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ON LOCALLY UNIFORMLY A-PSEUDOCONVEX ALGEBRAS

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ABSTRACT. Conditions when a unital locally uniformly A-pseudoconvex algebra (E,τ) is (or when there exists a topology τ' on E such that (E,τ') is) a locally p-convex algebra for some $p \in (0,1]$, are found. It is shown that on every unital advertibly complete locally uniformly A-pseudoconvex algebra E there exists a submultiplicative semi-norm $|\,.\,|$ such that $(E,|\,.\,|)$ is a Q-algebra.

1. Introduction. 1. Let (E,τ) be a locally pseudoconvex algebra over $\mathbf{C}$ with separately continuous multiplication (in short lpca) the topology τ of which has been given by a family $\{|\,.\,|_i : i \in I\}$ of p_i-homogeneous semi-norms $|\,.\,|_i$, where $0 < p_i \leq 1$ for each $i \in I$. In particular, when $p = \inf p_i > 0$, this lpca (E,τ) is a *locally p-convex* algebra (in short lp-ca) that is, an lpca in which every $p_i = p$.

If for any $x \in E$ there is a positive number $M(x)$ such that[1]

$$\max(|xy|_i, |yx|_i) \leqslant M(x)^{p_i}|y|_i \tag{1}$$

for each $y \in E$ and $i \in I$ (here $M(x)$ depends only on x, but not on i), then an lpca (E,τ) is a *locally uniformly A-pseudoconvex* algebra (in short lu$A$-pca) and if every semi-norm $|\,.\,|_i$ in the family $\{|\,.\,|_i : i \in I\}$ is *submultiplicative*, that is,

$$|xy|_i \leqslant |x|_i|y|_i$$

for each $x, y \in E$, then an lpca (A,τ) is a *locally multiplicatively pseudoconvex* (*or locally m-pseudoconvex*) *algebra* (in short lm-pca).

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2. A net $(x_\lambda)_\lambda$ in a unital topological algebra (E,τ) is said to be *advertibly convergent* if there exists an element $x \in E$ such that both nets $(xx_\lambda)_{\lambda\in\Lambda}$ and $(x_\lambda x)_{\lambda\in\Lambda}$ converge to the unit element e of E. A unital algebra (E,τ) is *advertibly complete* if every advertibly convergent Cauchy net converges in (E,τ), see [**12**, p. 45].

The set of all bounded subsets in a linear topological space (E,τ) is called the *von Neumann bornology* on (E,τ), see for example, [**9, 10**]. We will denote it as usual by $\mathcal{B}_\tau$.

3. By results of Ligaud [**11**] and Metzler [**13**], see also [**8**, pp. 102, 103], the von Neumann bornology $\mathcal{B}_\tau$ on a metrizable lpca (E,τ) has a countable basis only if (E,τ) is an lp-ca for some $p \in (0,1]$. We will show that on any unital lu$A$-pca (E,τ) there is a *bornology*[2] $\mathcal{B}$ with a countable basis such that $(E,\mathcal{B})$ is a *bornological algebra*[3]. On the other hand, we exhibit, that on any unital advertibly complete lu$A$-pca (E,τ) there exists a submultiplicative semi-norm $|\,.\,|$ such that $(E,|\,.\,|)$ is a Q-algebra and every unital lu$A$-pca has a stronger metrizable topology τ' such that (E,τ') is an lm-pca and an lu$A$-pca.

2. Structural results. Let (E,τ) be a unital lu$A$-pca. First we find conditions under which (E,τ) is an lp-ca for some $p \in (0,1]$.

Proposition 1. *Let (E,τ) be a unital luA-pca and $\{|\,.\,|_i : i \in I\}$ a family of p_i-homogeneous semi-norms on E which defines the topology τ. Then*

(i) *there is a bornology $\mathcal{B}$ on E with a countable basis (finer than the von Neumann bornology $\mathcal{B}_\tau$ on (E,τ)) such that $(E,\mathcal{B})$ is a bornological algebra*;

(ii) (E,τ) *is an lp-ca for some $p \in (0,1]$ if (E,τ) is metrizable and $\mathcal{B}_\tau = \mathcal{B}$.*

Proof. For each $x \in E$ and $i \in I$, let

$$\|x\|_i = \sup\{|xy|_i : y \in E \quad \text{and} \quad |y|_i \leqslant 1\}.$$

Then $\|\,.\,\|_i$ is a submultiplicative p_i-homogeneous semi-norm on E which satisfies the condition

$$\|xy\|_i \leqslant N(x)\|y\|_i \tag{2}$$

for each $i \in I$ and $y \in E$, where $N(x)$ does not depend on i. Now, for each $x \in E$ we put $\|x\| = \sup\{\|x\|_i : i \in I\}$. Then[4]

a) $\|x\| \leqslant N(x) < \infty$ for each $x \in A$;

b) $\|x\| = 0$ if and only if $x = \theta_E$;

c) $\|x+y\| \leqslant \|x\| + \|y\|$ for each $x, y \in E$;

d) $\|\alpha x\| \leqslant \max(|\alpha|, 1)\|x\|$ for each $\alpha \in \mathbf{C}$ and $x \in E$;

e) $\|xy\| \leqslant \|x\|\,\|y\|$ for each $x, y \in E$.

For any $n \in \mathbf{N}$ let $B_n = \{x \in E : \|x\| \leqslant n\}$. Then $\{B_n : n \in \mathbf{N}\}$ is a countable basis for a bornology $\mathcal{B}$ on E such that $(E, \mathcal{B})$ is a bornological algebra and $\mathcal{B} \subset \mathcal{B}_\tau$, because

$$|x|_i = |xe|_i \leqslant ||x||_i \leqslant ||x||$$

for each $x \in E$ and $i \in I$.

The statement (ii) holds by Theorems 1 and 2 from [**11**] (or by Theorem 1 and Proposition 3 from [**8**, pp. 102, 103]). □

Using another approach, we have

Proposition 2. *Let (E, τ) be a unital luA-pca and $\{|\,.\,|_i : i \in I\}$ a family of p_i-homogeneous semi-norms on E which defines the topology τ. Then*

(i) *there is a submultiplicative semi-norm $|\,.\,|$ on E;*

(ii) *$(E, |\,.\,|)$ is a Q-algebra if (E, τ) is advertibly complete.*

Proof. Let $\{\|\,.\,\|_i : i \in I\}$ be the family of submultiplicative p_i-homogeneous semi-norms on E, defined in the proof of Proposition 1, and $M(\tau)$ the topology on E defined by this family. Then

$$\|xy\|_i \leqslant M(x)^{p_i}\|y\|_i$$

for each $x, y \in E$. For each $i \in I$, let

$$\|x\|_{i,c} = \inf \sum_{k=1}^{n} \|x_k\|_i^{1/p_i},$$

where the infimum is taken over all decompositions $x = \sum_{k=1}^{n} x_k$ of x in E. Similarly as in [**7**], every $\| \,.\, \|_{i,c}$ is a submultiplicative semi-norm on E and

$$\|xy\|_{i,c} \leqslant M'(x)\|y\|_{i,c}$$

for each $y \in E$, where $M'(x) = \sum_{k=1}^{n} M(x_k)$. Therefore $| \,.\, |$, defined by

$$|x| = \sup\{\|x\|_{i,c} : i \in I\}$$

for each $x \in E$, is a submultiplicative semi-norm on E.

Let now (E, τ) be advertibly complete. Since $M(\tau)$ is finer that τ, then $(E, M(\tau))$ is also an advertibly complete algebra, see [**5**, Proposition 2]. Hence

$$\rho(x) = \sup\left\{\lim_{n\to\infty} \|x^n\|_i^{1/np_i} : i \in I\right\}$$

for each $x \in E$ (see[5] [**3**, Corollary 4.2]). Since

$$\lim_{n\to\infty} \|x^n\|_i^{1/np_i} = \lim_{n\to\infty} \|x^n\|_{i,c}^{1/n}$$

for each $i \in I$ by [**7**, Theorem 2], then $\rho(x) \leqslant |x|$ for each $x \in E$. Now $e - x$ is invertible in E if $|x| < 1$. Thus, the interior of the set of all invertible elements in $(E, | \,.\, |)$ is not empty. Consequently, $(E, | \,.\, |)$ is a Q-algebra, see [**12**, Lemma 6.4, pp. 43, 44]. □

Remark. If the topological dual space of (E, τ) separates the points of E, then $| \,.\, |$, introduced in the proof of Proposition 2, is a norm. In general, it is only a semi-norm. Take, for example, $\mathbf{C} \times L^p([0,1])$ with $0 < p \leqslant 1$, define the multiplication in $\mathbf{C} \times L^p([0,1])$ by $(\alpha, f)(\beta, g) = (\alpha\beta, fg)$ for each $\alpha, \beta \in \mathbf{C}$ and $f, g \in L^p([0,1])$ and the p-seminorm $\| \,.\, \|_p$ on $\mathbf{C} \times L^p([0,1])$ by $\|(\alpha, f)\|_p = |\alpha|^p + \|f\|_p$, where $\|f\|_p = \int_0^1 |f(t)|^p \, dt$.

In Proposition 2 the semi-norm $| \,.\, |$ defines a topology on E which is not necessarily stronger than τ. The following result is an analog of Proposition 4.6 from [**6**].

Proposition 3. *Let (E, τ) be a unital luA-pca and $\{| \,.\, |_i : i \in I\}$ a family of p_i-homogeneous semi-norms on E which defines the topology τ on E. Then*

(i) *there exists a topology* τ' *on* E, *stronger than* τ, *such that* (E, τ') *is a metrizable lm-pca and an luA-pca*;

(ii) (E, τ') *is advertibly complete if* (E, τ) *is advertibly complete*;

(iii) (E, τ') *is an lp-ca for some* $p \in (0, 1]$ *if every* τ*-bounded subset* B *of* E *is uniformly* τ'*-bounded*[6].

Proof. Let again $\{\|\,.\,\|_i : i \in I\}$ be the family of submultiplicative p_i-homogeneous semi-norms on E, introduced in the proof of Proposition 1, and let

$$\Lambda_n = \left\{ i \in I : \frac{1}{n} \leqslant p_i \right\}$$

for each $n \in \mathbf{N}$. Then $\Lambda_n \subset \Lambda_{n+1}$ for each $n \in \mathbf{N}$ and

$$I = \cup\{\Lambda_n : n \in \mathbf{N}\}.$$

Let now $q_n = \inf\{p_i : i \in \Lambda_n\}$ and

$$\|x\|_n = \sup\{\|x\|_i^{q_n/p_i} : i \in \Lambda_n\}$$

for each $n \in \mathbf{N}$. Then $q_n \in (0, 1]$ and $\|\,.\,\|_n$ is a q_n-homogeneous submultiplicative semi-norm on E for each $n \in \mathbf{N}$.

Let τ' be the topology on E which defines the countable family $\{\|\,.\,\|_n : n \in \mathbf{N}\}$ of semi-norms. Then τ' is stronger than τ on E and (E, τ') is a metrizable lm-pca and also an lu$A$-pca. by (2).

The statement (ii) is true by (i) and [**5**, Proposition 2], and the statement (iii) is true by (i) (similarly as statement (ii) in Proposition 1). $\square$

ENDNOTES

1. The first author of the present paper considered in [**1**] the case when $\inf p_i \neq 0$. Since $t^p \leq 1 + t$ for each $p \in (0, 1]$ if $t \geqslant 1$, then from (1) follows the condition $\max(|xy|_i, |yx|_i) \leqslant N(x)|y|_i$ of [**1**], where the positive number $N(x)$ depends again only on x, but not on i.

2. That is, $\mathcal{B}$ is a cover of E such that the union of every two elements of $\mathcal{B}$ and every subset of elements of $\mathcal{B}$ belong to $\mathcal{B}$, see for example, [**10**, p. 18]. In [**14**, p. 23], and in several other articles such a cover $\mathcal{B}$ of E is called also a *bound structure* or *boundedness* on E.

3. That is, an algebra in which all algebraic operations are bounded. In this case $\mathcal{B}$ is called an *algebra bornology*, see for example, [**8**, p. 21] or an *algebra boundedness*, see for example, [**14**, p. 24] or [**15**, p. 199] on E. Sometimes, instead of the term "bornological algebra" is used also the term "b-*algebra*." We prefer the term "bornological algebra," because "b-algebra" has also other meanings, see for example, [**15**, p. 199].

4. Here, and later on, θ_E denotes the zero element of E.

5. For complete locally m-pseudoconvex Hausdorff algebra this result has been proved in [**4**, Theorem 7.4.8], and for commutative unital advertibly complete locally m-pseudoconvex Hausdorff algebra in [**2**, Proposition 12].

6. That is, $\sup_{b\in B}\sup_{n\in\mathbf{N}}|b|_n < +\infty$ for each bounded subset B in (E,τ).

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