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ON A STATIONARY, TRIPLE-WISE INDEPENDENT, ABSOLUTELY REGULAR COUNTEREXAMPLE TO THE CENTRAL LIMIT THEOREM

RICHARD C. BRADLEY

ABSTRACT. An earlier paper gave a construction of a strictly stationary, finite-state, nondegenerate random sequence which satisfies pairwise independence and absolute regularity but fails to satisfy a central limit theorem. Here it will be shown that random sequence is in fact triple-wise independent (though not quadruple-wise independent).

1. Introduction. Etemadi [**7**] proved the strong law of large numbers for sequences of pairwise independent, identically distributed random variables with finite absolute first moment. Janson [**9**] constructed several classes of (nondegenerate) strictly stationary sequences of pairwise independent random variables with finite second moments such that the CLT (central limit theorem) fails to hold. Subsequently, the author [**2**] constructed such an example with the additional property of absolute regularity (defined below). For the examples in those two papers, as well as other examples of pairwise independent sequences, Cuesta and Matrán [**4**] examined the behavior of the partial sums further, e.g., in connection with the law of the iterated logarithm. For an arbitrary integer $N \geq 3$, Pruss [**12**] constructed a (not strictly stationary) sequence of N-tuple-wise independent, identically distributed random variables with finite second moment such that the CLT fails to hold. In that paper, for a given $N \geq 3$, the existence of such examples that are also strictly stationary was left as an open question. For $N = 3$, the answer is affirmative. The purpose of this note is to show that the example in [**2**] alluded to above is triple-wise independent.

Suppose $X := (X_k,\, k \in \mathbf{Z})$ is a strictly stationary sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$. For $-\infty \leq j \leq l \leq \infty$, let $\mathcal{F}_j^l$ denote the σ-field ($\subset \mathcal{F}$) of events generated by the random variables

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$X_k,\ j \le k \le l,\ k \in \mathbf{Z}$. For any two σ-fields $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{F}$, define the following two measures of dependence:

$$\alpha(\mathcal{A},\mathcal{B}) := \sup_{A\in\mathcal{A}, B\in\mathcal{B}} |P(A\cap B) - P(A)P(B)|;$$

and

$$\beta(\mathcal{A},\mathcal{B}) := \sup \frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{J} |P(A_i\cap B_j) - P(A_i)P(B_j)|,$$

where the latter supremum is taken over all pairs of partitions $\{A_1, \dots, A_I\}$ and $\{B_1, \dots, B_J\}$ of Ω such that $A_i \in \mathcal{A}$ for each i and $B_j \in \mathcal{B}$ for each j. For the given strictly stationary sequence X, define for each positive integer n the dependence coefficients

$$\alpha(n) := \alpha(\mathcal{F}_{-\infty}^{0}, \mathcal{F}_{n}^{\infty})$$

and

$$\beta(n) := \beta(\mathcal{F}_{-\infty}^{0}, \mathcal{F}_{n}^{\infty}).$$

As a consequence of strict stationarity, for every $n \ge 1$ and every $j \in \mathbf{Z}$, $\alpha(n) = \alpha(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty})$ and $\beta(n) = \beta(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty})$. The strictly stationary sequence X is said to be "strongly mixing" (or "α-mixing") [**13**] if $\alpha(n) \to 0$ as $n \to \infty$; and it is "absolutely regular" (or "β-mixing") [**14**] if $\beta(n) \to 0$ as $n \to \infty$. Obviously absolute regularity implies strong mixing.

For each $n \ge 1$, define the partial sum $S_n := X_1 + X_2 + \cdots + X_n$.

Here is our main result:

Theorem 1. *There exists a strictly stationary sequence* $X := (X_k,\ k \in \mathbf{Z})$ *of random variables with the following properties*:

(A) *The random variables* X_k *take just the three values* -1, 0 *and* 1, *with* $P(X_0 = -1) = P(X_0 = 1) = 1/4$ *and* $P(X_0 = 0) = 1/2$.

(B) *For every choice of three distinct integers* i, j *and* k, *the random variables* X_i, X_j *and* X_k *are independent.*

(C) *The sequence* X *satisfies absolute regularity with mixing rate* $\beta(n) = O(1/n)$ *as* $n \to \infty$.

(D) *One has that* $\inf_{n\geq 1} P(S_n = 0) > 0$.

(E) *The family of distributions of the partial sums* $(S_n,\ n \geq 1)$ *is tight.*

The author [**2**, Theorem 1] verified Theorem 1 except that, in property (B), only pairwise independence was verified. It turns out that the random sequence $X := (X_k,\ k \in \mathbf{Z})$ constructed in [**2**] for that theorem satisfies triple-wise independence, as stated in property (B) here. That will be shown in Section 3 below. As a key preliminary step, it will be shown in Section 2 that a strictly stationary, two-state, pairwise independent, ergodic counterexample used in [**2**] is in fact triple-wise independent.

In Theorem 1, the mixing rate in (C) was shown in [**2**] with an adaptation of mixing-rate calculations of Davydov [**5**]. It cannot be replaced by $o(1/n)$; see the CLT's in [**10**, Corollary 1.1(ii)] or [**3**, Theorem 10.3]. Properties (D) and (E) were adapted from the example of Herrndorf [**8**] (in which the random variables are uncorrelated).

It seems to still be an open question whether for a nondegenerate strictly stationary sequence $X := (X_k,\ k \in \mathbf{Z})$ with finite second moments, quadruple-wise independence implies the CLT. However, under the additional assumption of strong mixing, the CLT does hold. That is a special case of a well-known standard CLT under strong mixing given, e.g., in [**6**, Theorem 3], [**11**] or [**3**, Theorem 1.19]. The main task is, after the random variables are centered, to show that the family $(S_n^2/n,\ n \geq 1)$ is uniformly integrable. To accomplish that, one can truncate the X_k's at an arbitrarily high level, center the truncated random variables, and then apply a well known bound on fourth moments of sums of bounded random variables. (See, e.g., Billingsley [**1**, p. 85, eq. (6.2)]. The standard argument there uses just quadruple-wise independence.)

As a consequence, in Theorem 1, property (B) cannot be strengthened to quadruple-wise independence. (In particular, as one can also see with a direct, tedious calculation, the sequence X constructed here as in [**2**] for Theorem 1 fails to satisfy quadruple-wise independence.)

2. Preliminaries: a two-state sequence. Throughout, our probability space is $(\Omega, \mathcal{F}, P)$. Let us repeat here the construction in [**2**, p. 3, Definition 2.1].

Definition 2.1. (a) Let $\mathcal{S}$ denote the set of all ordered pairs (t, u) such that

$$t \in \{\pm 2^j,\ j \in \mathbf{N}\} \quad \text{and} \quad u \in \{0, 1, 2, \dots, |t| - 1\}.$$

(The first set is $\{\pm 2, \pm 4, \pm 8, \pm 16, \dots\}$; the letter $\mathbf{N}$ denotes the set of all positive integers.)

(b) Let μ denote the probability measure on $\mathcal{S}$ defined by $\mu(\{(t, u)\}) = 1/(2t^2)$ for $(t, u) \in \mathcal{S}$.

(c) Let $V := (V_k,\ k \in \mathbf{Z})$, with $V_k := (T_k, U_k)$, $k \in \mathbf{Z}$, be a strictly stationary Markov chain with state space $\mathcal{S}$, with invariant marginal probability measure μ, and with one-step transition probabilities (consistent with μ) given by the following equations:

$$P(V_1 = (t, |t| - 1) | V_0 = (T, 0)) = 3/(2t^2)$$
$$\forall\, t,\ T \in \{\pm 2, \pm 4, \pm 8, \pm 16, \dots\};$$

$$P(V_1 = (t, u - 1) | V_0 = (t, u)) = 1$$
$$\forall\, t = \pm 2, \pm 4, \pm 8, \dots,\ \forall\, u = 1, 2, \dots, |t| - 1;$$

$$P(V_1 = s_1 | V_0 = s_0) = 0$$
$$\text{for all other pairs of states } s_0, s_1 \in \mathcal{S}.$$

As a trivial technicality, we assume that for every $k \in \mathbf{Z}$ and every $\omega \in \Omega$, the ordered pair $(V_k, V_{k+1})(\omega)$ is either $((T, 0), (t, |t| - 1))$ for some numbers $T, t \in \{\pm 2, \pm 4, \pm 8, \dots\}$ or else $((t, u), (t, u-1))$ for some $t \in \{\pm 2, \pm 4, \pm 8, \dots\},\ u \in \{1, 2, \dots, |t| - 1\}$.

(d) Define the function $f : \mathcal{S} \to \{-1, 1\}$ as follows:

$$f((t, u)) := \begin{cases} 1 & \text{if } t \in \{2, 4, 8, \dots\} \text{ and } t/2 \le u < t \\ -1 & \text{if } t \in \{2, 4, 8, \dots\} \text{ and } 0 \le u < t/2 \\ -1 & \text{if } t \in \{-2, -4, -8, \dots\} \text{ and } |t|/2 \le u < |t| \\ 1 & \text{if } t \in \{-2, -4, -8, \dots\} \text{ and } 0 \le u < |t|/2. \end{cases}$$

(e) Define the (strictly stationary) sequence $W := (W_k, k \in \mathbf{Z})$ as follows: for all $k \in \mathbf{Z}$,

$$W_k = f(V_k).$$

Remark 2.2. As was noted in [**2**, Lemmas 2.4 and 2.6], the random variables W_k, $k \in \mathbf{Z}$ are pairwise independent, and for all $k \in \mathbf{Z}$,

$$P(W_k = 1) = P(W_k = -1) = 1/2. \tag{2.1}$$

That and the following definitions and observations will be useful:

(a) For any pair of integers $I < J$, define the event

$$D(I, J) := \{U_I = U_J = 0 \text{ and } U_k \neq 0 \;\forall\, k \in \{I+1, I+2, \dots, J-1\}\} \tag{2.2}$$

(with $D(I, I+1) := \{U_I = U_{I+1} = 0\}$). Then by Definition 2.1 (c), the event $D(I, J)$ is nonempty only if $J - I \in \{2, 4, 8, 16, \dots\}$. That fact plays a central role throughout Section 2.

(b) As was noted in [**2**, equation (2.1)], $P(U_0 = 0) = 1/3$.

(c) From [**2**, Lemma 2.3] and strict stationarity, one has that if $I \in \mathbf{Z}$, $A \in \sigma(V_k,\ k \leq I)$, $B \in \sigma(V_k,\ k \geq I+1)$ and $P(A \cap \{U_I = 0\}) > 0$, then $P(B | A \cap \{U_I = 0\}) = P(B | U_I = 0)$. Thus for a given $I \in \mathbf{Z}$, the "one-sided" Markov chains $(V_I, V_{I-1}, V_{I-2}, \dots)$ and $(V_{I+1}, V_{I+2}, V_{I+3}, \dots)$ are conditionally independent given $\{U_I = 0\}$.

(Here and below, the notation $\sigma(\dots)$ means the σ-field generated by $(\dots)$.)

(d) By (c) above and Definition 2.1 (c), if $I < J$ are integers and $J - I \in \{2, 4, 8, 16, \dots\}$, then

$$\begin{aligned} P(D(I, J) | U_I = 0) &= P(V_{I+1} = (J - I, J - I - 1) | U_I = 0) \\ &\quad + P(V_{I+1} = (-(J - I), J - I - 1) \mid U_I = 0) \end{aligned}$$

and each term in the right-hand side is $3/[2(J - I)^2]$. Hence by (b) above and a simple calculation, if $I < J$ and $J - I \in \{2, 4, 8, 16, \dots\}$, then

$$P(D(I,J)) = 1/(J-I)^2, \tag{2.3}$$

and $P(V_{I+1} = (z(J-I),\ J-I-1) \mid D(I,J)) = 1/2$ for $z \in \{-1,1\}$.

(e) From that last equality and Definition 2.1 (c), (d) and (e), if $I < J$ and $J - I \in \{2,4,8,16,\dots\}$, then for any $\omega \in D(I,J)$, $(W_{I+1}, W_{I+2}, \dots, W_J)(\omega)$ is either $(1,\dots,1,-1,\dots,-1)$ or $(-1,\dots,-1,1,\dots,1)$, with $(J-I)/2$ 1's and $(J-I)/2$ -1's in either case; and, in fact,

(2.4)

$$P((W_{I+1},\dots,W_J) = (1,\dots,1,-1,\dots,-1) \mid D(I,J)) = 1/2$$

and

$$P((W_{I+1},\dots,W_J) = (-1,\dots,-1,1,\dots,1) \mid D(I,J)) = 1/2.$$

(f) By systematic use of (c) above, one has the following: If $I(0) < I(1) < \cdots < I(n)$, $n \ge 2$, are integers and $I(u) - I(u-1) \in \{2,4,8,16,\dots\}$ for each $u = 1,\dots,n$, then the event $D := \cap_{u=1}^{n} D(I(u-1),\ I(u))$ satisfies $P(D) > 0$, and the random vectors $(V_{I(u-1)+1}, V_{I(u-1)+2}, \dots, V_{I(u)})$, $u = 1,\dots,n$, are conditionally independent given D, and for each $u = 1,\dots,n$, the conditional distribution of $(V_{I(u-1)+1},\dots,V_{I(u)})$ given D is the same as its conditional distribution given $D(I(u-1),\ I(u))$. By Definition 2.1 (e), the same comments hold for the random vectors $(W_{I(u-1)+1},\dots,W_{I(u)})$, $u = 1,\dots,n$.

(g) By systematic use of (e) and (f) above, one has the following: Suppose $I < J$ and $J - I \in \{2,4,8,16,\dots\}$.

(i) If i, j and k are integers such that $i \le I$, $j \in \{I+1,\dots,J\}$, and $k > J$, then the random variables W_i, W_j and W_k are conditionally independent given $D(I,J)$, and $P(W_a = z|D(I,J)) = 1/2$ for $a = i,j,k$ and $z = 1,-1$.

(ii) If i, j and k are integers such that $i \le I$ or $i > J$, and $j \in \{I+1,\dots,(I+J)/2\}$ and $k \in \{(I+J)/2+1,\dots,J\}$, then the random variable W_i and the random vector (W_j, W_k) are conditionally independent given $D(I,J)$, with $P(W_i = z|D(I,J)) = 1/2$

for $z = 1, -1$, and $P((W_j, W_k) = (1, -1)|D(I,J)) = P((W_j, W_k) = (-1, 1)|D(I,J)) = 1/2$. (Of course one can say more, but this is what will be used below.)

Theorem 2.3. *For any three distinct integers a, b and c, the random variables W_a, W_b, and W_c are independent.*

From this theorem, equation (2.1), Definition 2.1 and (say) [**2**, Lemmas 2.12 and 2.13], W is a (nondegenerate) strictly stationary, two-state, ergodic, triple-wise independent random sequence that fails to satisfy the CLT. (To see that W is ergodic, note that the strictly stationary, countable-state Markov chain V is irreducible and therefore ergodic.)

Theorem 2.3 will play a key role in the proof, in Section 3, of property (B) in Theorem 1.

Proof of Theorem 2.3. By Definition 2.1 (c), (d) and (e), the random sequence W is strictly stationary. Hence, to prove Theorem 2.3, it suffices to prove for arbitrary positive integers J and L that the three random variables W_{-J}, W_0 and W_L are independent. The proofs for the two cases $J \geq L$ and $J \leq L$ are essentially identical. We shall give the proof in the latter case.

Accordingly, let J and L be arbitrary fixed integers such that

$$1 \leq J \leq L. \tag{2.5}$$

To prove that W_{-J}, W_0 and W_L are independent, and thereby complete the proof of Theorem 2.3, our task is to show that, see(2.1), for all $\alpha, \beta, \gamma \in \{-1, 1\}$,

$$P(W_{-J} = \alpha, \quad W_0 = \beta, \quad W_L = \gamma) = 1/8. \tag{2.6}$$

Recall from [**2**, Lemma 2.6] that the random variables W_k, $k \in \mathbf{Z}$, are pairwise independent. If (2.6) holds for a given ordered triplet $(\alpha, \beta, \gamma) \in \{-1, 1\}^3$, then it also holds for $(\alpha, \beta, -\gamma)$, since by (2.1),

$$\begin{aligned} P(W_{-J} = \alpha, \quad W_0 = \beta, \quad W_L = -\gamma) \\ &= P(W_{-J} = \alpha,\ W_0 = \beta) \\ &\quad - P(W_{-J} = \alpha,\ W_0 = \beta,\ W_L = \gamma) \\ &= 1/4 - 1/8 = 1/8; \end{aligned}$$

and the same comment applies to $(\alpha, -\beta, \gamma)$ and to $(-\alpha, \beta, \gamma)$. Hence (2.6) needs to be verified for only one ordered triplet $(\alpha, \beta, \gamma) \in \{-1, 1\}^3$. We shall use what seems to be the easiest one to work with.

Define the event

$$E := \{W_{-J} = 1,\ W_0 = -1,\ W_L = 1\}. \tag{2.7}$$

To complete the proof of Theorem 2.3, it suffices to show that

$$P(E) = 1/8. \tag{2.8}$$

Refer to (2.2). If $\omega \in \Omega$, then $\omega \in D(q, r)$ where $q := \max\{k \leq -1,\ U_k(\omega) = 0\}$ and $r := \min\{k \geq 0 : U_k(\omega) = 0\}$. The events $D(q, r)$, where $q \leq -1,\ r \geq 0$, and $r - q \in \{2, 4, 8, 16, \dots\}$, form a countable partition of Ω. For what follows, a different "coordinate system" will be easier to work with. For integers $q < r$ such that $r - q \in \{2, 4, 8, 16, \dots\}$, the numbers $m := (q + r)/2$ (the midpoint) and $I := (r - q)/2$ are integers, with $I \in \{1, 2, 4, 8, \dots\}$, and $q = m - I$ and $r = m + I$. The sample space Ω is thereby partitioned into the events $D(m - I, m + I),\ (m, I) \in \Gamma$, where

$$\Gamma := \{(m, I) \in \mathbf{Z} \times \{1, 2, 4, 8, \dots\} : m - I \leq -1 \text{ and } m + I \geq 0\}. \tag{2.9}$$

Hence, see (2.7), the event E is partitioned into the events $E \cap D(m - I, m + I)$, $(m, I) \in \Gamma$. To simplify, we need to show that some of these events are empty.

Define the following six sets:

(2.10)

$$\begin{aligned}
\Lambda(1) &:= \{(m, I) \in \mathbf{Z} \times \{1, 2, 4, 8, \dots\} : -J \le m - I \le -1 \\
&\qquad \text{and } 0 \le m + I \le L - 1\}; \\
\Lambda(2) &:= \{(m, I) \in \mathbf{Z} \times \{1, 2, 4, 8, \dots\} : -J \le m - I \le -1,\ m + I \ge L, \\
&\qquad \text{and } 0 \le m \le L - 1\}; \\
\Lambda^*(2) &:= \{(m, I) \in \mathbf{Z} \times \{1, 2, 4, 8, \dots\} : -J \le m - I \le -1,\ m + I \ge L, \\
&\qquad \text{and } m \notin \{0, \dots, L - 1\}\}; \\
\Lambda(3) &:= \{(m, I) \in \mathbf{Z} \times \{1, 2, 4, 8, \dots\} :\ m - I \le -J - 1, \\
&\qquad 0 \le m + I \le L - 1, \text{ and } -J \le m \le -1\}; \\
\Lambda^*(3) &:= \{(m, I) \in \mathbf{Z} \times \{1, 2, 4, 8, \dots\} :\ m - I \le -J - 1, \\
&\qquad 0 \le m + I \le L - 1, \text{ and } m \notin \{-J, \dots, -1\}\}; \\
\Lambda(4) &:= \{(m, I) \in \mathbf{Z} \times \{1, 2, 4, 8, \dots\} :\ m - I \le -J - 1 \\
&\qquad \text{and } m + I \ge L\}.
\end{aligned}$$

These six sets together form a partition of the set Γ in (2.9).

For any element $(m, I) \in \Lambda(4)$, the event $E \cap D(m-I, m+I)$ is empty. Let us verify that. Suppose $\omega \in E \cap D(m-I, m+I)$. By Definition 2.1 (d), (e), the numbers $W_k(\omega)$, $k \in \{m-I+1, m-I+2, \dots, m+I\}$ change sign only once (from $+1$'s to -1's or from -1's to $+1$'s). However, see (2.5), all three indices $-J$, 0, and L are in that set $\{m-I+1, \dots, m+I\}$, and hence, by (2.7), those numbers $W_k(\omega)$ change sign at least twice. Thus a contradiction occurs.

For any element $(m, I) \in \Lambda^*(2)$, the event $E \cap D(m - I, m + 1)$ is empty. Let us verify that. Suppose $\omega \in E \cap D(m - I, m + I)$. Then, see (2.5), the indices 0 and L are both in $\{m - I + 1, \dots, m + I\}$. By (2.7) and Remark 2.2 (e), 0 and L cannot be in the same "half," $\{m - I + 1, \dots, m\}$ or $\{m + 1, \dots, m + I\}$, hence 0 is in the first "half" and L in the second "half," hence $0 \le m \le L - 1$, contradicting the definition of the set $\Lambda^*(2)$.

Similarly, for any element $(m, I) \in \Lambda^*(3)$, the event $E \cap D(m - I, m + I)$ is empty. If $\omega \in E \cap D(m - I, m + I)$ were to exist, then the indices $-J$ and 0, see (2.5), would be in $\{m - I + 1, \dots, m + I\}$, but, see (2.7), in opposite "halves," forcing $-J \le m \le -1$ and a contradiction.

By the three preceding three paragraphs, the sentence after (2.10), and the sentence after (2.9), the event E is partitioned into the events $E \cap D(m-I, m+I)$, $(m,I) \in \Lambda(1) \cup \Lambda(2) \cup \Lambda(3)$, and one has that

$$P(E) = \sum_{s=1}^{3} \sum_{(m,I)\in\Lambda(s)} P(E \cap D(m-I, m+I)). \tag{2.11}$$

Here and below, an "empty sum" $\sum_{\varnothing}$ (anything) is defined to be 0.

Let us simplify (2.11) a little. For a given $(m,I) \in \Lambda(1)$, one has by (2.10) that $-J \le m-I$, $0 \in \{m-I+1, \dots, m+I\}$, and $L > m+I$, and hence by (2.7) and Remark 2.2 (g), (i),

$$\begin{aligned} &P(E|D(m-I, m+I)) \\ &\quad= \prod_{u\in\{-J,0,L\}} P(W_u = z_u \mid D(m-I, m+I)) = (1/2)^3 = 1/8, \end{aligned}$$

where $z_{-J} = z_L = 1$ and $z_0 = -1$. For a given $(m,I) \in \Lambda(2)$, one has by (2.10) that $-J \le m-I$, $0 \in \{m-I+1, \dots, m\}$, and $L \in \{m+1, \dots, m+I\}$, and hence by (2.7) and Remark 2.2 (g) (ii),

$$\begin{aligned} P(E|D(m-I, m+I)) &= P(W_{-J} = 1|D(m-I, m+I)) \\ &\qquad \times P((W_0, W_L) = (-1,1)|D(m-I, m+I)) \\ &= 1/2 \cdot 1/2 = 1/4. \end{aligned}$$

Similarly, for a given $(m,I) \in \Lambda(3)$, one has that $L > m+1$, $-J \in \{m-I+1, \dots, m\}$, and $0 \in \{m+1, \dots, m+I\}$, and by Remark 2.2 (g) (ii), $P(E|D(m-I, m+I)) = 1/4$. Hence by (2.11) and (2.3),

$$\begin{aligned} P(E) &= \sum_{(m,I)\in\Lambda(1)} P(D(m-I, m+I)) \cdot 1/8 \\ &\quad + \sum_{s\in\{2,3\}} \sum_{(m,I)\in\Lambda(s)} P(D(m-I, m+I)) \cdot 1/4 \\ &= \frac{1}{8} \sum_{(m,I)\in\Lambda(1)} 1/(2I)^2 + \frac{1}{4} \sum_{s\in\{2,3\}} \sum_{(m,I)\in\Lambda(s)} 1/(2I)^2. \end{aligned} \tag{2.12}$$

For each $s \in \{1,2,3\}$ and each $I \in \{1,2,4,8,\dots\}$, define the nonnegative integer

$$c(s,I) := \text{card}\,\{m \in \mathbf{Z} : (m,I) \in \Lambda(s)\}. \tag{2.13}$$

Then, by (2.12),

(2.14)

$$P(E) = \sum_{I\in\{1,2,4,8,\dots\}} (2I)^{-2}[(1/8)c(1,I) + (1/4)c(2,I) + (1/4)c(3,I)].$$

The integers $c(s, I)$ will be calculated in the next three lemmas.

Lemma 2.4. *Suppose $I \in \{1, 2, 4, 8, \dots\}$. Then, under the assumption of* (2.5)*,*

$$c(1,I) = \begin{cases} 2I & \textit{if } 1 \le I \le J/2 \\ J & \textit{if } J/2 \le I \le L/2 \\ L + J - 2I & \textit{if } L/2 \le I \le (L + J)/2 \\ 0 & \textit{if } (L + J)/2 \le I. \end{cases} \tag{2.15}$$

Proof. Suppose $I \in \{1, 2, 4, 8, \dots\}$. In order for $m \in \mathbf{Z}$ to be such that $(m, I) \in \Lambda(1)$, the restrictions on m are, see (2.10),

$$m \ge I - J, \quad m \ge -I, \quad \text{and} \quad m \le I - 1, \quad m \le L - I - 1. \tag{2.16}$$

If $1 \le I \le J/2$, then $2I \le J \le L$ by (2.5), hence $I - J \le -I$ and $I \le L-I$, and the restrictions on m in (2.16) are simply $-I \le m \le I-1$, and hence $c(1, I) = 2I$. If $J/2 \le I \le L/2$, then $J \le 2I \le L$, hence $-I \le I - J$ and $I \le L - I$, and the restrictions on m in (2.16) are simply $I - J \le m \le I - 1$. If $L/2 \le I < (L + J)/2$, then $J \le L \le 2I$ by (2.5), hence $-I \le I - J$ and $L - I \le I$, and the restrictions on m in (2.16) are simply $I - J \le m \le L - I - 1$ (satisfied by $L + J - 2I$ integers m). If $(L + J)/2 \le I$, then $L + J \le 2I$, hence $L - I \le I - J$, and the first and last inequalities in (2.16) conflict. Thus all cases of (2.15) hold. (The third case in (2.15) was not formally shown above for $I = (L + J)/2$, but holds for that I by the fourth case in (2.15).) □

Lemma 2.5. *Suppose $I \in \{1, 2, 4, 8, \dots\}$. Then, under the assumption of* (2.5)*,*

$$c(2, I) = \begin{cases} 0 & \text{if } 1 \le I \le L/2 \\ 2I - L & \text{if } L/2 \le I \le (L + J)/2 \\ J & \text{if } (L + J)/2 \le I \le L \\ J + L - I & \text{if } L \le I \le L + J \\ 0 & \text{if } L + J \le I \end{cases} \tag{2.17}$$

Proof. Suppose $I \in \{1, 2, 4, 8, \dots\}$. In order for $m \in \mathbf{Z}$ to be such that $(m, I) \in \Lambda(2)$, the restrictions on m are, see (2.10),

$$m \ge I - J, \quad m \ge L - I, \quad m \ge 0, \quad \text{and} \quad m \le I - 1, \quad m \le L - 1. \tag{2.18}$$

If $1 \le I \le L/2$, then $I \le L - I$, and the second and fourth inequalities in (2.18) conflict. If $L/2 < I \le (L + J)/2$, then $L < 2I \le L + J \le 2L$, and thus $I \le L$, by (2.5), hence $0 \le L - I$ and $I - J \le L - I$, and the restrictions on m in (2.18) are simply $L - I \le m \le I - 1$. If $(L + J)/2 \le I \le L$, then $0 \le L - I \le I - J$, and the restrictions on m in (2.18) are $I - J \le m \le I - 1$. If $L \le I < L + J$, then $J \le I$ by (2.5), hence $L - I \le 0 \le I - J$, and the restrictions on m in (2.18) are $I - J \le m \le L - 1$. If $I \ge L + J$, then the first and last inequalities in (2.18) conflict. Thus all cases of (2.17) hold. (The second case of (2.17) for $I = L/2$, and the fourth case for $I = L + J$, hold by the first and last cases.) □

Lemma 2.6. *Suppose $I \in \{1, 2, 4, 8, \dots\}$. Then, under the assumption of* (2.5)*,*

$$c(3, I) = \begin{cases} 0 & \text{if } 1 \le I \le J/2 \\ 2I - J & \text{if } J/2 \le I \le J \\ J & \text{if } J \le I \le L \\ L + J - I & \text{if } L \le I \le L + J \\ 0 & \text{if } L + J \le I \end{cases} \tag{2.19}$$

Proof. Suppose $I \in \{1, 2, 4, 8, \dots\}$. In order for $m \in \mathbf{Z}$ to be such that $(m, I) \in \Lambda(3)$, the restrictions on m are, see (2.10),

(2.20)
$$m \geq -I, \quad m \geq -J, \quad \text{and} \quad m \leq I-J-1, \quad m \leq L-I-1, \quad m \leq -1.$$

If $1 \leq I \leq J/2$, then $I - J \leq -I$, and the first and third inequalities in (2.20) conflict. If $J/2 < I \leq J$, then $-I \geq -J$ and $I \leq L$ (by (2.5)) and $I - J - 1 \leq -1 \leq L - I - 1$, and the restrictions on m in (2.20) are $-I \leq m \leq I - J - 1$. If $J \leq I \leq L$, then $-J \geq -I$ and $-1 \leq I - J - 1$ and $-1 \leq L - I - 1$, and the restrictions on m in (2.20) are $-J \leq m \leq -1$. If $L \leq I < L + J$, then $J \leq I$ by (2.5), and $-J \geq -I$ and $L - I - 1 \leq -1 \leq I - J - 1$, and the restrictions on m in (2.20) are $-J \leq m \leq L-I-1$. If $I \geq L+J$, then the second and fourth inequalities in (2.20) conflict. Thus all cases of (2.19) hold. (The second case of (2.19) for $I = J/2$, and the fourth case for $I = L + J$, hold by the first and last cases.) That completes the proof of Lemma 2.6. □

Now let us represent the elements $I \in \{1, 2, 4, 8, \dots\}$ by $I = 2^n$, $n \in \{0, 1, 2, 3, \dots\}$. Equation (2.14) can be rewritten as

(2.21)
$$P(E) = \sum_{n=0}^{\infty} 4^{-n-1} \cdot \left[(1/8)c(1, 2^n) + (1/4)c(2, 2^n) + (1/4)c(3, 2^n)\right].$$

For any interval $\mathcal{I} \subset (0, \infty)$ (open, closed, or half-open), define the (possibly empty) set

$$Q\mathcal{I} := \{n \in \{0, 1, 2, \dots\} : 2^n \in \mathcal{I}\}. \tag{2.22}$$

(In the calculations below, interpret $(x, x] := \varnothing$, $[x, y] := \varnothing$ if $y < x$, and $Q\varnothing := \varnothing$.) With that notation, let us "decompose" the right side of (2.21). By Lemma 2.4,

(2.23)
$$\sum_{n=0}^{\infty} 4^{-n-1} c(1, 2^n) = \sum_{n \in Q[1, J/2]} 4^{-n-1} \cdot (2 \cdot 2^n) + \sum_{n \in Q(J/2, L/2]} 4^{-n-1} J$$
$$+ \sum_{n \in Q(L/2, (J+L)/2)} 4^{-n-1} \cdot (L + J - 2 \cdot 2^n).$$

By Lemma 2.5,

$$\sum_{n=0}^{\infty} 4^{-n-1} c(2, 2^n) = \sum_{n \in Q(L/2,(J+L)/2)} 4^{-n-1} \cdot (2 \cdot 2^n - L) + \sum_{n \in Q[(J+L)/2,L]} 4^{-n-1} J + \sum_{n \in Q(L,L+J)} 4^{-n-1} (J + L - 2^n). \tag{2.24}$$

By Lemma 2.6,

$$\sum_{n=0}^{\infty} 4^{-n-1} c(3, 2^n) = \sum_{n \in Q(J/2,J]} 4^{-n-1} (2 \cdot 2^n - J) + \sum_{n \in Q(J,L]} 4^{-n-1} J + \sum_{n \in Q(L,L+J)} 4^{-n-1} (L + J - 2^n). \tag{2.25}$$

Referring to (2.5), let G and H denote the nonnegative integers such that $2^G \le J < 2^{G+1}$ and $2^H \le L < 2^{H+1}$. Then, see (2.5),

$$0 \le G \le H, \quad 2^{G-1} \le J/2 < 2^G \le J, \tag{2.26}$$

and

$$2^{H-1} \le L/2 < 2^H \le L < 2^{H+1}.$$

By (2.5), (2.22), and (2.26), the index sets in the right sides of (2.23)–(2.25) are as follows:

$$\begin{aligned} Q[1, J/2] &= \begin{cases} \varnothing & \text{if } G = 0 \\ \{0, \dots, G-1\} & \text{if } G \ge 1, \end{cases} \\ Q(J/2, J] &= \{G\}, \\ Q(J/2, L/2] &= \begin{cases} \varnothing & \text{if } G = H \\ \{G, \dots, H-1\} & \text{if } G < H \end{cases} \\ Q(J, L] &= \begin{cases} \varnothing & \text{if } G = H \\ \{G+1, \dots, H\} & \text{if } G < H, \end{cases} \end{aligned} \tag{2.27}$$

$$Q(L/2,(J+L)/2) = \begin{cases} \varnothing & \text{if } J+L \le 2^{H+1} \\ \{H\} & \text{if } J+L > 2^{H+1}, \end{cases}$$
$$Q[(J+L)/2,L] = \begin{cases} \{H\} & \text{if } J+L \le 2^{H+1} \\ \varnothing & \text{if } J+L > 2^{H+1}, \end{cases}$$
$$Q(L,L+J) = \begin{cases} \varnothing & \text{if } J+L \le 2^{H+1} \\ \{H+1\} & \text{if } J+L > 2^{H+1}. \end{cases}$$

Let us evaluate the first two sums in the right side of (2.23). If $G \ge 1$, then by (2.27), $\sum_{n\in Q[1,J/2]} 4^{-n-1} \cdot (2 \cdot 2^n) = \sum_{n=0}^{G-1} 2^{-n-1} = 1 - 2^{-G}$. If instead $G = 0$, then by (2.27), one still has $\sum_{n\in Q[1,J/2]} 4^{-n-1} = 1 - 2^{-G}$, with both sides being 0. If $G < H$, then by (2.27), $\sum_{n\in Q(J/2,L/2]} 4^{-n-1}J = \sum_{n=G}^{H-1} 4^{-n-1}J = (J/3)(4^{-G} - 4^{-H})$. If instead $G = H$, then by (2.27), one still has $\sum_{n\in Q(J/2,L/2]} 4^{-n-1}J = (J/3)\cdot(4^{-G} - 4^{-H})$, with both sides being 0. Hence, regardless of the values of G and H, subject to (2.26), by (2.23),

$$\begin{aligned}\sum_{n=0}^{\infty} 4^{-n-1}c(1,2^n) &= 1 - 2^{-G} + (J/3)(4^{-G} - 4^{-H}) \\ &\quad + \sum_{n\in Q(L/2,(J+L)/2)} 4^{-n-1}(L+J-2^{n+1}).\end{aligned} \tag{2.28}$$

Next let us evaluate the first two sums in the right side of (2.25). By (2.27), $\sum_{n\in Q(J/2,J]} 4^{-n-1}(2\cdot 2^n - J) = 2^{-G-1} - 4^{-G-1}J$. By (2.27) and a calculation like the one just prior to (2.28), $\sum_{n\in Q(J,L]} 4^{-n-1}J = (J/3)(4^{-G-1} - 4^{-H-1})$, regardless of whether, see (2.26), $G < H$ or $G = H$. Hence, by (2.25),

$$\begin{aligned}\sum_{n=0}^{\infty} 4^{-n-1}c(3,2^n) &= 2^{-G-1} - 4^{-G-1}J + (J/3)(4^{-G-1} - 4^{-H-1}) \\ &\quad + \sum_{n\in Q(L,L+J)} 4^{-n-1}(L+J-2^n).\end{aligned} \tag{2.29}$$

Now we are ready to prove (2.8), and thereby complete the proof of Theorem 2.3.

Consider first the case where $J+L \le 2^{H+1}$. Then, by (2.27), the sets $Q(L/2,(J+L)/2)$ and $Q(L,L+J)$ are empty, $Q[(J+L)/2,L] = \{H\}$, and hence by (2.24), $\sum_{n=0}^{\infty} 4^{-n-1}c(2,2^n) = 4^{-H-1}J$. Substituting that and (2.28) and (2.29) into (2.21), one obtains after some arithmetic,

$$P(E) = (1/8)\cdot\left[1-2^{-G}+(J/3)(4^{-G}-4^{-H})+0\right]+(1/4)\cdot 4^{-H-1}J \\ +(1/4)\cdot\left[2^{-G-1}-4^{-G-1}J+(J/3)(4^{-G-1}-4^{-H-1})+0\right].$$

After some arithmetic, one obtains $P(E)=1/8$; everything else cancels out. Thus (2.8) holds in the case where $J+L \le 2^{H+1}$.

Now consider the case where $J+L > 2^{H+1}$. By (2.27), the set $Q[(J+L)/2,L]$ is empty, $Q(L/2,(J+L)/2) = \{H\}$, and $Q(L,J+L) = \{H+1\}$. Hence by substituting (2.28), (2.24), and (2.29) (in that order) into (2.21), one obtains

$$\begin{aligned} &P(E) \\ &= (1/8)\cdot\left[1-2^{-G}+(J/3)(4^{-G}-4^{-H})+4^{-H-1}(L+J-2^{H+1})\right] \\ &\quad+(1/4)\cdot\left[4^{-H-1}(2\cdot 2^H-L)+0+4^{-(H+1)-1}(L+J-2^{H+1})\right] \\ &\quad+(1/4)\cdot[2^{-G-1}-4^{-G-1}J+(J/3)(4^{-G-1}-4^{-H-1}) \\ &\quad+4^{-(H+1)-1}(L+J-2^{H+1})]. \end{aligned}$$

After some arithmetic, one obtains $P(E)=1/8$; everything else cancels out. Thus (2.8) holds in the case where $J+L > 2^{H+1}$. That completes the proof of (2.8) and of Theorem 2.3. □

3. Proof of Theorem 1 (property (B)). Let us repeat here the construction of the random sequence $X := (X_k, k \in \mathbf{Z})$ in [**2**, Definition 3.1]. We retain all of the definitions, remarks, and random sequences in Section 2.

Definition 3.1. (a) Let $\varepsilon := (\varepsilon_k,\ k \in \mathbf{Z})$ be a sequence of i.i.d. random variables taking only the values 0 and 1, with $P(\varepsilon_0 = 0) = P(\varepsilon_0 = 1) = 1/2$, with this sequence ε being independent of $(T_k, U_k, V_k, W_k,\ k \in \mathbf{Z})$. As a trivial technical formality, assume that for every $\omega \in \Omega$, $\varepsilon_k(\omega) = 1$ for infinitely many negative integers k and infinitely many positive integers k.

(b) Define the random integers $\dots, I_{-1}, I_0, I_1, \dots$ by the conditions

$$\cdots < I_{-2} < I_{-1} < I_0 \le 0 < 1 \le I_1 < I_2 < I_3 < \cdots$$

and for all $\omega \in \Omega$,

$$\{k \in \mathbf{Z} : \varepsilon_k(\omega) = 1\} = \{\dots, I_{-1}(\omega), I_0(\omega), I_1(\omega), \dots\}.$$

(c) Define the random sequence $X := (X_k,\ k \in \mathbf{Z})$ as follows: For all $k \in \mathbf{Z}$ and all $\omega \in \Omega$,

$$X_k(\omega) := \begin{cases} W_j(\omega) & \text{if } k = I_j(\omega) \text{ for some } j \in \mathbf{Z} \\ 0 & \text{if } k \notin \{\dots, I_{-1}(\omega), I_0(\omega), I_1(\omega), \dots\}. \end{cases}$$

Note that for all $k \in \mathbf{Z}$,

$$\{X_k = 0\} = \{\varepsilon_k = 0\} \quad \text{and} \quad \{X_k = -1 \text{ or } 1\} = \{\varepsilon_k = 1\}. \tag{3.1}$$

Also note that the σ-fields $\sigma(\varepsilon_k, I_k, i \in \mathbf{Z})$ and $\sigma(W_k, k \in \mathbf{Z})$ are independent. Also, for any given $\omega \in \Omega$, $I_k(\omega) \le k$ for all $k \le 0$ and $I_k(\omega) \ge k$ for all $k \ge 1$. If $\varepsilon_0(\omega) = 0$, then $I_0(\omega) \le -1$ and (by induction) $I_k(\omega) \le k-1$ for all $k \le 0$.

From [**2**, Theorem 1 and Section 3], one has that the sequence X is strictly stationary and satisfies properties (A), (C), (D) and (E) in Theorem 1 as well as pairwise independence. Our task is just to prove property (B) in Theorem 1, that is, to show that if a, b and c are distinct integers then the three random variables X_a, X_b and X_c are independent. By strict stationarity, it suffices to show this for $a = -J$, $b = 0$, and $c = L$ for arbitrary positive integers J and L. Thus our task is to show that, if $J \ge 1$, $L \ge 1$, and $\alpha, \beta, \gamma \in \{-1, 0, 1\}$, then

$$\begin{aligned} P(X_{-J} = \alpha, X_0 = \beta, X_L = \gamma) \\ = P(X_{-J} = \alpha) \cdot P(X_0 = \beta) \cdot P(X_L = \gamma). \end{aligned} \tag{3.2}$$

There are 27 choices of α, β and γ. We shall verify (3.2) for two choices: (i) $\alpha = 1$ and $\beta = \gamma = 0$, and (ii) $\alpha = \beta = -1$ and $\gamma = 1$. The proofs of (3.2) for the other twenty-five choices are similar.

By Definition 3.1, equation (3.1) and the subsequent observations, equation (2.1), and property (A) in Theorem 1,

$$\begin{aligned}
P(X_{-J} = 1, X_0 = 0, X_L = 0) &= P(X_{-J} = 1, \varepsilon_{-J} = 1, \varepsilon_0 = 0, \varepsilon_L = 0) \\
&= \sum_{j=0}^{J-1} P(X_{-J} = 1, I_{-j} = -J, \varepsilon_0 = 0, \varepsilon_L = 0) \\
&= \sum_{j=0}^{J-1} P(W_{-j} = 1, I_{-j} = -J, \varepsilon_0 = 0, \varepsilon_L = 0) \\
&= \sum_{j=0}^{J-1} P(W_{-j} = 1) \cdot P(I_{-j} = -J, \varepsilon_0 = 0, \varepsilon_L = 0) \\
&= \sum_{j=0}^{J-1} (1/2) \cdot P(I_{-j} = -J, \varepsilon_0 = 0, \varepsilon_L = 0) \\
&= (1/2) \cdot P(\varepsilon_{-J} = 1, \varepsilon_0 = 0, \varepsilon_L = 0) = (1/2) \cdot (1/8) \\
&= (1/4) \cdot (1/2)^2 = P(X_{-J} = -1) \cdot P(X_0 = 0) \cdot P(X_L = 0).
\end{aligned}$$

Thus (3.2) holds for $\alpha = 1$ and $\beta = \gamma = 0$. By a similar argument, using Theorem 2.3,

$$\begin{aligned}
&P(X_{-J} = -1, X_0 = -1, X_L = 1) \\
&\quad = P(X_{-J} = -1, X_0 = -1, X_L = 1, \varepsilon_{-J} = 1, \varepsilon_0 = 1, \varepsilon_L = 1) \\
&\quad = \sum_{j=1}^{J} \sum_{l=1}^{L} P(X_{-J} = -1, X_0 = -1, X_L = 1, I_{-j} = -J, I_0 = 0, I_l = L) \\
&\quad = \sum_{j=1}^{J} \sum_{l=1}^{L} (W_{-j} = -1, W_0 = -1, W_l = 1, I_{-j} = -J, I_0 = 0, I_l = L) \\
&\quad = \sum_{j=1}^{J} \sum_{l=1}^{L} P(W_{-j} = -1, W_0 = -1, W_l = 1) P(I_{-j} = -J, I_0 = 0, I_l = L) \\
&\quad = \sum_{j=1}^{J} \sum_{\ell=1}^{L} (1/8) \cdot P(I_{-j} = -J, I_0 = 0, I_\ell = L) \\
&\quad = (1/8) \cdot P(\varepsilon_{-J} = 1, \varepsilon_0 = 1, \varepsilon_L = 1) = (1/8) \cdot (1/8) \\
&\quad = (1/4)^3 = P(X_{-J} = -1) \cdot P(X_0 = -1) \cdot P(X_L = 1).
\end{aligned}$$

Thus (3.2) holds for $\alpha = \beta = -1$ and $\gamma = 1$. The proofs of (3.2) for the other 25 choices of α, β, and γ are similar and are left to the reader. That completes the proof of property (B) in Theorem 1. $\square$

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REFERENCES

1. P. Billingsley, *Probability and measure*, 3rd ed., Wiley, New York, 1995.

2. R.C. Bradley, *A stationary, pairwise independent, absolutely regular sequence for which the central limit theorem fails*, Probab. Theory Related Fields **81** (1989), 1–10.

3. ———, *Introduction to strong mixing conditions*, Vol. 1, Technical Report, Dept. of Mathematics, Indiana University, Bloomington, Custom Publ. of I.U., Bloomington, 2005.

4. J.A. Cuesta and C. Matrán, *On the asymptotic behavior of sums of pairwise independent random variables*, Statist. Probab. Lett. **11** (1991), 201–210 (Correction Ibid. **12** (1991), 183).

5. Yu.A. Davydov, *Mixing conditions for Markov chains*, Theory Probab. Appl. **18** (1973), 312–328.

6. M. Denker, *Uniform integrability and the central limit theorem for strongly mixing processes*, in *Dependence in probability and statistics* (E. Eberlein and M.S. Taqqu, eds.), Birkhäuser, Boston, 1986.

7. N. Etemadi, *An elementary proof of the strong law of large numbers*, Z. Wahrsch. Verw. Gebeite **55** (1981), 119–122.

8. N. Herrndorf, *Stationary strongly mixing sequences not satisfying the central limit theorem*, Ann. Probab. **11** (1983), 809–813.

9. S. Janson, *Some pairwise independent sequences for which the central limit theorem fails*, Stochastics **23** (1988), 439–448.

10. F. Merlevède and M. Peligrad, *The functional central limit theorem under the strong mixing condition*, Ann. Probab. **28** (2000), 1336–1352.

11. T. Mori and K. Yoshihara, *A note on the central limit theorem for stationary strong-mixing sequences*, Yokohama Math J. **34** (1986), 143–146.

12. A.R. Pruss, *A bounded N-tuplewise independent and identically distributed counterexample to the* CLT, Probab. Theory Related Fields **111** (1998), 323–332.

13. M. Rosenblatt, *A central limit theorem and a strong mixing condition*, Proc. Nat. Acad. Sci. USA **42** (1956), 43–47.

14. V.A. Volkonskii and Yu.A. Rozanov, *Some limit theorems for random functions* I, Theory Probab. Appl. **4** (1959), 178–197.

Department of Mathematics, Indiana University, Bloomington, Indiana 47405
E-mail address: bradleyr@indiana.edu