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WHEN DIVISIBILITY BY AN ELEMENT IMPLIES INVERTIBILITY

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ABSTRACT. Let R be a commutative ring with unity and M_R a unital right R-module. Let $x \in R$ and $\rho_x : M_R \to M_R$ be given by $\rho_x(m) = mx$ for all $m \in M_R$. Rings in which every nonzero module M has the property that if ρ_x is surjective then x is invertible in R are fully characterized.

1. Introduction. Throughout, R will denote a commutative ring with unity and M_R will denote a unitary right R-module. We will use M for M_R when the coefficient ring is obvious. We will also denote the right-multiplication map by an element $x \in R$ with $\rho_x : M_R \to M_R$ with $\rho_x(m) = mx$ for all $m \in M$. When ρ_x is surjective then $Mx = M$ and we say that M is *divisible by* x.

Maxson presented the following situation. If R is nonlocal, then there exist noninvertible elements r and s such that $r + s = 1$. Suppose that $f : M_R \to M_R$ is a homogeneous function (preserving scalar multiplication) and f is linear on submodules Mr and Ms. Calculations show that f will also be linear on M. A collection of proper submodules is said to *force linearity* if every homogeneous map which is linear on the collection of submodules is also linear on M. The *forcing linearity number* of M, is the minimum integer n, if one exists, such that a collection of n proper submodules forces linearity on M. Thus, assuming that Mr and Ms are both proper submodules, then in this case, M will have forcing linearity number of at most two. Maxson asked if one can describe when right multiplication by a ring element *onto* a module implies that the element is invertible. Hence, in this case, if R satisfied such a property then Mr and Ms would have to be proper submodules. To study this situation, the following terms are defined.

Definition 1. Let $0 \neq M_R$ have the property that for all $x \in R$, if ρ_x is surjective, then x is invertible in R. Then M is an *OI R-module*.

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Definition 2. If every nonzero module of R is OI, then we say that R is an *OI ring*.

The first observation that needs to be made is that if x is invertible in R, then ρ_x is an isomorphism for any R-module. Thus, one could consider OI modules a generalization of Hopfian modules, that is, the class of modules in which every epimorphism is an isomorphism. In fact, the term OI comes from "onto implies invertible." Also, if R is an OI ring and $x \in R$ such that there exists a nonzero module for which ρ_x is surjective, then ρ_x is an isomorphism, and hence, surjective, on every R-module.

1. Examples. Clearly, the class of rings satisfying Definition 2 is not trivial because the class of fields falls into this category. Since the zero map is never a surjective map for nonzero modules, fields satisfy the property by default. The next example shows a nonfield, in fact a nondomain, which satisfies the property.

Example 1.1. Let $0 \neq M$ be a $\mathbf{Z}_4$-module. Then for any $m \in M$, $4m = 4(m\bar{1}) = m\bar{4} = 0$. So the order of m divides 4. Since M is a nonzero module, there must be at least one element of even order, not 1, so let y be an element of maximal even order. Thus, if $y = m \cdot \bar{2} \in M \cdot \bar{2}$, then m would have an even order greater than the order of y (since the order of y is half the order of m). Since this contradicts y having maximal even order, $y \notin M \cdot \bar{2}$. Thus $\rho_{\bar{2}}$ is not surjective. Clearly the zero map is not surjective. Thus only $\rho_{\bar{1}}$ and $\rho_{\bar{3}}$ could be surjective right multiplication maps. Since $\bar{1}$ and $\bar{3}$ are both invertible in $\mathbf{Z}_4$, then $\mathbf{Z}_4$ is OI. A similar argument can be used to show that $\mathbf{Z}_{p^n}$ is OI for any prime p.

Example 1.2. Consider $\mathbf{Q}$ as a $\mathbf{Z}$-module. Since for all $q \in \mathbf{Q}$, $q/2 \cdot 2 = q$, we have that ρ_2 is surjective. However, 2 is not invertible in $\mathbf{Z}$, so $\mathbf{Q}$ is not OI over $\mathbf{Z}$ and therefore $\mathbf{Z}$ is not OI.

Clearly, R is always an OI module over itself. The following example also illustrates an OI module over a ring which is not necessarily an OI ring.

Example 1.3. Let $M_{\mathbf{Z}} = \sum \oplus \mathbf{Z}_p$ where p runs over all primes. Then, ρ_x is not surjective on M as long as x is divisible by some prime. Hence, the only surjective multiplication maps are ρ_1 and ρ_{-1}. Since 1 and -1 are both invertible in $\mathbf{Z}$, M is OI over $\mathbf{Z}$.

2. OI rings. The following section fully characterizes OI rings. Inherent in the theory is the class of *quasilocal* rings, the class of rings which have a unique maximal ideal. Thus, one may think of these as local but not Noetherian. In a quasilocal ring, every element is either in the maximal ideal or is invertible.

In the following proposition, we make the observation that a nilpotent element can never produce a surjective multiplication map.

Proposition 2.1. *Let R be a quasilocal ring such that the maximal ideal is nil. Then R is an OI ring.*

Proof. Let $x \in R$, and let M_R be a nonzero R-module. Since R is quasilocal, either x is invertible or x is nilpotent. If x is nilpotent such that $x^n = 0$, then ρ_x being surjective implies $M = Mx = Mx^n = 0$. Thus, ρ_x is never surjective for a nilpotent x. Thus, if ρ_x is surjective on M, then x must be invertible and R is OI. $\square$

Proposition 2.2. *Let R be an* OI *ring. Then $N = \{x \in R \mid$ x is not invertible $\}$ is an ideal of R.*

Proof. Let $x, y \in N$ and $r \in R$. Then if $xr \notin N$, then $xrz = 1$ for some $z \in R$. Hence, x is invertible which contradicts $x \in N$. Thus $xr \in N$. Suppose $x - y \notin N$. Since $x \in N$, $\rho_x : Rx_R \to Rx_R$ is not surjective, that is, $Rx^2 \subset Rx$, provided that $Rx \neq 0$. Suppose $0 \neq M = Rx/Rx^2$. Let $\overline{rx} \in M$. Then $(\overline{-rx(x-y)^{-1}})y = \overline{-rxy(x-y)^{-1}} = \overline{rx^2(x-y)^{-1}} - \overline{rxy(x-y)^{-1}} = \overline{rx(x-y)(x-y)^{-1}} = \overline{rx}$. Thus $\rho_y : M \to M$, is surjective and hence $y \notin N$. To avoid contradiction, we have that either $x - y \in N$ or $Rx = 0$. If the latter, then we can repeat the same argument with $M = Ry/Ry^2$ provided that $Ry \neq 0$. Thus, either $x - y \in N$ or $Rx = Ry = 0$. In the latter case we have that

$R(x-y)=0$ and hence $x-y$ is clearly not invertible. Thus $x-y \in N$ and $N \triangleleft R$. $\square$

Proposition 2.3. *Let R be an* OI *ring. Then R is quasilocal.*

Proof. Let $I \triangleleft R$ with $I \nsubseteq N$. Then for any $x \in I - N$, x is invertible. Hence, $1 \in I$ and $I = R$. So either $I = R$ or $I = N$. Thus, N is a maximal ideal of R. If we let I be another maximal ideal, then the previous argument shows that $I = N$ and N is the unique maximal ideal. $\square$

Theorem 2.4. *A ring R is* OI *if and only if R is quasilocal with a nil maximal ideal.*

Proof. Let R be OI, and let $u \in R$. Let I be the ideal of $R[x]$ generated by $1-ux$, and define $M = R[x]/I$ as a quotient of rings. For any $\overline{p(x)} \in M$ with $p(x) = a_n x^n + \cdots + a_0$, we have that $p(x) = a_n x^{n+1}u + a_n x^n(1-ux) + a_{n-1}x^n u + a_{n-1}x^{n-1}(1-ux) + \cdots + a_0 xu + a_0(1-ux)$. Thus, $\overline{p(x)} = \overline{a_n x^{n+1}u + a_{n-1}x^n u + \cdots + a_0 xu} \in Mu$. Hence, $\rho_u : M \to M$ is a surjective map. Since R is OI, either u is invertible, or $M = 0$. So suppose that $\langle 1-ux \rangle_R = R[x]$. Since for all $p(x) \in R[x]$, $p(x) = q(x)(1-ux)$, we have that $q(x)(1-ux) = 1$ for some $q(x) \in R[x]$. So $(a_0 + \cdots + a_n x^n)(1-ux) = a_0 + (a_1 - a_0 u)x + \cdots + (a_n - a_{n-1}u)x^n - (a_n u)x^{n+1} = 1$. So $a_0 = 1$, $a_1 = u$, $a_2 = u^2$, and so forth, so that $a_n = u^n$ and $u^{n+1} = 0$. Thus u is nilpotent. Hence, we have that R is quasilocal, and if u is not invertible, then it is nilpotent, making the maximal ideal, N a nil ideal. The converse is given by Proposition 2.1. $\square$

This completes the characterization of OI rings. A classification in terms of R can still be made of OI modules.

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