.ac.kr, ksnam@math.vanderbilt.edu