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PAIRS OF TOPOLOGICAL ALGEBRAS

MART ABEL AND MATI ABEL

ABSTRACT. Let (A, B) be a pair of topological algebras A and B. Conditions for A, respectively B, to be a Gelfand-Mazur algebra or an exponentially galbed algebra, if B, respectively A, is one, are given. It is shown that $\hom A$, the set of all nonzero continuous homomorphisms from A onto $\mathbf{K}$ endowed with Gelfand topology, and $\hom B$ are homeomorphic if either $\hom A$ is equicontinuous or $\hom B$ is locally equicontinuous. Topological algebras A with jointly continuous multiplication for which a) the completion $\tilde{A}$ is a Gelfand-Mazur algebra or exponentially galbed algebra or b) $\hom A$ and $\hom \tilde{A}$ are homeomorphic are described.

1. Introduction. Let A be an associative topological algebra over the field $\mathbf{K}$ (of real or complex numbers) with separately continuous multiplication (in the sequel, a topological algebra), $m(A)$ the set of such closed regular two-sided ideals of A which are maximal as left or right ideals and $\hom A$ the set of all nonzero continuous homomorphisms from A onto $\mathbf{K}$ endowed, as usual, with the topology in which a base of neighborhoods of $\varphi_0 \in \hom A$ consists of sets

$$O(\varphi_0; a_1, \dots, a_n, \varepsilon) = \bigcap_{k=1}^{n} \{\varphi \in \hom A : |(\varphi - \varphi_0)(a_k)| < \varepsilon\}$$

for some $n \in \mathbf{N}$, $\varepsilon > 0$ and $a_1, \dots, a_n \in A$. The set $\hom A$ is *equicontinuous* if, for any $\varepsilon > 0$, there is a neighborhood O of zero in A such that $|\varphi(a)| < \varepsilon$ for each $a \in O$ and $\varphi \in \hom A$ and $\hom A$ is *locally equicontinuous* if every $\varphi_0 \in \hom A$ has an equicontinuous neighborhood. It is known (see, for example, [**19**, p. 75]) that $\hom A$ is equicontinuous if A is a Q-*algebra*, that is, a topological algebra in which the set of quasi-invertible elements is open.

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A topological algebra A is *locally pseudoconvex* if it has a base $\{U_\lambda : \lambda \in \Lambda\}$ of neighborhoods of zero consisting of balanced and pseudoconvex sets, that is, of sets U for which $\mu U \subset U$, whenever $|\mu| \leqslant 1$, and $U + U \subset \rho U$ for a $\rho \geqslant 2$. In particular, when every U_λ in $\{U_\lambda : \lambda \in \Lambda\}$ is idempotent, that is, $U_\lambda U_\lambda \subset U_\lambda$, then A is called a *locally m-pseudoconvex algebra*. It is well known, see [**24**, p. 4], that the locally pseudoconvex (locally m-pseudoconvex) topology on A we can give by a family $\{p_\lambda : \lambda \in \Lambda\}$ of k_λ-homogeneous semi-norms, respectively of k_λ-homogeneous submultiplicative semi-norms, where $k_\lambda \in (0, 1]$ for each $\lambda \in \Lambda$. In particular, when $k_\lambda = 1$ for each $\lambda \in \Lambda$, then A is a *locally convex*, respectively *locally m-convex algebra*, and when the topology of A has been defined by only one k-homogeneous semi-norm with $k \in (0, 1]$, then A is a *locally bounded algebra*. Examples of locally m-pseudoconvex algebras[1] have been given in [**13**, pp. 209–213]; of locally m-convex algebras[2] in several books, see, for example, [**14, 19, 20, 26**] and of locally bounded algebras, in particular Banach algebras, in [**25**] and [**26**].

A topological algebra A is called a *Gelfand-Mazur algebra* (see[3], for example, [**1, 2, 4, 5, 8, 10**]) if A/M is topologically isomorphic with $\mathbf{K}$ for each $M \in m(A)$. In this case every $M \in m(A)$ defines a $\varphi_M \in \hom A$ such that $M = \ker \varphi_M$. Herewith, the set $m(A)$ can be empty both in case of commutative topological algebras, see [**17**, pp. 124–125] and of noncommutative topological algebras, even in case of noncommutative Banach algebras, see [**17**, p. 706]. Since every topological algebra A, for which the set $m(A)$ is empty, is a Gelfand-Mazur algebra, then it is of interest to study only these topological algebras A for which the set $m(A)$ is not empty.

A topological algebra A is an *exponentially galbed algebra*, (see[4], for example, [**1, 2, 4, 5, 8, 23**]), if every neighborhood O of A defines another neighborhood U of zero such that

$$\left\{ \sum_{k=1}^{n} \frac{a_k}{2^k} : a_1, \dots, a_n \in U \right\} \subset O$$

for each $n \in \mathbf{N}$. Besides, A is a *simplicial*[5] topological algebra, see [**6**, p. 15] or *normal* topological algebra in the sense of Michael, see [**20**, p. 68], if every closed regular left (right or two-sided) ideal of A is contained in some closed maximal regular left, respectively right or two-sided, ideal of A and A is a *strongly simplicial topological algebra*, if every closed regular two-sided ideal of A is contained in some ideal $M \in m(A)$. It is known that all locally pseudoconvex algebras, in particular, locally convex and locally bounded algebras, are exponentially galbed algebras and all exponentially galbed algebras A over $\mathbf{C}$ (see, for example, [**5**, Corollary 2] or [**8**, Theorem 2]) are Gelfand-Mazur algebras if all elements in A are *bounded*, see [**12**, p. 400], i.e., for any $a \in A$ there is a number $\lambda \in \mathbf{C} \setminus \{0\}$ such that the set

$$\left\{ \left(\frac{a}{\lambda} \right)^n : n \in \mathbf{N} \right\}$$

is bounded in A. Moreover, all commutative locally m-pseudoconvex, in particular locally m-convex, Hausdorff algebras over $\mathbf{C}$ are simplicial algebras, see [**9**, Corollary 3]; in the complete case, see [**7**, Proposition 2]; [**13**, p. 300] and in the locally m-convex case, see [**14**, p. 321], and $m(A)$ is not empty if A is a commutative unital simplicial Gelfand-Mazur algebra, see [**9**, Corollary 2].

A net $(a_\lambda)_{\lambda\in\Lambda}$ of elements of a topological algebra A is *advertibly convergent* in A, see [**6**, p. 15], if there exists an element $a \in A$ such that $(a \circ a_\lambda)_{\lambda\in\Lambda}$ and $(a_\lambda \circ a)_{\lambda\in\Lambda}$ converge in A to the zero element. In the case when every advertibly convergent Cauchy net of A converges in A, then A is an *advertibly complete* topological algebra. It is known, see [**19**, p. 45] that every complete algebra and every Q-algebra is an advertibly complete topological algebra.[6] Moreover, a topological algebra A is a *topological algebra with functional spectrum* if the spectrum $\mathrm{sp}_A(a)$ of element a coincides with the set $\{\varphi(a) : \varphi \in \hom A\}$ for each $a \in A$. For example, every complex commutative locally m-pseudoconvex Q-algebra with unit, see [**6**, Proposition 11], in particular, every Banach algebra is a topological algebra with functional spectrum. In this case the *spectral radius* $\mathrm{r}_A(a)$ of a is equal to

$$\sup\{|\varphi(a)| : \varphi \in \hom A\}.$$

We will say that two topological algebras (A, τ_A) and (B, τ_B) form a *pair of topological algebras* and denote it by (A, B), if a) B is a dense subalgebra of (A, τ_A); b) the topology τ_B is not weaker than the topology $\tau_A|_B$ induced on B by τ_A.

Properties of pairs (A, B) in case of commutative unital Banach algebras have been considered in [**22**] and in [**21**, Chapter 11] and in case of topological algebras in [**20**, see Appendix B]. The study of properties of pairs of topological algebras, more general than Banach algebras, is continued in the present paper.

2. Pairs of Gelfand-Mazur algebras. Let (A, B) be a pair of topological algebras A and B. To describe the case when one of algebras A or B is a Gelfand-Mazur algebra, we need the following results.

Proposition 1. a) *If A is a Gelfand-Mazur algebra, $M \in m(A)$ and u is a unit of A modulo (meaning that $a - ua \in M$ and $a - au \in M$ for each $a \in A$) M, then every element $a \in A$ is representable in the form $a = \lambda u + m$ for some $\lambda \in \mathbf{K}$ and $m \in M$.*

b) *Let A be a topological algebra, M a closed regular two-sided ideal of A and u a unit of A modulo M. If every $a \in A$ is representable in the form $a = \lambda u + m$ for some $\lambda \in \mathbf{K}$ and $m \in M$, then $M \in m(A)$.*

Proof. a) Let A be a Gelfand-Mazur algebra and $M \in m(A)$. Then there is a $\varphi_M \in \hom A$ such that $M = \ker\varphi_M$ and $\varphi_M(u) = 1$. Since $a - \varphi_M(a)u \in \ker\varphi_M$ for each $a \in A$, then every $a \in A$ is representable in the form $a = \lambda u + m$ for some $\lambda \in \mathbf{K}$ and $m \in M$.

b) Let A be a topological algebra, M a closed regular two-sided ideal of A, π_M the canonical homomorphism from A onto A/M and J a left (right) ideal of A such that $M \subset J$. Then $\pi_M(J) \neq A/M$. Indeed, if $\pi_M(J) = A/M$, then from $\pi_M(u) \in \pi_M(J)$ it follows that $\pi_M(u) = \pi_M(j)$ for some $j \in J$. Therefore, $u - j \in M \subset J$. Hence, $u = (u - j) + j \in J$ but it is not possible. Consequently, $\pi_M(J)$ is a left (respectively, right) ideal of A/M. Since every $x \in A/M$ is representable in the form $x = \pi_M(a)$ for some $a \in A$ and $a = \lambda_a u + m_a$ for some $\lambda_a \in \mathbf{K}$ and $m_a \in M$, by assumption, then $x = \lambda_a \pi_M(u)$, where $\pi_M(u)$ is a unit element of A/M. It means that the map ν_M from A/M onto $\mathbf{K}$, defined by $\nu_M(\pi_M(a)) = \lambda_a$ for each $a \in A$, is an isomorphism. Hence, $\pi_M(J) = \{\theta_{A/M}\}$. (Here, and later on, θ_A denotes the zero element of A.) Taking this into account, from

$$J \subset \pi_M^{-1}(\pi_M(J)) = \pi_M^{-1}(\{\theta_{A/M}\}) = M \subset J$$

it follows that $M = J$. Consequently, $M \in m(A)$. □

Proposition 2. *Let (A, B) be a pair of topological algebras A and B, $M \in m(A)$ and $u \in B$ a unit element of A modulo M. If A is a Gelfand-Mazur algebra, then $M \cap B \in m(B)$ and $\mathrm{cl}_A(M \cap B) \in m(A)$.*

Proof. Let A be a Gelfand-Mazur algebra, $b \in B$, $M \in m(A)$, $\varphi_M \in \hom A$ such that $M = \ker \varphi_M$, and let $\lambda = \varphi_M(b)$. Since $M \cap B \neq B$, then $M \cap B$ is a closed regular two-sided ideal of B, u is a unit of B modulo $M \cap B$ and $b - \lambda u \in M \cap B$. Therefore, every $b \in B$ is representable in the form $b = \lambda u + m$ for some $m \in M \cap B$. Hence, $M \cap B \in m(B)$, by Proposition 1 b).

Let now a be an arbitrary element of A. Since B is dense in A, then there is a net $(b_\alpha)_{\alpha \in \mathcal{A}}$ in B which converges to a. As above, every $b_\alpha \in B$ defines a number $\lambda_\alpha \in \mathbf{K}$ and an element $m_\alpha \in M \cap B$ such that $b_\alpha = \lambda_\alpha u + m_\alpha$. Since $\varphi_M(b_\alpha) = \lambda_\alpha$ for each $\alpha \in \mathcal{A}$ and φ_M is continuous, then the convergence of $(\varphi_M(b_\alpha))_{\alpha \in \mathcal{A}}$ to $\varphi_M(a)$ means that $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$ converges to $\lambda_a = \varphi_M(a)$. Hence, the net $(m_\alpha)_{\alpha \in \mathcal{A}}$ converges to $a - \lambda_a u \in \mathrm{cl}_A(M \cap B)$. Thus $a = \lambda u + m$ for some $\lambda \in \mathbf{K}$ and $m \in \mathrm{cl}_A(M \cap B)$. Since $\mathrm{cl}_A(M \cap B) \subset M \neq A$, then $\mathrm{cl}_A(M \cap B)$ is a closed regular two-sided ideal of A. Therefore, $\mathrm{cl}_A(M \cap B) \in m(A)$, by Proposition 1 b). □

Corollary 1. *Let (A, B) be a pair of topological algebras A and B with the same unit e. Then*

a) $\mathrm{cl}_A(M \cap B) = M$ *for each* $M \in m(A)$ *if* A *is a Gelfand-Mazur algebra.*

b) $\mathrm{cl}_A(M) \cap B = M$ for each $M \in m(B)$ if A and B are Gelfand-Mazur algebras and $\tau_B = \tau_A|_B$.

Proof. If $M \in m(A)$, then $\mathrm{cl}_A(M \cap B) \subset M$. Therefore the statement a) holds by Proposition 2. Let now $M \in m(B)$. If $e \in \mathrm{cl}_A(M)$, then there exists a net $(m_\lambda)_{\lambda \in \Lambda}$ in M which converges in A to e. Since $\tau_B = \tau_A|_B$, then every neighborhood O_B of zero in B defines a neighborhood O_A of zero in A such that $O_B = O_A \cap B$. Now O_A defines a number $\lambda_0 \in \Lambda$ such that $m_\lambda - e \in O_A$ for every $\lambda > \lambda_0$. Since $m_\lambda - e \in B$ for every $\lambda > \lambda_0$, then $(m_\lambda)_{\lambda \in \Lambda}$ converges also in B to e. But this means that $e \in \mathrm{cl}_B(M) = M$, which is not possible. Hence, $\mathrm{cl}_A(M)$ is a closed two-sided ideal in A.

Let now $a \in A$ be an arbitrary element of A. Then there is a net $(b_\alpha)_{\alpha\in\mathcal{A}}$ in B, which converges to a in the topology of A. If B is a Gelfand-Mazur algebra, then $M = \ker\varphi_M$ for some $\varphi_M \in \hom B$ and every b_α is representable in the form $b_\alpha = \lambda_\alpha e + m_\alpha$ for some $\lambda_\alpha \in \mathbf{K}$ and $m_\alpha \in M$, by Proposition 1 a). As $\varphi_M(b_\alpha) = \lambda_\alpha$ for each $\alpha \in \mathcal{A}$ and $\varphi_M(b_\alpha)_{\alpha\in\mathcal{A}}$ converges to $\varphi_M(a)$, then $(m_\alpha)_{\alpha\in\mathcal{A}}$ converges to $a - \varphi_M(a)e \in \mathrm{cl}_A(M)$. Hence, $a = \varphi_M(a)e + m$ for some $m \in \mathrm{cl}_A(M)$. Consequently, $\mathrm{cl}_A(M) \in m(A)$, by Proposition 1 b), and $\mathrm{cl}_A(M) \cap B \in m(B)$, by Proposition 2 because A is a Gelfand-Mazur algebra. Therefore, from $M \subset \mathrm{cl}_A(M) \cap B$ follows that $M = \mathrm{cl}_A(M) \cap B$. □

Proposition 3. *Let A be a topological algebra with jointly continuous multiplication and B a subalgebra of A endowed with the topology $\tau_A|_B$. If $\hom B$ is not empty, then every $\varphi \in \hom B$ defines a $\bar\varphi \in \hom \mathrm{cl}_A(B)$ such that $\bar\varphi(b) = \varphi(b)$ for each $b \in B$.*

Proof. It is known, see [**18**, pp. 129–131] that every $\varphi \in \hom B$ has a uniformly continuous linear extension $\bar\varphi$ of φ to $\mathrm{cl}_A(B)$. Herewith $\bar\varphi$ is nonzero. To show that $\bar\varphi$ is multiplicative, let $a_1, a_2 \in \mathrm{cl}_A(B)$, $\mu_1 = |\bar\varphi(a_1)|$, $\mu_2 = |\bar\varphi(a_2)|$, $\varepsilon > 0$ and $\delta > 0$ be such that

$$\delta^2 + \delta(\mu_1 + \mu_2 + 1) < \varepsilon.$$

Since $\bar\varphi$ is uniformly continuous on $\mathrm{cl}_A(B)$, then there exists in A a neighborhood U of zero such that $|\bar\varphi(a) - \bar\varphi(a')| < \delta$ if $a - a' \in U$. By the continuity of the multiplication in A, there exists in A a balanced neighborhood V of zero and, by density of B in A, elements $b_1, b_2 \in B$ such that $V \subset U$, $Va_2 + a_1V + V^2 \subset U$, $b_1 - a_1 \in V$ and $b_2 - a_2 \in V$. Now by

$$a_1a_2 - b_1b_2 = (a_1 - b_1)a_2 + a_1(a_2 - b_2) - (a_1 - b_1)(a_2 - b_2) \in U \cap \mathrm{cl}_A(B)$$

we have that

$$\begin{aligned}|\bar\varphi(a_1)\bar\varphi(a_2) - \bar\varphi(a_1a_2)| &\leqslant |\bar\varphi(a_1) - \bar\varphi(b_1)|\,|\bar\varphi(a_2)| + |\bar\varphi(a_1)|\,|\bar\varphi(a_2) - \bar\varphi(b_2)| \\ &\quad + |\bar\varphi(a_1) - \bar\varphi(b_1)|\,|\bar\varphi(a_2) - \bar\varphi(b_2)| \\ &\quad + |\bar\varphi(a_1a_2) - \bar\varphi(b_1b_2)| \\ &< \delta\mu_2 + \delta\mu_1 + \delta^2 + \delta < \varepsilon.\end{aligned}$$

Consequently, $\bar{\varphi}(a_1)\bar{\varphi}(a_2) = \bar{\varphi}(a_1a_2)$ for each $a_1, a_2 \in \mathrm{cl}_A(B)$. Thus, the extension $\bar{\varphi} \in \hom \mathrm{cl}_M(B)$. □

Theorem 1. *Let (A, B) be a pair of topological algebras A and B with the same unit e. If the multiplication in A is jointly continuous, B is a Gelfand-Mazur algebra and $\tau_B = \tau_A|_B$, then A is a Gelfand-Mazur algebra if and only if $M \cap B \in m(B)$ for every $M \in m(A)$.*

Proof. Let (A, B) be a pair of topological algebras A and B with the same unit e. If herewith A is a Gelfand-Mazur algebra, then $M \cap B \in m(B)$ for every $M \in m(A)$, by Proposition 2.

Let now A be a topological algebra with jointly continuous multiplication. If the set $m(A)$ is empty, then A is a Gelfand-Mazur algebra. Therefore, we assume that there is an ideal $M \in m(A)$. Let B be a Gelfand-Mazur algebra and $M \cap B \in m(B)$. Then $M \cap B = \ker \varphi_M$ for some $\varphi_M \in \hom B$ and every $b \in B$ is representable in the form $b = \varphi_M(b)e + m$ for some $m \in M \cap B$, by Proposition 1 a). Since the multiplication in A is jointly continuous then, by Proposition 3, there is an extension $\bar{\varphi}_M$ of φ_M such that $\bar{\varphi}_M \in \hom A$ and $\bar{\varphi}_M(b) = \varphi_M(b)$ for each $b \in B$. As B is dense in A, then every $a \in A$ defines a net $(b_\alpha)_{\alpha \in \mathcal{A}}$ in B which converges to a in the topology of A. Now for each $\alpha \in \mathcal{A}$, there is an element $m_\alpha \in M \cap B$ such that $b_\alpha = \bar{\varphi}_M(b_\alpha)e + m_\alpha$. Since the net $(\bar{\varphi}_M(b_\alpha))_{\alpha \in \mathcal{A}}$ converges to $\bar{\varphi}_M(a)$ (because $\bar{\varphi}_M$ is continuous) and $b_\alpha - \bar{\varphi}_M(b_\alpha)e \in B$ for each $\alpha \in \mathcal{A}$, then $(b_\alpha - \bar{\varphi}_M(b_\alpha)e)_{\alpha \in \mathcal{A}}$ converges to $a - \bar{\varphi}_M(a)e \in \mathrm{cl}_A(M \cap B)$. From $M \cap B = \ker \varphi_M$ follows that $\mathrm{cl}_A(M \cap B) \subset \ker \bar{\varphi}_M \neq A$. Therefore, $\mathrm{cl}_A(M \cap B)$ is a closed (regular) ideal of A and every element $a \in A$ is representable in the form $a = \lambda_a e + m_a$, where $\lambda_a = \bar{\varphi}_M(a)$ and $m_a \in \mathrm{cl}_A(M \cap B)$. It means that $\mathrm{cl}_A(M \cap B) \in m(A)$, by Proposition 1 b). Thus,

$$M = \mathrm{cl}_A(M \cap B) = \ker \bar{\varphi}_M$$

Let now π_M be the canonical homomorphism of A onto A/M, $\tau_{A/M}$ the quotient topology on A/M and ν_M the isomorphism from A/M onto $\mathbf{K}$ defined by $\nu_M(\pi_M(a)) = \bar{\varphi}_M(a)$ for each $a \in A$. Then $\pi_M(a) = \bar{\varphi}_M(a)\pi_M(e)$ for each $a \in A$, where $\pi_M(e)$ is the unit element of A/M. Since $(A/M, \tau_{A/M})$ is a topological algebra, then ν_M^{-1} is continuous. To show the continuity of ν_M in the topology $\tau_{A/M}$, let

O be a neighborhood of zero in $\mathbf{K}$. Then there is a number $\varepsilon > 0$ such that $O_\varepsilon = \{\lambda \in \mathbf{K} : |\lambda| < \varepsilon\} \subset O$. If $\lambda_0 \in O_\varepsilon \setminus \{0\}$, then $\lambda_0 \pi_M(e) \neq \theta_{A/M}$. Hence, there is a balanced neighborhood U of zero in $(A/M, \tau_{A/M})$ such that $\lambda_0 \pi_M(e) \notin U$ (because $(A/M, \tau_{A/M})$ is a Hausdorff space). If now $|\bar{\varphi}_M(a)| \geqslant |\lambda_0|$, then $|\lambda_0 \bar{\varphi}_M(a)^{-1}| \leqslant 1$. Therefore $\lambda_0 \pi_M(e) = (\lambda_0 \bar{\varphi}_M(a)^{-1}) \pi_M(a) \in U$ for each $\pi_M(a) \in U$. Since this is not possible, then $\bar{\varphi}_M(a) \in O$ for each $\pi_M(a) \in U$ because of which ν_M is continuous. It means that $(A/M, \tau_{A/M})$ and $\mathbf{K}$ are topologically isomorphic for each $M \in m(A)$. Consequently, A is a Gelfand-Mazur algebra. $\square$

Theorem 2. *Let (A, B) be a pair of topological algebras A and B. If A is a Gelfand-Mazur algebra for which for every $M \in m(B)$ there is $M_A \in m(A)$ such that $\mathrm{cl}_A(M) \subset M_A$, then B is also a Gelfand-Mazur algebra.*

Proof. If B is a topological algebra for which the set $m(B)$ is empty, then B is a Gelfand-Mazur algebra. Therefore, we assume that $M \in m(B)$. Then there is $M_A \in m(A)$ such that $\mathrm{cl}_A(M) \subset M_A$. Since A is a Gelfand-Mazur algebra, then $M_A = \ker \psi$ for some $\psi \in \hom A$. Now $\varphi = \psi|_B \in \hom B$ because B is dense in A and $M \subset \ker \varphi$. Thus $M = \ker \varphi$ and B/M and $\mathbf{K}$ are topologically isomorphic (see the proof of Theorem 1). It means that B is a Gelfand-Mazur algebra. $\square$

Corollary 2. *Let (A, B) be a pair of topological algebras A and B. If A is a strongly simplicial, in particular, commutative and simplicial, Gelfand-Mazur algebra, then B is also a Gelfand-Mazur algebra in the topology $\tau_A|_B$.*

Proof. Let $M \in m(B)$ and u be a unit of B modulo M. Then $\mathrm{cl}_A(M) \neq A$. Indeed, if $\mathrm{cl}_A(M) = A$, then there exists a net $(m_\lambda)_{\lambda \in \Lambda}$ in M which converges to u in the topology of A. Let O_B be a neighborhood of u in B. Then there is a neighborhood O_A of u in A such that $O_B = O_A \cap B$. Now O_A defines an index $\lambda_0 \in \Lambda$ such that $m_\lambda - u \in O_A$ whenever $\lambda > \lambda_0$. Since $m_\lambda - u \in B$ for each $\lambda \in \Lambda$, then $m_\lambda - u \in O_B$ whenever $\lambda > \lambda_0$. It means that $(m_\lambda)_{\lambda \in \Lambda}$ converges to u in B. Since M is closed in B, then $u \in M$ but it is not possible.

Hence, $I = \mathrm{cl}_A(M)$ is a closed regular two-sided ideal in A. Since A is strongly simplicial, then there is an ideal $M_A \in m(A)$ such that $I \subset M_A$. Consequently, B is a Gelfand-Mazur algebra by Theorem 2. □

3. Pairs of exponentially galbed algebras. To show that A in the pair (A, B) of topological algebras A and B is exponentially galbed if and only if B is exponentially galbed we use the following

Lemma 1. *Let (A, B) be a pair of topological algebras A and B. If $\tau_B = \tau_A|_B$, then $\mathrm{cl}_A(O_B)$ is a neighborhood of zero in A for each open neighborhood O_B of zero in B.*

Proof. Let O_B be an open neighborhood of zero in B and O_A an open neighborhood of zero in A such that $O_B = O_A \cap B$. If $a \in O_A$ and $O(a)$ is an arbitrary neighborhood of a in A, then $O_A \cap O(a)$ is also a neighborhood of a in A. Since B is dense in A, then $(O_A \cap O(a)) \cap B$ is not empty. Hence, $O(a) \cap O_B = O(a) \cap (O_A \cap B)$ is also not empty. Therefore, $a \in \mathrm{cl}_A(O_B)$ for each $a \in O_A$. It means that $\mathrm{cl}_A(O_B)$ is a neighborhood of zero in A. □

Theorem 3. *Let (A, B) be a pair of topological algebras A and B. If $\tau_B = \tau_A|_B$, then A is an exponentially galbed algebra if and only if B is an exponentially galbed algebra.*

Proof. Let A be an exponentially galbed algebra, B a topological algebra, O_B a neighborhood of zero in B, U_B a closed neighborhood of zero in B such that $U_B \subset O_B$ and V_B an open neighborhood of zero in B such that $V_B \subset U_B$. Then $\mathrm{cl}_A(V_B)$ is a neighborhood of zero in A, by Lemma 1, and there is a neighborhood W_A in A such that

$$\left\{ \sum_{k=1}^{n} \frac{a_k}{2^k} : a_1, \dots, a_n \in W_A \right\} \subset \mathrm{cl}_A(V_B)$$

for each $n \in \mathbf{N}$. Let $W_B = W_A \cap B$, $n \in \mathbf{N}$ and $b_1, \dots, b_n \in W_B$. Then

$$\sum_{k=1}^{n} \frac{b_k}{2^k} \in \mathrm{cl}_A(V_B) \cap B = \mathrm{cl}_B(V_B) \subset U_B \subset O_B$$

for each $n \in \mathbf{N}$, because $\tau_B = \tau_A|_B$. Consequently, B is also an exponentially galbed algebra.

Let now A be a topological algebra, B an exponentially galbed algebra and O_A a neighborhood of zero in A. Then there is a closed neighborhood U_A of zero in A such that $U_A \subset O_A$, $U_B = U_A \cap B$ is a closed neighborhood of zero in B and there is an open neighborhood V_B of zero in B such that

$$\left\{ \sum_{k=1}^{n} \frac{b_k}{2^k} : b_1, \dots, b_n \in V_B \right\} \subset U_B$$

for each $n \in \mathbf{N}$. Since $\tau_B = \tau_A|_B$, then $\mathrm{cl}_A(V_B)$ is a neighborhood of zero in A, by Lemma 1.

Let now $n \in \mathbf{N}$ and $a_1, \dots, a_n \in \mathrm{cl}_A(V_B)$. Then for each $k \in \{1, \dots, n\}$ there is a net $(b(k)_\alpha)_{\alpha \in \mathcal{A}}$ in V_B which converges to a_k in the topology of A. Hence

$$\sum_{k=1}^{n} \frac{a_k}{2^k} = \lim_{\alpha} \sum_{k=1}^{n} \frac{b(k)_\alpha}{2^k} \in U_A \subset O_A.$$

It means that A is also an exponentially galbed algebra. $\square$

4. Pairs of topological algebras A and B for which $\hom A$ and $\hom B$ are homeomorphic. The next result describes such pairs (A, B) of topological algebras A and B for which $\hom A$ and $\hom B$ are homeomorphic.

Theorem 4. *Let (A, B) be a pair of such topological algebras A and B for which the multiplication in A is jointly continuous, $\tau_B = \tau_A|_B$ and $\hom B$ is not empty. Then there is a bijection Λ from $\hom B$ onto $\hom A$ such that Λ^{-1} is continuous. If, in addition, $\hom A$ is equicontinuous or $\hom B$ is locally equicontinuous, then $\hom A$ and $\hom B$ are homeomorphic.*

Proof. Let (A, B) be a pair of topological algebras A and B. If A and B are such as described in the formulation of Theorem 4, then every $\varphi \in \hom B$ defines a $\bar{\varphi} \in \hom A$ such that $\bar{\varphi}(b) = \varphi(b)$ for each

$b \in B$, by Proposition 3, because B is dense in A. Let Λ be a map from $\hom B$ into $\hom A$ defined by $\Lambda(\varphi) = \bar{\varphi}$ for each $\varphi \in \hom B$. Then Λ is a bijection by density of B in A.

To show that Λ^{-1} is continuous, let $O(\varphi_0)$ be a neighborhood of φ_0 in $\hom B$. Then there exist $n \in \mathbf{N}$, $\varepsilon > 0$ and $b_1, \dots, b_n \in B$ such that $U = O(\varphi_0; b_1, \dots, b_n, \varepsilon) \subset O(\varphi_0)$. Since $V = O(\bar{\varphi}_0; b_1, \dots, b_n, \varepsilon)$ is a neighborhood of $\bar{\varphi}_0$ in $\hom A$ and $\Lambda(U) = V$, then Λ^{-1} is continuous.

To show the continuity of Λ, let $\psi_0 \in \hom A$ and $O(\psi_0)$ be a neighborhood of ψ_0 in $\hom A$. Then there exist $n \in \mathbf{N}$, $\varepsilon > 0$ and $a_1, \dots, a_n \in A$ such that $U = O(\psi_0; a_1, \dots, a_n, \varepsilon) \subset O(\psi_0)$. If $\hom A$ is equicontinuous, then there is a neighborhood O of zero in A such that $|\psi(a)| < \varepsilon/4$ for each $a \in O$ and $\psi \in \hom A$. For each $k \in \{1, \dots, n\}$, let $b_k \in B$ be such that $b_k - a_k \in O$ (because B is dense in A). Then $V = O(\psi_0; b_1, \dots, b_k, \varepsilon/4)$ is a neighborhood of ψ_0 in $\hom A$. Since

$$|(\psi - \psi_0)(a_k)| \leqslant |\psi(b_k - a_k)| + |(\psi - \psi_0)(b_k)| + |\psi_0(b_k - a_k)| < \frac{3\varepsilon}{4} < \varepsilon$$

for each $\psi \in V$, then $V \subset U \subset O(\psi_0)$. If $\varphi_0 = \psi_0|_B$ and $W = O(\varphi_0; b_1, \dots, b_n, \varepsilon/4)$, then $\varphi_0 \in \hom B$ (because B is dense in A), W is a neighborhood of φ_0 in $\hom B$ and $\Lambda(W) \subset V \subset O(\psi_0)$. Hence, Λ is continuous.

Let now $\varphi_0 \in \hom B$, $\bar{\varphi}_0 \in \hom A$ be the extension of φ_0, defined by Proposition 3, and $O(\bar{\varphi}_0)$ a neighborhood of $\bar{\varphi}_0$ in $\hom A$. Then there exist $n \in \mathbf{N}$, $\varepsilon > 0$ and $a_1, \dots, a_n \in A$ such that $U = O(\bar{\varphi}_0; a_1, \dots, a_n, \varepsilon) \subset O(\bar{\varphi}_0)$. If $\hom B$ is locally equicontinuous, then φ_0 has an equicontinuous neighborhood $O(\varphi_0)$. Therefore, there is an open neighborhood of zero O_B in B such that $|\varphi(b)| < \varepsilon/3$ for each $b \in O_B$ and $\varphi \in O(\varphi_0)$. Since $\bar{\varphi}$ is continuous for every $\varphi \in O(\varphi_0)$, then $\bar{\varphi}(\mathrm{cl}_A(O_B)) \subset \mathrm{cl}_{\mathbf{K}}(\bar{\varphi}(O_B)) = \mathrm{cl}_{\mathbf{K}}(\varphi(O_B))$. It means that $|\bar{\varphi}(a)| \leqslant \varepsilon/3$ for each $a \in \mathrm{cl}_A(O_B)$ and $\varphi \in O(\varphi_0)$. Herewith, $\mathrm{cl}_A(O_B)$ is a neighborhood of zero in A, by Lemma 1, because $\tau_B = \tau_A|_B$. Now for each $k \in \{1, \dots, n\}$ there is an element $b_k \in (a_k + \mathrm{cl}_A(O_B)) \cap B$. Hence, $|\bar{\varphi}(b_k - a_k)| \leqslant \varepsilon/3$ for each $k \in \{1, \dots, n\}$ and $\varphi \in O(\varphi_0)$. Taking this into account,

$$|(\bar{\varphi} - \bar{\varphi}_0)(a_k)| \leqslant |\bar{\varphi}(a_k - b_k)| + |(\bar{\varphi} - \bar{\varphi}_0)(b_k)| + |\bar{\varphi}_0(a_k - b_k)| < \varepsilon$$

for each $\varphi \in V = O(\varphi_0) \cap O(\varphi_0; b_1, \dots, b_k, \varepsilon/3)$. As V is a neighborhood of φ_0 in $\hom B$ and $\Lambda(V) \subset U \subset O(\bar{\varphi}_0)$, then Λ is continuous. Consequently, $\hom B$ and $\hom A$ are homeomorphic. $\square$

Corollary 3. *Let (A, B) be a pair of such topological algebras A and B that the multiplication in A is jointly continuous, $\tau_B = \tau_A|_B$ and* $\hom A$ *and* $\hom B$ *are not empty. Then* $\hom B$ *is equicontinuous if and only if* $\hom A$ *is equicontinuous.*

Proof. Let (A, B) be a pair of topological algebras A and B described in the formulation of Corollary 3. Let $\varepsilon > 0$ and $\delta \in (0, \varepsilon)$. If $\hom B$ is equicontinuous, then there is an open neighborhood O_B of zero in B such that $|\varphi(b)| < \delta$ for each $b \in O_B$ and $\varphi \in \hom B$. Since $O_A = \mathrm{cl}_A(O_B)$ is a neighborhood of zero in A, by Lemma 1, and $\bar{\varphi}(O_A) \subset \mathrm{cl}_{\mathbf{K}}\varphi(O_B)$ for each $\varphi \in \hom A$, then $|\varphi(a)| \leqslant \delta < \varepsilon$ for each $a \in O_A$ and $\varphi \in \hom A$. Hence, $\hom A$ is equicontinuous. On the other hand, if $\hom A$ is equicontinuous, then the sets $\hom B$ and $\hom A$ are homeomorphic, by Theorem 4. Therefore, $\hom B$ is also equicontinuous. □

5. Properties of the completion of a topological algebra. Let A be a topological Hausdorff algebra. Then A has the completion $\tilde{A}$ which is a linear topological Hausdorff space, see [**18**, p. 131], but not necessarily an algebra, see [**16**, p. 311] or [**11**, the example in Remark 3.3]. In particular, when the multiplication in A is jointly continuous, then $\tilde{A}$ is a topological algebra with jointly continuous multiplication, see [**19**, p. 22] or [**13**, Theorem 2.3.14], and there is a topological isomorphism ν from A into $\tilde{A}$ such that $\nu(A)$ is dense in $\tilde{A}$ and $\tau_{\nu(A)} = \tau_{\tilde{A}}|_{\nu(A)}$. Hence, $(\tilde{A}, \nu(A))$ is a pair of topological Hausdorff algebras. Next we apply results proved above to the pair $(\tilde{A}, \nu(A))$. By Theorems 1 and 3 and Corollary 2, we have

Theorem 5. a) *Let A be a unital Gelfand-Mazur algebra with jointly continuous multiplication. Then the completion $\tilde{A}$ of A is also a Gelfand-Mazur algebra if and only if $M \cap \nu(A) \in m(\nu(A))$ for each $M \in m(\tilde{A})$.*

b) *A topological algebra with jointly continuous multiplication is a Gelfand-Mazur algebra if the completion $\tilde{A}$ of A is a strongly simplicial* (*in particular, a commutative simplicial*) *Gelfand-Mazur algebra.*

c) *A topological algebra A is an exponentially galbed algebra if and only if the completion $\tilde{A}$ of A is an exponentially galbed algebra.*

Theorem 6. *Let A be a topological algebra with jointly continuous multiplication. If the set* $\hom A$ *is not empty, then*

a) *the sets* $\hom A$ *and* $\hom \tilde{A}$ *are homeomorphic if either* $\hom \tilde{A}$ *is equicontinuous or* $\hom A$ *is locally equicontinuous.*

b) *the set* $\hom A$ *is equicontinuous if and only if the set* $\hom \tilde{A}$ *is equicontinuous.*

Corollary 4. *Let A be a topological algebra with jointly continuous multiplication. If* $\hom A$ *is not empty and $\tilde{A}$ is a Q-algebra, then* $\hom A$ *and* $\hom \tilde{A}$ *are homeomorphic.* (*Since $\tilde{A}$ is a Q-algebra, then* $\hom \tilde{A}$ *is equicontinuous.*)

Theorem 7. *Let A be an advertibly complete topological Hausdorff algebra over* $\mathbf{C}$ *and the completion $\tilde{A}$ of A a topological algebra with functional spectrum. Then A is a Q-algebra if and only if $\tilde{A}$ is a Q-algebra.*

Proof. Let A be an advertibly complete topological Hausdorff algebra over $\mathbf{C}$ the completion $\tilde{A}$ of which is a topological algebra with functional spectrum. Then A is also a topological algebra with functional spectrum, see [**6**, Corollary 7]. If $\tilde{A}$ is a Q-algebra, then the set $\hom \tilde{A}$ is equicontinuous. Hence, the set $\hom A$ is equicontinuous too, by Corollary 3. It means that there is a neighborhood O of zero in A such that $|\varphi(a)| < 1$ for each $a \in O$ and $\varphi \in \hom A$. Thus, $\mathrm{r}_A(a) \leqslant 1$ for each $a \in O$ because of which $\{a \in A : \mathrm{r}_A(a) \leqslant 1\}$ is a neighborhood of zero in A. Consequently, (see [**19**, Lemma II.4.2] or [**26**, Proposition 12.19]) A is a Q-algebra.

Let now A be a Q-algebra. Then $\hom A$ is equicontinuous. Therefore, the set $\hom \tilde{A}$ is equicontinuous too, by Corollary 3, and similarly as in the above we have that $\tilde{A}$ is a Q-algebra (because $\tilde{A}$ is a topological algebra with functional spectrum). $\square$

Corollary 5. *Let A be a commutative advertibly complete locally m-pseudoconvex Hausdorff algebra over* $\mathbf{C}$. *Then A is a Q-algebra if and only if $\tilde{A}$ is a Q-algebra.*

Proof. By the assumption of Corollary 5, $\tilde{A}$ is a commutative advertibly complete (because $\tilde{A}$ is complete) locally m-pseudoconvex Hausdorff algebra over $\mathbf{C}$. Therefore, $\tilde{A}$ has functional spectrum, see the proof of Proposition 11 in [**6**]. Hence, Corollary 5 is true, by Theorem 7. □

Remark. Corollary 1 has been proved in [**21**, Chapter III, part 11] in case of commutative Banach algebras with unit, a part of Theorem 6 in [**19**, Theorem 2.1, p. 150, Lemma 2.2, p. 146] and Corollaries 4 and 5 in [**19**, pp. 150–151] in the case of commutative locally m-convex algebras.

ENDNOTES

1. One of the simplest examples of locally m-pseudoconvex algebra is $C(\mathbf{K};(k_n))$ with $0 < k_n \leqslant 1$ of all $\mathbf{K}$-valued continuous functions f on $\mathbf{K}$ with respect to the point-wise algebraic operations and the topology defined by the system $\{p_n : n \in \mathbf{N}\}$ of k_n-homogeneous semi-norms, where

$$p_n(f) = \sup_{|x|\leqslant n} |f(t)|^{k_n} \quad \text{for each } f \in C(\mathbf{K};(k_n)).$$

2. One of the simplest examples of locally m-convex algebra is $C(X,\mathbf{K})$ of all $\mathbf{K}$-valued continuous functions on a topological space X with respect to point-wise algebraic operations and the uniform topology on compact subsets of X.

3. The class of Gelfand-Mazur algebras is very large. In addition to Banach algebras, it contains all locally m-pseudoconvex, in particular, locally m-convex and locally bounded, algebras, all locally pseudoconvex Fréchet, in particular p-Banach, algebras and many other topological algebras, see [**5, 8**]. Moreover, there exist topological algebras (see, for example, [**26**, p. 86]) which are not Gelfand-Mazur algebras.

4. It is known, see [**3**, Proposition 5] that the algebra $l^{(\rho_n)}$, with coordinate-wise algebraic operations, of all sequences (x_n) of complex numbers such that $\sum |x_n|^{\rho_n} < \infty$, is not exponentially galbed if $0 < \rho_n \leqslant 1$ and (ρ_n) converges to zero.

5. For example, $C(\mathbf{K};(k_n))$ and $C(X,\mathbf{K})$ are simplicial topological algebras.

6. It is known (see, for example, [**15**, Example 3]) that the algebra of all measurable functions f on $[0,1]$, endowed with the topology defined by the system $\{p_k : k_0 < k < 1,\ k_0 \in (0,1]\}$ of k-norms, where

$$p_k(f) = \int_{[0,1]} |f(t)|^k \, dt$$

for each measurable f on $[0,1]$, is a sequentially advertibly complete algebra, which is neither a Q-algebra, a complete algebra nor a locally m-pseudoconvex algebra.

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2 Liivi Str. 615, Institute of Pure Mathematics, University of Tartu, 50409 Tartu, Estonia
E-mail address: `mart.abel@ut.ee`

2 Liivi Str. 614, Institute of Pure Mathematics, University of Tartu, 50409 Tartu, Estonia
E-mail address: `mati.abel@ut.ee`