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REAL RANK OF C^*-TENSOR PRODUCTS WITH THE C^*-ALGEBRA OF BOUNDED OPERATORS

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ABSTRACT. We show under an assumption on the real rank zero that the real rank of the minimal C^*-tensor products of unital exact C^*-algebras with the C^*-algebra of bounded operators is less than or equal to one. Moreover, several consequences of this result are obtained.

1. Introduction. The real rank for C^*-algebras was introduced by Brown and Pedersen [**3**]. This notion has been quite important in the theory of C^*-algebras such as the classification theory of C^*-algebras, cf. [**9**] and its reference. On the other hand, some basic formulas for the real rank has been obtained by [**1, 3, 6, 10, 11, 15**], etc. However, it is hard to compute the real rank of C^*-algebras in some general situations so that some desirable formulas for the real rank has not been proven yet. For example, the real rank formula for C^*-tensor products has not been obtained completely.

In this paper we obtain a real rank formula for the minimal C^*-tensor products of unital exact C^*-algebras with the C^*-algebra of bounded operators under an assumption on the real rank zero. The main idea of the proof is a modification (to the real rank case) of Rieffel's proof for the stable rank formula [**16**, Theorem 6.4] for C^*-tensor products by the C^*-algebra of compact operators. However, the process of the real rank case is more complicated than the stable rank case as shown in Theorem 1. As a consequence, several results of the real rank of C^*-tensor products are obtained by using the results of Kodaka-Osaka [**10, 11, 15**], Zhang [**24**] and Lin [**13**]. Also, the real rank formula in Theorem 1 would be useful in other situations in the future. See [**5, 17–22**] for some related works.

Notation. Let $\mathbf{B}(H)$ be the C^*-algebra of all bounded operators on a separable infinite-dimensional Hilbert space H, and let $\mathbf{K}$ be the C^*-

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algebra of all compact operators on H. Denote by $Q(H) = \mathbf{B}(H)/\mathbf{K}$ the Calkin algebra. The symbol $\otimes$ means the minimal (or unique) C^*-tensor product throughout this paper. For a unital C^*-algebra $\mathfrak{A}$, or the unitization $\mathfrak{A}^+$ of a nonunital C^*-algebra $\mathfrak{A}$, we denote by $\mathrm{RR}\,(\mathfrak{A})$ the real rank of $\mathfrak{A}$, cf. [**3**]. By definition, $\mathrm{RR}\,(\mathfrak{A}) \in \{0, 1, \dots, \infty\}$ and $\mathrm{RR}\,(\mathfrak{A}) \leq n$ if and only if for any $\varepsilon > 0$ and $(a_j) \in \mathfrak{A}^{n+1}$ with $a_j^* = a_j$, there exists $(b_j) \in \mathfrak{A}^{n+1}$ with $b_j^* = b_j$ such that $\|a_j - b_j\| < \varepsilon$, $1 \leq j \leq n+1$, and $\sum_{j=1}^{n+1} b_j^2$ is invertible in $\mathfrak{A}$ (this condition is equivalent to that there exists $(c_j) \in \mathfrak{A}^{n+1}$ such that $\sum_{j=1}^{n+1} c_j b_j$ is invertible in $\mathfrak{A}$).

Theorem 1. *Let $\mathfrak{A}$ be a unital exact C^*-algebra with* $\mathrm{RR}\,(\mathfrak{A} \otimes Q(H)) = 0$. *Then we have* $\mathrm{RR}\,(\mathfrak{A} \otimes \mathbf{B}(H)) \leq 1$.

Proof. Since $\mathfrak{A}$ is exact, the following exact sequence is obtained, cf. [**7**]:

$$0 \longrightarrow \mathfrak{A} \otimes \mathbf{K} \longrightarrow \mathfrak{A} \otimes \mathbf{B}(H) \xrightarrow{\pi} \mathfrak{A} \otimes Q(H) \longrightarrow 0.$$

Let a, b be two self-adjoint elements of $\mathfrak{A} \otimes \mathbf{B}(H)$. Then $\pi(a)$ and $\pi(b)$ can be approximated by invertible self-adjoint elements s, t of $\mathfrak{A} \otimes Q(H)$ by assumption, respectively. Let $c, d, c' \in \mathfrak{A} \otimes \mathbf{B}(H)$ be self-adjoint lifts of s, t, s^{-1} respectively. Then there exist self-adjoint elements $l, l' \in \mathfrak{A} \otimes \mathbf{K}$ such that the norms of $a - c - l$ and $b - d - l'$ are small enough, and there exists $k \in \mathfrak{A} \otimes \mathbf{K}$ such that $1 - k = c'c$. We may replace l, l' with self-adjoint finite sums $\sum l_j \otimes n_j$, $\sum l_j' \otimes n_j'$ of simple tensors $l_j \otimes n_j$ and $l_j' \otimes n_j'$ such that all the ranges of the factors n_j, n_j' in $\mathbf{K}$ are finite dimensional. By the following multiplication, we have

$$(c'(c + l), d + l') = (1 - k + c'l, d + l') \in (\mathfrak{A} \otimes \mathbf{K})^+ \oplus (\mathfrak{A} \otimes \mathbf{B}(H)).$$

By the following matrix operation, we have

$$\begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \begin{pmatrix} 1 - k + c'l \\ d + l' \end{pmatrix} = \begin{pmatrix} 1 - k + c'l \\ dk - dc'l + l' \end{pmatrix} \in \oplus^2 (\mathfrak{A} \otimes \mathbf{K})^+$$

where we identify the units between $\mathfrak{A} \otimes \mathbf{B}(H)$ and $(\mathfrak{A} \otimes \mathbf{K})^+$. Since $k, dk \in \mathfrak{A} \otimes \mathbf{K}$, there exist finite sums $m = \sum m_j^1 \otimes m_j^2$ and $n = \sum n_j^1 \otimes n_j^2$ of simple tensors of $\mathfrak{A} \otimes \mathbf{K}$ such that all the ranges of m_j^2 and n_j^2 are finite dimensional, and the norms $\|k-m\|$ and $\|dk-n\|$ are small enough. In particular, we may let $\|dk-n\| < \varepsilon^2$, where $\varepsilon > 0$ is fixed later.

Let $P = 1 \otimes p$ be a projection of $\mathfrak{A} \otimes \mathbf{K}$, where p is a finite rank projection with its range containing all the ranges of the factors (m_j^2, n_j^2 and n_j') in $\mathbf{K}$ of simple tensors of m, n and l' (finite sums of simple tensors), and all the spaces obtained by restricting (or reducing) the ranges of $c'l, dc'l$ to H. Let $1 \otimes q$ be a projection of $\mathfrak{A} \otimes \mathbf{K}$, where q is orthogonal and equivalent to p. Let $U = 1 \otimes u$, $V = 1 \otimes v$ be partial isometries of $\mathfrak{A} \otimes \mathbf{K}$ such that $uv = p$ and $vu = q$. Since l' has no effect from the above multiplication and matrix operation, we may replace l' with $l' + \varepsilon(V + V^*)$ for $\varepsilon > 0$ small enough. Then, it follows that

$$\begin{aligned}(1-P)(1-(k-m+m)+c'l)&\\ &+\varepsilon^{-1}U(dk-n+n-dc'l\\ &+l'+\varepsilon(V+V^*))\\ &=1-P-(k-m+m)+c'l\\ &\quad+P(k-m)-P(-m+c'l)\\ &\quad+\varepsilon^{-1}U(dk-n)+\varepsilon^{-1}U(n-dc'l+l')+U(V+V^*)\\ &=1-P-(k-m)+P(k-m)+\varepsilon^{-1}U(dk-n)+0+P\\ &=1-(k-m)+P(k-m)+\varepsilon^{-1}U(dk-n).\end{aligned}$$

Since the norms of $k-m$, $P(k-m)$ are small enough, and $\|\varepsilon^{-1}U(dk-n)\| < \varepsilon$, the last expression in the above calculation is invertible in $(\mathfrak{A} \otimes \mathbf{K})^+$. This is equivalent to that $(1-k+c'l)^2 + (dk - dc'l + l' + \varepsilon(V+V^*))^2$ is invertible in $(\mathfrak{A} \otimes \mathbf{K})^+ \subset \mathfrak{A} \otimes \mathbf{B}(H)$. Since the matrix in the above matrix operation is invertible, we deduce that there exist $r, r' \in \mathfrak{A} \otimes \mathbf{B}(H)$ such that $rc'(c+l) + r'(d+l'+\varepsilon(V+V^*))$ is invertible in $\mathfrak{A} \otimes \mathbf{B}(H)$, cf. [**16**, Proposition 4.1]. Moreover, this is equivalent to that $(c+l)^2 + (d+l'+\varepsilon(V+V^*))^2$ is invertible in $\mathfrak{A} \otimes \mathbf{B}(H)$. Therefore, it is concluded that $RR(\mathfrak{A} \otimes \mathbf{B}(H)) \leq 1$. $\square$

Remark. If $\mathfrak{A} \otimes Q(H)$ is unital, simple and purely infinite, then it has the real rank zero, cf. [**3**, Proposition 3.9]. Especially, we can take the Cuntz algebras O_n for $2 \leq n \leq \infty$ as $\mathfrak{A}$ in Theorem 1. In fact, O_n is nuclear, and $\mathrm{RR}(O_n \otimes Q(H)) = 0$ since $O_n \otimes Q(H)$ is simple and purely infinite, cf. [**9**, Proposition 4.5 and Theorem 5.11], [**15**, Corollary 2.3]. On the other hand, we can take all AF-algebras as $\mathfrak{A}$ in Theorem 1.

Remark. For $\mathfrak{A}$ a nonunital C^*-algebra, the assumption in Theorem 1 should be replaced by $\mathrm{RR}\,(\mathfrak{A}^+ \otimes Q(H)) = 0$. Then $\mathrm{RR}\,(\mathfrak{A} \otimes \mathbf{B}(H)) \leq 1$ is deduced from Theorem 1 and that $\mathfrak{A} \otimes \mathbf{B}(H)$ is a closed ideal of $\mathfrak{A}^+ \otimes \mathbf{B}(H)$, cf. [**6**, Theorem 1.4].

Moreover, the following theorem is obtained:

Theorem 2. *Let $\mathfrak{A}$ be a unital exact C^*-algebra with* $\mathrm{RR}\,(\mathfrak{A} \otimes Q(H)) = 0$ *and* $K_1(\mathfrak{A}) \neq 0$. *Then we have* $\mathrm{RR}\,(\mathfrak{A} \otimes \mathbf{B}(H)) = 1$.

Proof. If $\mathfrak{A}$ is a unital exact C^*-algebra with $K_1(\mathfrak{A}) \neq 0$, then $\mathrm{RR}\,(\mathfrak{A} \otimes \mathbf{B}(H)) \geq 1$ by Kodaka and Osaka ([**10**], [**15**, Proposition 1.1]). Combining this result with Theorem 1, the conclusion is obtained. $\square$

Remark. We can take $\mathfrak{BD} \otimes O_n$ and $\mathfrak{A}_\theta \otimes O_n$, $2 \leq n \leq \infty$, as $\mathfrak{A}$ in Theorem 2, where $\mathfrak{BD}$ is one of the Bunce-Deddens algebras and $\mathfrak{A}_\theta$ is one of the irrational rotation algebras. In fact, $\mathfrak{BD} \otimes O_n$ and $\mathfrak{A}_\theta \otimes O_n$ are simple and purely infinite with $K_1(\mathfrak{BD} \otimes O_n) \neq 0$ and $K_1(\mathfrak{A}_\theta \otimes O_n) \neq 0$, cf. [**8**], [**15**, Remark 1.3], [**4**, V.3 and V.7], [**2**, 10.11.4 and 10.11.8] and [**23**, 9.3.3 and 12.3]. However, it is known that $K_1(O_n) = 0$ for $2 \leq n \leq \infty$. It is obtained by [**15**, Corollary 2.3] that $\mathrm{RR}\,(O_n \otimes \mathbf{B}(H)) = 0$ for $2 \leq n \leq \infty$.

For simple C^*-algebras, the following theorem is obtained:

Theorem 3. *Let $\mathfrak{A}$ be a unital, simple, separable, purely infinite, nuclear C^*-algebra with $K_1(\mathfrak{A}) \neq 0$. Then* $\mathrm{RR}\,(\mathfrak{A} \otimes \mathbf{B}(H)) = 1$.

Remark. If $\mathfrak{A}$ is a unital, simple, separable, purely infinite, nuclear C^*-algebra, then $\mathfrak{A} \otimes Q(H)$ is always purely infinite by [**9**], cf. [**8**]. See [**15**, Corollary 2.3 and its proof].

It is obtained by the same way as Theorem 1 that

Theorem 4. *Let $M(\mathfrak{A}\otimes\mathbf{K})$ be the multiplier algebra of $\mathfrak{A}\otimes\mathbf{K}$ for $\mathfrak{A}$ either a σ-unital purely infinite, simple C^*-algebra or a σ-unital simple C^*-algebra with $\mathrm{RR}\,(\mathfrak{A}) = 0$ and stable rank one. Then*

$$\mathrm{RR}\,(M(\mathfrak{A}\otimes\mathbf{K})) = \begin{cases} 0 & \text{if } K_1(\mathfrak{A}) = 0, \\ 1 & \text{if } K_1(\mathfrak{A}) \neq 0. \end{cases}$$

Proof. Note that the following exact sequence is obtained:

$$0 \longrightarrow \mathfrak{A}\otimes\mathbf{K} \longrightarrow M(\mathfrak{A}\otimes\mathbf{K}) \longrightarrow M(\mathfrak{A}\otimes\mathbf{K})/\mathfrak{A}\otimes\mathbf{K} \longrightarrow 0.$$

By [**24**, Corollary 2.6] or [**13**, Theorem 15], $\mathrm{RR}\,(M(\mathfrak{A}\otimes\mathbf{K})/\mathfrak{A}\otimes\mathbf{K}) = 0$. Note that $\mathfrak{A}\otimes\mathbf{K}$ has real rank zero and stable rank one by [**3**, Corollary 3.3] and [**16**, Theorem 3.6]. Moreover, it is obtained by [**24**, Corollary 2.6] that $\mathrm{RR}\,(M(\mathfrak{A}\otimes\mathbf{K})) = 0$ if and only if $K_1(\mathfrak{A}) = 0$. Thus, if $K_1(\mathfrak{A}) \neq 0$, then $\mathrm{RR}\,(M(\mathfrak{A}\otimes\mathbf{K})) \geq 1$. $\square$

Remark. See [**15**, Corollary 2.4] for the same result in the case of $\mathfrak{A}$ a nonunital, σ-unital purely infinite simple C^*-algebra. Also see [**12**, Theorem 3.2] as a related result on extremally rich C^*-algebras. On the other hand, it is deduced from [**24**, Examples 2.7] and Theorem 4 that

$$\mathrm{RR}(M(\mathbf{K}\otimes Q(H)) = 1, \quad \text{and} \quad \mathrm{RR}(M(\mathbf{K}\otimes O_A)) = 1,$$

where O_A is the Cuntz-Krieger algebra for A an irreducible matrix such that $\det(I-A) = 0$. Moreover, it is obtained from [**24**, Corollary 3.6] that

$$\begin{cases} \mathrm{RR}\,(M(C_{(2m-1)}(\mathfrak{A})\otimes\mathbf{K})) = 1 & \text{if } K_0(\mathfrak{A}) \neq 0, \\ \mathrm{RR}\,(M(C_{(2m)}(\mathfrak{A})\otimes\mathbf{K})) = 1 & \text{if } K_1(\mathfrak{A}) \neq 0 \end{cases}$$

for $\mathfrak{A}$ a σ-unital, nonunital purely infinite, simple C^*-algebra, where $C_{(n+1)}(\mathfrak{A}) = M(C_{(n)}(\mathfrak{A})\otimes\mathbf{K})/C_{(n)}(\mathfrak{A})\otimes\mathbf{K}$ for $n \geq 1$, with $C_{(1)}(\mathfrak{A}) = M(\mathfrak{A})/\mathfrak{A}$.

As a remarkable generalization of Theorem 1, the following is obtained:

Theorem 5. *Let $\mathcal{E}$ be an extension of a C^*-algebra $\mathfrak{B}$ with* $\mathrm{RR}\,(\mathfrak{B}) = 0$ *by $\mathfrak{A} \otimes \mathbf{K}$ for $\mathfrak{A}$ a C^*-algebra. Then* $\mathrm{RR}\,(\mathcal{E}) \leq 1$.

Proof. Note that $0 \to \mathfrak{A} \otimes \mathbf{K} \to \mathcal{E} \to \mathfrak{B} \to 0$. If $\mathcal{E}$ is nonunital, we have $0 \to \mathfrak{A} \otimes \mathbf{K} \to \mathcal{E}^+ \to \mathfrak{B}^+ \to 0$, with $\mathrm{RR}\,(\mathfrak{B}^+) = \mathrm{RR}\,(\mathfrak{B}) = 0$. The the rest of the proof is the same as the proof of Theorem 1. $\square$

Remark. This result would be useful in the extension theory of C^*-algebras with real rank zero. Note that $\mathrm{RR}\,(\mathcal{E}) = 1$ when $\mathrm{RR}\,(\mathfrak{A} \otimes \mathbf{K}) = 1$. For example, we may let $\mathfrak{A} = C([0,1])$ the C^*-algebra of continuous functions on $[0,1]$, cf. [**14**, Proposition 5.1]. On the other hand, we obtain $\mathrm{RR}\,(\mathcal{E}) = 0$ when $\mathfrak{A} = \mathbf{C}$ and $\mathfrak{B} = O_n$ or $\mathbf{B}(H)$ by [**11**, Lemma 1] or [**14**, Proposition 1.6].

Finally, we state the following question:

Question. Is it true that $\mathrm{RR}\,(\mathfrak{A} \otimes \mathbf{B}(H)) \leq 1$ for any C^*-algebra $\mathfrak{A}$?

Remark. If this question is true, we obtain $\mathrm{RR}\,(\mathbf{B}(H) \otimes \mathbf{B}(H)) \leq 1$, which answers Osaka's question in [**15**]. Unfortunately, $\mathbf{B}(H)$ is nonexact ([**7**]), so that our Theorem 1 is not available to this case. On the other hand, $\mathrm{RR}\,(\mathfrak{A} \otimes \mathbf{K}) \leq 1$ for any C^*-algebra $\mathfrak{A}$ by [**1**].

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