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TOPOLOGICAL TRANSITIVITY AND MIXING NOTIONS FOR GROUP ACTIONS

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ABSTRACT. This paper surveys six notions of dynamical transitivity and mixing, in the context of group actions on topological spaces. We discuss the relations between these notions, and the manner in which they are inherited by subgroups, by taking products, and when passing to the induced action on hyperspace, i.e., the space of compact subsets. The focus of the paper is on the fact that certain standard notions, which are equivalent in the classical theory of the dynamics of flows and the iteration of single maps, are distinct for general group actions. The paper examines how the notions coalesce (a) for actions of abelian groups and (b) for chaotic actions.

0. Introduction. Consider an action of an infinite group G on a Hausdorff topological space M. This paper surveys six notions of dynamical transitivity and mixing for the action of G. We don't assume any particular topology on G, but we assume that the action is "continuous" in the sense that, for each group element g, the corresponding map $g : M \to M$ is a homeomorphism.

The fundamental transitivity and mixing notions are:

Definition 1. The action of G on M is:

(a) *topologically transitive* if, for every pair of nonempty open subsets U and V of M, there is an element $g \in G$ such that $gU \cap V \neq \varnothing$.

(b) *strongly topologically mixing* if for any pair of nonempty open subsets U and V of M, the set $\{g \in G;\ gU \cap V = \varnothing\}$ is finite.

(c) *topologically k-transitive for $k \in \mathbf{N}$*, if the induced action of G on the k-fold Cartesian product M^k is topologically transitive. Topological 2-transitivity is also called *weak topological mixing*.

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We emphasize that we are not assuming any topology on G. For actions of topological groups, it is more natural, and in keeping with tradition, to define *strong topologically mixing* by the condition: for any pair of nonempty open subsets U, V of M, the set $\{g \in G;\ gU \cap V = \varnothing\}$ is compact.

For brevity, we will drop the adjective *topological* wherever there is no risk of confusion. In particular, transitivity will mean topological transitivity, and not the group theoretic sense of point-transitivity.

Apart from the above fundamental notions, there are also two related notions that we will consider in this paper:

Definition 2. The action of G on M is:

(a) *totally transitive* if every subgroup of finite index is transitive on M.

(b) *elastic* if for every $n \in \mathbf{N}$ and any finite collection of nonempty open sets $U, V_1, \dots, V_n$, there exists $g \in G$ such that $gU \cap V_i \neq \varnothing$, for all $i \in \{1, \dots, n\}$.

The above notions of transitive, and strongly and weakly mixing actions are classical, see [**19, 20, 24**]. The notions of totally transitive and elastic actions are less common; they are both generalizations of corresponding notions of the dynamics of flows and the iterates of a single map. The term "totally transitive" is reasonably well established. A continuous map $f : M \to M$ is said to be totally transitive if, for all natural numbers k, the kth iterate f^k is transitive, see [**3, 8, 15, 34**]. When f is a homeomorphism, this is the same as demanding that the finite index subgroups of the group $\langle f \rangle = \{f^k;\ k \in \mathbf{Z}\}$ are transitive. The notion of an elastic action has appeared occasionally in the literature, but it doesn't usually have a separate name since it coincides with weak mixing for flows and single maps, see Theorem 2 below. For single maps, elastic was termed "strongly transitive" by Banks in [**7**]. However the terminology "strongly transitive" is already employed to mean two distinct things in topological dynamics, see [**10, 23, 30**]. For this reason we have felt it necessary to introduce the term "elastic".

For flows and single maps, the three notions k-transitive for $k \in \mathbf{N}$, weak mixing and elastic are all equivalent, while we will see that, for general group actions, the three conditions are distinct. The aim of this paper is to give an essentially self-contained discussion of the relations between the six conditions: strongly mixing, k-transitive for all k, weakly mixing, elastic, totally transitive and transitive. We also examine their inheritance properties, and the special cases of actions of abelian groups and actions which are chaotic in the sense that they are transitive and the points with finite orbit form a dense set. See Theorems 1, 2 and 3. The general conclusion is that, for abelian groups, the relations between the six conditions have the same equivalences as they do for flows and single maps, while the assumption that an action (of a not necessarily abelian group) is chaotic has an even greater combining effect on the conditions.

The sections of this paper are: Section 1, Brief review of transitivity, Section 2, Logical implications between the notions, Section 3, Examples, Section 4, Inheritance of notions (a) under semi-conjugacy, (b) from and by subgroups, (c) when taking products, (d) when passing to hyperspace, Section 5, Actions of abelian groups and Section 6, Chaotic actions.

Important note. Throughout this paper, we consider a continuous action of an infinite group G on a Hausdorff space M. We denote the image of $x \in M$ under $g \in G$ simply by gx. We denote the orbit $\{gx;\ g \in G\}$ of $x \in M$ by Gx. We denote by id the identity element of G. For brevity, instead of saying that the action of G on M is transitive, elastic, etc., we will simply say that G is transitive, elastic, etc.

1. Brief review of transitivity. For general properties of transitivity in the case of a single map, see [**2, 5, 25**]. The following elementary lemma is quite useful, see [**20**, Remark 9.10] for further equivalent conditions.

Lemma 1. *For the action of G on M, the following conditions are equivalent*:

(a) *G is transitive*,

(b) *every nonempty G-invariant open subset of M is dense*,

(c) *every G-invariant subset $U \subseteq M$ is either dense or nowhere dense*,

(d) *M does not possess two disjoint G-invariant nonempty open subsets.*

Notice that if there exists a point $x \in M$ such that the orbit Gx of x is dense in M, then G is transitive. For the converse, one needs some additional hypotheses on M. Recall that a subset $A \subseteq M$ is said to be *meager* (or of *first category*) if A is the union of countably many nowhere dense subsets of M. (For more information on meager sets, see the exercises in [**9**, Chapter IX.5].) A G_δ set is a subset of M that can be written as the intersection of countably many open sets. A subset $A \subseteq M$ is said to be *residual*, or *generic*, if its complement is meager. A topological space M is a *Baire space* (or a space of *second category*) if every countable intersection of open dense subsets of M is dense in M. Every locally compact Hausdorff space is a Baire space, as is every complete metric space; this latter fact is Baire's category theorem. A good brief account of Baire spaces is given in [**9**, Chapter IX.5.3]. A key fact for us is that Baire spaces are not themselves meager, and more generally:

Lemma 2. *In Baire spaces, meager sets have empty interiors.*

The next result is well known, see [**20**, Theorem 9.22], and is often presented for continuous maps of spaces, see for example, [**29**, Prop. I.11.4]:

Proposition 1. *If G is transitive and M is a second countable Baire space M, then there exists a point $x \in M$ with dense orbit in M and, in fact, the set of points with dense orbit is a residual set.*

Proof. If M is second countable, then there exists a countable open base $\{U_i;\ i \in \mathbf{Z}\}$. Let $V_i = G(U_i)$ for each i, and set $V = \cap V_i$. The sets V_i are dense, by Lemma 1, and so V is residual and hence dense, as M is a Baire space. We claim that each element of V has a dense orbit. Indeed, if $W \subseteq M$ is open, one has $U_i \subseteq W$ for some i. Thus, if $x \in V$, we have $x \in V_i$ and it follows that $gx \in U_i$ for some $g \in G$. $\square$

Recall that a topological space M is *dense in itself* if it contains no isolated points. The following lemma was shown for compact metric spaces in [**24**]; their short argument also applies to any second countable Baire space M which is dense in itself. We present the result in slightly more generality:

Lemma 3. *If G is transitive and M is dense in itself, then for every pair of nonempty open subsets U, V of M, the set $\{g \in G;\ gU \cap V \neq \varnothing\}$ is infinite.*

Proof. Let U, V be as in the statement of the lemma, and let $k \in \mathbf{N}$. We will show that the set $\{g \in G;\ gU \cap V \neq \varnothing\}$ has at least k elements. First note that there exist k pair-wise disjoint nonempty subsets $V_1, \dots, V_k \subseteq V$. Indeed, as M is Hausdorff and dense in itself, there exist nonempty disjoint open subsets V_1 and W_1 of V. Similarly, there exist nonempty disjoint open subsets V_2 and W_2 of W_1, and nonempty disjoint open subsets V_3 and W_3 of W_2, and so on.

Let $U_1 = U$. As the action is transitive, there exists $g_1 \in G$ such that $g_1U_1 \cap V_1 \neq \varnothing$. Let $U_2 = U_1 \cap g_1^{-1}V_1$. Then there exists $g_2 \in G$ such that $g_2U_2 \cap V_2 \neq \varnothing$. As $V_1 \cap V_2 = \varnothing$, we have $g_2 \neq g_1$. Let $U_3 = U_2 \cap g_2^{-1}V_2$. There exists $g_3 \in G$ such that $g_3U_3 \cap V_3 \neq \varnothing$. As $V_1 \cap V_3 = \varnothing$ and $V_2 \cap V_3 = \varnothing$, we have $g_3 \neq g_1$ and $g_3 \neq g_2$. Continuing is this way, we obtain k distinct elements $g_1, \dots, g_k \in \{g \in G;\ gU \cap V \neq \varnothing\}$. □

An action of G on M is said to be *non-wandering* if the set $\{g \in G;\ g \neq \mathrm{id}\,,\ gU \cap U \neq \varnothing\}$ is not empty for all nonempty open sets U. A transitive flow on an infinite space M is non-wandering if and only if M is dense in itself [**16**, Proposition II.4.10]. Similarly, by Lemma 3, every transitive action of a group on a dense in itself Hausdorff space is non-wandering. However, there are transitive non-wandering group actions on spaces which are not dense in themselves. For example, consider the usual action of the infinite dihedral group D_∞ on $\mathbf{Z}$, equipped with the discrete topology; D_∞ is generated by a translation $x \mapsto x+1$ and a reflection $x \mapsto -x$. Obviously, this action is non-wandering, but every point of $\mathbf{Z}$ is an isolated point. Provided the Hausdorff space M has at least two elements, it is easy to see that M is dense in itself if M admits an elastic action of some group. The same is true if M admits a weakly mixing action, see [**24**]. However, it is easy to construct examples of totally transitive actions on spaces which are not dense in themselves; see the third action in Example 2 below, and consider a group with discrete topology.

Lemma 3 has a useful extension which we will use later:

Lemma 4. *If G is elastic on M, then for all nonempty open subsets $U, V_1, \dots, V_k$ of M, the set $\{g \in G;\ gU \cap V_i \neq \varnothing$ for all $i = 1, \dots, k\}$ is infinite.*

Proof. As we have just remarked, if G is elastic, then M is dense in itself. Consider nonempty open subsets $U, V_1, \dots, V_k$. Since G is transitive, the set $S_{U,U} = \{g \in G;\ gU \cap U \neq \varnothing\}$ is infinite, by Lemma 3. Let

$$S = \{g \in G;\ gU \cap V_i \neq \varnothing \text{ for all } i = 1, \dots, k\}$$

and assume that S is finite: $S = \{h_1, \dots, h_n\}$, say. Let $g \in S_{U,U}$ and $\widetilde{U} = gU \cap U$. As G is elastic, there exists $h \in G$ such that $h\widetilde{U} \cap V_i \neq \varnothing$ for all i. One has $hgU \cap hU \cap V_i \neq \varnothing$ and so $hg, h \in S$. Then $hg = h_i$ and $h = h_j$ for some $i, j \in \{1, \dots, k\}$ and $g = h_j^{-1} h_i$. Thus, the set $S_{U,U}$ contains at most n^2 elements, which gives the required contradiction. □

Now recall:

Definition 3. A subset $A \subseteq M$ of a topological space M is said to have the *Baire property* if there are meager sets B, C and an open set U such that $A = B \cup (U \backslash C)$.

It is easy to see that the class of sets having the Baire property is a σ-algebra. We state the following simple lemma without proof:

Lemma 5. *Suppose that M is a Baire space and that $A \subseteq M$ has the Baire property, with $A = B \cup (U \backslash C)$ where B, C are meager and U is open. Then A is meager if and only if $U = \varnothing$.*

For more information on sets with the Baire property, see [**32**], [**33**, Chapter 4], [**9**, Chapter XII.8] and Exercise 6 of [**9**, Chapter IX.5].

Note that for sets A with the Baire property, the presentation $A = B \cup (U \backslash C)$ is not unique. Obviously one can suppose in general that $B \cap U = \varnothing$ and that $C \subseteq U$, but this still doesn't eliminate the lack of uniqueness. In particular, there exist sets A with the Baire property for which one can write $A = B \cup (U \backslash C)$ and $A = B' \cup (U' \backslash C')$, where B, B', C, C' are meager and U, U' are open, and $U' \neq U$. For example, the interval $A = (0, 1)$ can be written as $\varnothing \cup A$, and as $\{1/2\} \cup U$, where $U = (0, (1/2)) \cup ((1/2), 1)$. Nevertheless, there is a "canonical" way to write a set with the Baire property, as the following lemma shows:

Lemma 6. *Suppose that M is a topological space and that $A \subseteq M$ has the Baire property. Then there are unique sets B, C, U such that the following conditions hold*:

(a) *B, C are meager, U is open and $A = B \cup (U \backslash C)$,*

(b) *$B \cap U = \varnothing$ and $C \subseteq U$,*

(c) *if B', C', U' verify the analogous conditions to parts* (a) *and* (b)*, then $U' \subseteq U$, $B \subseteq B'$ and $C \subseteq C'$.*

Proof. Consider all the possible ways of writing $A = B' \cup (U' \backslash C')$, where B', C', U' verify the analogous conditions to part (a) and (b), and let U be the union of the U', and B, C be the intersection of B', C' respectively. It is easy to verify that B, C satisfy the required properties. □

We will require the following:

Lemma 7. *Suppose that G acts on M and that $A \subseteq M$ is a G-invariant set with the Baire property. Write $A = B \cup (U \backslash C)$, where B, C, U have the properties of Lemma* 6*. Then the open set U is G-invariant.*

Proof. Let $g \in G$. As A is G-invariant, we have $A = gA = gB \cup (gU \backslash gC)$. Thus, by Lemma 6, $gU \subseteq U$. For the same reason, $g^{-1}U \subseteq U$, and thus $gU = U$. □

Definition 4. We say that an action of G on M is *topologically ergodic* if every G-invariant set with the Baire property is either meager or residual.

The following result is probably well known, but we could not find its proof in the literature. It was stated without proof by Oxtoby in [**33**].

Proposition 2. *A continuous action on a Baire space is transitive if and only if it is topologically ergodic.*

Proof. Suppose that G acts continuously on a Baire space M. Suppose first that the action is not transitive. By Lemma 1 (d), M possesses two nonempty disjoint G-invariant open subsets, U_1, U_2. As U_1, U_2 are open, they have the Baire property. Moreover, by Lemma 5, neither of them is meager and consequently, as each is contained in the complement of the other, neither of them is residual. Hence, the action is not topologically ergodic.

Conversely, if the action is not topologically ergodic, then M possesses two disjoint G-invariant subsets, A_1, A_2, each having the Baire property, such that neither is meager. As the A_i have the Baire property, we can write $A_i = B_i \cup (U_i \backslash C_i)$, where B_i, C_i, U_i have the properties of Lemma 6. Since the sets A_i are not meager, the open sets U_i are not empty, by Lemma 5. As A_1, A_2 are disjoint, one has $U_1 \cap U_2 \subseteq C_1 \cup C_2$. In particular, the open set $U_1 \cap U_2$ is meager and hence empty by Lemma 2. Thus, the U_i are disjoint and nonempty and, moreover, they are G-invariant by Lemma 7. Hence, the action is not transitive. □

Transitivity is a topological version of the measure theoretic notion of ergodicity. We finish this brief review by describing this connection. Recall that an action of a group G on a measure space M by measure preserving transformations is *ergodic* if M does not contain two disjoint G-invariant measurable sets each of positive measure. The following classical result provides a useful source of transitive actions, cf. [**29**, Proposition II.2.6]:

Proposition 3. *On a measure space in which nonempty open sets have positive measure, every ergodic action is transitive.*

Proof. Suppose that a group G acts on a measure space M by measure preserving transformations. If the action is not transitive, then by part Lemma 1 (d), M possesses two nonempty disjoint G-invariant open subsets, U_1, U_2. As the U_i are open and nonempty, they have positive measure. So the action is not ergodic. □

2. Logical implications between the notions. Most of the implications in the following theorem are generalizations of known results for flows or single maps. The only exception is the result that weak mixing implies total transitivity. For actions of abelian groups this is a consequence of well known simple arguments, see Section 5. However, for non-abelian groups, one requires a different approach.

Theorem 1. *On second countable Baire spaces, one has the following implications*:

$$\begin{tikzcd}
 & \text{strongly mixing} \arrow[d] & \\
 & k\text{-transitive for all } k \arrow[dl] \arrow[dr] & \\
\text{weakly mixing} \arrow[dr] & & \text{elastic} \arrow[dl] \\
 & \text{totally transitive} \arrow[d] & \\
 & \text{transitive} &
\end{tikzcd}$$

and the "second countable Baire" hypothesis is only used in the proof that weak mixing implies total transitivity.

Proof. It is obvious that k-transitivity for all k implies both weak mixing and elasticity. It is also obvious that total transitivity implies transitivity. So there remain 3 implications to prove:

Strongly mixing $\Rightarrow$ k-transitive for all k. Indeed, let $k \in \mathbf{N}$ and suppose the action of G is strongly mixing, and let $U_1, \dots, U_k, V_1, \dots, V_k$ be nonempty open subsets of M. Let $A_i = \{g \in G;\ gU_i \cap V_i = \varnothing\}$ for $i \in \{1, \dots, k\}$ and $A = \cup_i A_i$. Since the action is strongly mixing, A is finite and so there exists $g \in G \backslash A$. Then $gU_i \cap V_i \neq \varnothing$ for all i and so G is k-transitive.

Elastic $\Rightarrow$ totally transitive. Suppose that H is a subgroup of finite index in G. Consider a left transversal $\{g_1, g_2, \dots, g_k\}$ of H in G, i.e., suppose that $g_i H$ are the distinct left cosets of H. Let U and V be nonempty open sets of M. Since G is elastic, there exists $g \in G$ such that $gU \cap g_i V \neq \varnothing$, for all $i \in \{1, \dots, k\}$. Notice that $g = g_i h$ for some i and some $h \in H$. Then $g_i hU \cap g_i V \neq \varnothing$ implies $hU \cap V \neq \varnothing$. Hence H is transitive. So G is totally transitive.

Weakly mixing $\Rightarrow$ totally transitive. We will prove the contrapositive, i.e., if the action of a group G on a second countable Baire space M is not totally transitive, then G is not weakly mixing. That is, we suppose that G acts transitively, and that the induced action of a finite index subgroup H is not transitive; we will show that the action of G is not weakly mixing.

For each $x \in M$, let V_x denote the interior of the closure of the H-orbit of x; $V_x = \operatorname{Int}(\overline{Hx})$. Let D denote the set of points x for which Gx is dense in M. By Proposition 1, D is nonempty.

Lemma 8. *For each $x \in D$, we have*:

(a) *V_x is H-invariant, i.e., for all $h \in H$, $hV_x = V_x$,*

(b) $x \in V_x$,

(c) $\overline{V_x} = \overline{Hx}$,

(d) *For all $g \in G$, if $V_x \cap V_{gx} \neq \varnothing$, then $V_{gx} = V_x$.*

Proof. Let $x \in D$. Part (a) is immediate from the definition of V_x.

(b) Consider a left transversal $\{g_1 = \mathrm{id}, g_2, \dots, g_k\}$ of H in G. We have $Gx = \cup_{i=1}^k g_i Hx$, and so

$$M = \overline{Gx} = \bigcup_{i=1}^{k} \overline{g_i Hx} = \bigcup_{i=1}^{k} g_i \overline{Hx}.$$

Since M is a Baire space and M is the finite union of the homeomorphic sets $g_i\overline{Hx}$, the set $g_1\overline{Hx} = \overline{Hx}$ must have nonempty interior, i.e., V_x is nonempty. Since V_x is a nonempty open subset of $\overline{Hx}$ and the H-orbit of x is dense in $\overline{Hx}$, there exists $h \in H$ such that $hx \in V_x$. Thus, by part (a), $x \in V_x$.

(c) It follows from (a) and (b) that $Hx \subseteq V_x$ and so $\overline{Hx} \subseteq \overline{V_x}$. Conversely, by definition, $V_x \subseteq \overline{Hx}$ and so $\overline{V_x} \subseteq \overline{Hx}$.

(d) If $V_x \cap V_{gx} \neq \varnothing$, then $V_x \cap V_{gx}$ is a nonempty open subset of $\overline{Hx}$. Thus, as the H-orbit of x is dense in $\overline{Hx}$, there exists $h \in H$ such that $hx \in V_{gx}$. Then, by part (a), $Hx \subseteq V_{gx}$ and so $\overline{Hx} \subseteq \overline{V_{gx}} = \overline{H(gx)}$, by part (c). Thus, taking interiors, $V_x \subseteq V_{gx}$. Since $x \in D$, we have $gx \in D$. So the previous argument gives $V_{gx} \subseteq V_{g^{-1}gx} = V_x$. Hence, $V_x = V_{gx}$. $\square$

Returning to the proof of Theorem 1, let $x \in D$. As H is not transitive, by hypothesis, $\overline{Hx} \neq M$. As $M \backslash \overline{Hx}$ is open, there exists $g \in G$ such that $gx \in M \backslash \overline{Hx}$. If G is weakly mixing, then there exists $f \in G$ such that $fV_x \cap V_x \neq \varnothing$ and $fV_x \cap V_{gx} \neq \varnothing$. But then, by Lemma 8 (d), one would have $fV_x = V_x$ and $fV_x = V_{gx}$. But then $V_x = V_{gx}$, which is impossible as $gx \in V_{gx}$ by Lemma 8 (b) and $gx \notin V_x$, by our choice of g. $\square$

3. Examples. In this section we give some simple examples which show that the implications of Theorem 1 are strict.

Example 1. *Transitive $\not\Rightarrow$ totally transitive.* The action by multiplication on $\mathbf{R}$ of the group $\mathbf{R}_*$ of nonzero reals is transitive, but the induced action of the positive reals $\mathbf{R}_*^+$ is not.

Example 2. *Totally transitive $\not\Rightarrow$ weakly mixing nor elastic.* We give three examples. First, the linear action of $SL(2, \mathbf{Z})$ on $\mathbf{R}^2$ is totally transitive [**13**], but it is neither weakly mixing nor elastic. Indeed, consider a small disc U centered at the origin, and small discs V_1, V_2 centered on the standard basis vectors e_1, e_2. Because $SL(2, \mathbf{Z})$ preserves area, there is no matrix $A \in SL(2, \mathbf{Z})$ for which $AU \cap V_1$ and $AU \cap V_2$ are both nonempty.

Second, for an action of an abelian group, consider an irrational rotation R_α on the circle S^1. The cyclic group of homeomorphisms generated by R_α is totally transitive since R_α and every iterate $R_\alpha^n = R_{n\alpha}$ with $n \in \mathbf{Z}\backslash\{0\}$, is transitive. However, being an isometry, R_α is clearly neither weakly mixing nor elastic.

Our third example emphasizes just how weak the totally transitive hypothesis is. Let G be an infinite topological group which is simple as an abstract group, for instance $G = SO(3, \mathbf{R})$. The action of G on itself by left-translation is obviously transitive, and is neither weakly mixing nor elastic. Moreover, it is totally transitive since G has no nontrivial finite index subgroups. Indeed, recall that if G had a finite index subgroup H, with left transversal $\{g_1, g_2, \dots, g_l\}$, then its core, $\mathrm{Core}\,(H) = \cap_{i=1}^{l} g_i H g_i^{-1}$, is the largest normal subgroup of G that is contained in H. Each of the conjugates $g_i H g_i^{-1}$ has the same index in G as H and, since the intersection of finitely many finite index subgroups has finite index (by Poincaré lemma [**36**]), $\mathrm{Core}\,(H)$ would also have finite index. Thus, G would have a finite index normal subgroup, contradicting the assumption that G is simple.

Example 3. *Elastic $\not\Rightarrow$ weakly mixing.* Let G be the group of all orientation preserving homeomorphisms of $\mathbf{R}$. It is clear that G is elastic, but it is not weakly mixing, since no element of G can reverse the order of two disjoint subintervals.

Example 4. *Weakly mixing $\not\Rightarrow$ elastic.* For $n > 2$ the linear action of $SL(n, \mathbf{Z})$ on $\mathbf{R}^n$ is weakly mixing [**13**], but it is not elastic. Indeed, arguing as in Example 2, consider a small disc U centered at the origin and small discs $V_1, \dots, V_n$ centered on the standard basis vectors $e_1, \dots, e_n$. Because $SL(n, \mathbf{Z})$ preserves volume, there is no matrix $A \in SL(n, \mathbf{Z})$ for which the $AU \cap V_i$ are all nonempty.

Example 5. *Weakly mixing + elastic $\not\Rightarrow$ k-transitive for all k.* Let G be the group of homeomorphisms of $\mathbf{R}$. It is clear that G is elastic and weakly mixing, but it is not 3-transitive, since G cannot perform all permutations of three disjoint subintervals.

Example 6. *k-transitive for all $k \not\Rightarrow$ strongly mixing.* It is easy to see that the group of homeomorphisms of $\mathbf{R}^n$ for $n \geq 2$, or in fact any manifold of dimension $n \geq 2$, is k-transitive for all k but not strongly mixing. For more examples, see subsection 4.3 and Section 6.

Example 7. *Strongly mixing examples.* Every infinite group G has a strongly mixing action on a compact metric space. Indeed, let $M = \{0,1\}^G$ be equipped with the product topology. The natural action of G on $\{0,1\}^G$ is given by

$$g(f)(x) = f(g^{-1}x),$$

for all $g, x \in G$ and $f : G \to \{0,1\}$. It is well known and not difficult to see that this action is strongly mixing [**24**].

4. Inheritance of notions.

4.1 *Inheritance under semi-conjugacy.* Recall that if G acts on two topological spaces M_1 and M_2, a continuous map $f : M_1 \to M_2$ is called a *semi-conjugacy* if f is surjective and G-equivariant, i.e., $gf(x) = f(gx)$ for all $g \in G,\ x \in M_1$. If f is a semi-conjugacy, then it is obvious that if the action of G on M_1 is strongly mixing, k-transitive for all k, weakly mixing, elastic or transitive, then the action of G on M_2 also enjoys the same property. Of course, properties of the action on M_2 are not in general inherited by the action on M_1.

4.2 *Inheritance from and by subgroups.* Suppose that H is a subgroup of G. We first consider what properties of the action of H are inherited by G.

Proposition 4. *If H is a subgroup of G, then*

(a) *H strongly mixing $\Rightarrow$ G strongly mixing, provided H has finite index in G,*

(b) *H k-transitive for all k $\Rightarrow$ G k-transitive for all k,*

(c) *H weakly mixing $\Rightarrow$ G weakly mixing,*

(d) *H elastic $\Rightarrow$ G elastic,*

(e) *H totally transitive $\Rightarrow$ G totally transitive,*

(f) *H transitive $\Rightarrow$ G transitive.*

Proof. Parts (b), (c), (d) and (f) are obvious. For (e), note that if H is totally transitive, and K is a finite index subgroup of G, then $H \cap K$ has finite index in H and so $H \cap K$ is transitive and thus K is transitive.

For (a), choose a right transversal $\{g_1 = \mathrm{id}, g_2, \dots, g_l\}$ of H in G;

$$G = Hg_1 \cup \cdots \cup Hg_l.$$

Let U and V be nonempty open subsets of M and consider the set

$$\begin{aligned}\{g \in G;\ gU \cap V = \varnothing\} &= \bigcup_{i=1,\dots,l} \{g \in Hg_i;\ gU \cap V = \varnothing\} \\ &= \bigcup_{i=1,\dots,l} \{hg_i;\ h \in H \text{ and } h(g_iU) \cap V = \varnothing\} \\ &= \bigcup_{i=1,\dots,l} \{h \in H;\ h(g_iU) \cap V = \varnothing\} g_i.\end{aligned}$$

As H is strongly mixing, the sets $\{h \in H;\ h(g_iU) \cap V = \varnothing\}$ are finite. So $\{g \in G;\ gU \cap V = \varnothing\}$ is finite; thus G is strongly mixing. □

We now consider what properties of the action of G are inherited by H.

Proposition 5. *If H is a finite index subgroup of G, then*

(a) *G strongly mixing $\Rightarrow$ H strongly mixing,*

(b) *G k-transitive for all $k \Rightarrow H$ k-transitive for all k,*

(c) *G weakly mixing $\not\Rightarrow$ H weakly mixing,*

(d) *G elastic $\Rightarrow$ H elastic,*

(e) *G totally transitive $\Rightarrow$ H totally transitive,*

(f) *G transitive $\not\Rightarrow$ H transitive.*

Proof. Parts (a) and (e) are obvious; part (a) is given in [**24**]. Parts (b) and (d) follow from the following:

Lemma 9. *If G is kl-transitive, then every subgroup of index l is k-transitive.*

Proof. Consider a left transversal of the index l subgroup H in G: $\{g_1, g_2, \dots, g_l\}$. Let $U_1, \dots, U_k$ and $V_1, \dots, V_k$ be nonempty open sets. If G is kl-transitive, there exists $g \in G$ such that

$$gU_j \cap g_i V_j \neq \varnothing,$$

for all $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, k\}$. Notice that $g = g_i h$ for some g_i and some $h \in H$. Then $g_i h U_j \cap g_i V_j \neq \varnothing$ implies $hU_j \cap V_j \neq \varnothing$. Hence, H is k-transitive. $\square$

Parts (c) and (d) are well known. For (c), let G be the group of homeomorphisms of $\mathbf{R}$, and let H be the subgroup of orientation preserving homeomorphisms; see Examples 3 and 4. For (f), let G be the group $\mathbf{R}_*$ of nonzero reals, acting by multiplication on the real line, and let H be the positive reals $\mathbf{R}_*^+$, see Example 1. $\square$

Remark 1. The assumption that H has finite index is not required in part (a); it suffices to assume that H is infinite.

4.3 *Inheritance when taking products.* The inheritance of dynamical properties under the taking of products is a traditional problem, see [**19, 24**], [**16**, Section II.4]. It is obvious that if infinite groups G and H act on spaces M and N, and if both actions are k-transitive for all k, respectively weakly mixing, respectively elastic, respectively totally transitive, respectively transitive, then the obvious product action of $G \times H$ on $M \times N$ also has the same property. Notice however that, except in the trivial case of actions on singleton sets, strong mixing is *never* preserved by taking products. Indeed, for nonempty open sets $U_1, U_2 \subseteq M, V_1, V_2 \in N$, one has:

$$\begin{aligned}\{(g,h) \in G \times H;\ ((g,h)(U_1 \times V_1)) \cap (U_2 \times V_2) = \varnothing\} \\ = \{g \in G;\ gU_1 \cap V_1 = \varnothing\} \\ \times H \ \cup\ G \times \{h \in H;\ hU_2 \cap V_2 = \varnothing\}.\end{aligned}$$

Provided M or N has at least two elements, one of the last two sets is infinite; so the action of $G \times H$ on $M \times N$ is not strongly mixing. Note that this gives a simple means of constructing examples of actions that are k-transitive for all k but not strongly mixing.

We now turn to actions on products in a different sense; we consider actions of a group G on spaces M and N and we consider the *diagonal action* of G on $M \times N$ defined by $g(x, y) = (g(x), g(y))$ for all $(x, y) \in M \times N$. There are examples where G is weakly mixing on M and N, but the diagonal action on $M \times N$ is not transitive [**28**]. Note that the group of all orientation preserving homeomorphisms of $\mathbf{R}$, see Example 3, is elastic, but its diagonal action on $\mathbf{R}^2$ is not transitive. On the other hand, the following proposition holds (parts (a) and (f) were stated in [**24**] for compact spaces):

Proposition 6. *Suppose that G acts on M and on a dense in itself Hausdorff space N. If the action of G on M is strongly mixing, then*:

(a) *G strongly mixing on $N \Rightarrow G$ strongly mixing on $M \times N$,*

(b) *G k-transitive for all k on $N \Rightarrow G$ k-transitive for all k on $M \times N$,*

(c) *G weakly mixing on $N \Rightarrow G$ weakly mixing on $M \times N$,*

(d) *G elastic on $N \Rightarrow G$ elastic on $M \times N$,*

(e) *G totally transitive on $N \Rightarrow G$ totally transitive on $M \times N$,*

(f) *G transitive on $N \Rightarrow G$ transitive on $M \times N$.*

Proof. Assume that the action of G on M is strongly mixing and consider nonempty open sets $U_1, U_2 \subseteq M$, $V_1, V_2 \in N$.

(a) As G is strongly mixing on N, the sets $\{g \in G;\ gU_1 \cap U_2 = \varnothing\}$ and $\{g \in G;\ gV_1 \cap V_2 = \varnothing\}$ are finite and so $\{g \in G;\ gU_1 \cap U_2 = \varnothing \text{ or } gV_1 \cap V_2 = \varnothing\}$ is finite. Hence the diagonal action of G on $M \times N$ is also strongly mixing.

(f) The set $\{g \in G;\ gU_1 \cap U_2 = \varnothing\}$ is finite and if G is transitive on N, the set $\{g \in G;\ gV_1 \cap V_2 \neq \varnothing\}$ is infinite, by Lemma 3. So there exists $g \in G$ with $gU_1 \cap U_2 \neq \varnothing$ and $gV_1 \cap V_2 \neq \varnothing$. Hence, G acts transitively on $M \times N$.

(d) Consider nonempty open sets $U, U_1, \dots, U_k \subseteq M$, $V, V_1, \dots, V_k \in N$. As G is strongly mixing on M, the set $\cup_i \{g \in G;\ gU \cap U_i = \varnothing\}$ is finite while, by Lemma 4, the set $\{g \in G;\ gV \cap V_i \neq \varnothing \text{ for all } i\}$ is infinite. Thus G is elastic on $M \times N$.

(e) Suppose that G is totally transitive on N, and let H be a finite index subgroup of G. Then H is strongly mixing on M, by Proposition 5, and H is transitive on N; so by (f), H is transitive on $M \times N$. Thus, G is totally transitive on $M \times N$.

(b), (c) Suppose the action of G on N is k-transitive, i.e., G is transitive on N^k. As G is strongly mixing on M, we have from (a) that G is strongly mixing on M^k. Thus, by (f), G is transitive on $(M \times N)^k$, i.e., G is k-transitive on $M \times N$. □

Note that the above proposition gives examples which show that Proposition 4 (a) fails without the finite index hypothesis. Indeed, suppose that G is strongly mixing M. Then G^2 is not strongly mixing on M^2, as we remarked above, but the diagonal action of G is strongly mixing on M^2 by Proposition 6(a). That is, on M^2, the diagonal subgroup $\{(g, g);\ g \in G\} \leq G^2$ is strongly mixing, but the group G^2 isn't.

Remark 2. If G is k-transitive for all k on M, then obviously the diagonal action of G on M^2 is also k-transitive for all k. More precisely, if G is nk-transitive on M, then the action of G on M^n is k-transitive.

4.4 *Inheritance when passing to hyperspace.* Let (M, d) be a metric space. For each $x \in M$ and $\varepsilon > 0$, let $B_\varepsilon(x)$ denote the open ball of radius ε centered at x, and for each $K \subseteq M$ and $\varepsilon > 0$, let $D_\varepsilon(K)$ denote the union of all $B_\varepsilon(x)$ with $x \in K$. Let $\mathcal{K}(M)$ be the set of all nonempty compact subsets of M. For each pair $K, L \in \mathcal{K}(M)$, the *Hausdorff distance* from K to L is

$$h(K, L) = \inf\{\varepsilon;\ K \subseteq D_\varepsilon(L),\ L \subseteq D_\varepsilon(K)\}.$$

It is easy to verify that $(\mathcal{K}(M), h)$ is a metric space, called the *hyperspace* of (M, d). If (M, d) is connected, compact, or complete then $(\mathcal{K}(M), h)$ has the corresponding property, see [**22, 31**].

Let $f : M \to M$ be a continuous function, and let $\hat{f} : \mathcal{K}(M) \to \mathcal{K}(M)$ be the *image function* which takes $A \in \mathcal{K}(M)$ to $f(A)$. Using a modification of the proof of Heine's theorem on uniform continuity [**4**, Theorem 4-24], it is not difficult to see that $\hat{f}$ is continuous [**31**, Corollary 4.8]. (For more information about the properties of $\hat{f}$, see

[**14**]). Let Φ be a continuous action of a group G on a metric space (M, d), and let $\hat{\Phi}$ be the induced *image action* of G on $\mathcal{K}(M)$. The subspace of $(\mathcal{K}(M), h)$ consisting of the singleton subsets of M is an isometric copy (M, d), and the action of G on this subspace is a copy of the continuous action Φ.

It is easy to see that in $(\mathcal{K}(M), h)$, the open ball of radius ε centered at K is

$$B_\varepsilon(K) = \{L \in \mathcal{K}(M);\ L \subseteq D_\varepsilon(K),\ K \subseteq D_\varepsilon(L)\}.$$

In particular, if $L \subseteq K$, then $L \in B_\varepsilon(K)$ if and only if $K \subseteq D_\varepsilon(L)$. Notice that for each $K \in \mathcal{K}(M)$ and $\varepsilon > 0$ there is a finite (therefore compact) subset $A_\varepsilon \subseteq K$ such that $K \subseteq D_\varepsilon(A_\varepsilon)$, that is, such that $A_\varepsilon \in B_\varepsilon(K)$. If $x \in M$, the open ball $B_\varepsilon(\{x\})$ in $\mathcal{K}(M)$ is

$$\begin{aligned} &\{A \in \mathcal{K}(M);\ A \subseteq D_\varepsilon(\{x\}),\ \{x\} \subseteq D_\varepsilon(A)\} \\ &\qquad\qquad\qquad\qquad = \{A \in \mathcal{K}(M);\ A \subseteq D_\varepsilon(\{x\})\}, \end{aligned}$$

that is, just the nonempty compact subsets contained in the open ball $B_\varepsilon(x)$ of (M, d). Similarly, for points $x_1, \dots, x_n \in M$, the open ball $B_\varepsilon(\{x_1, \dots, x_n\})$ in $\mathcal{K}(M)$ is

$$\begin{aligned} &\{A \in \mathcal{K}(M);\ A \subseteq D_\varepsilon(\{x_1, \dots, x_n\}),\ \{x_1, \dots, x_n\} \subseteq D_\varepsilon(A)\} \\ &\qquad = \Big\{A \in \mathcal{K}(M);\ A \subseteq \bigcup_i B_\varepsilon(x_i),\ x_i \in D_\varepsilon(A),\ \forall\, i\Big\}, \\ &\qquad = \Big\{A \in \mathcal{K}(M);\ A \subseteq \bigcup_i B_\varepsilon(x_i),\ A \cap B_\varepsilon(x_i) \neq \varnothing,\ \forall\, i\Big\}. \end{aligned}$$

The following result generalizes results for the case of a single map in [**7, 37**], see [**17, 21, 28**] for related results.

Proposition 7. *On second countable complete metric spaces, one has the following implications*:

$$
\begin{tikzcd}
 & \Phi \textit{ strongly mixing} \iff \hat{\Phi} \textit{ strongly mixing} \arrow[d] \\
 & \Phi\ k\textit{-transitive for all } k \iff \hat{\Phi}\ k\textit{-transitive for all } k \arrow[dl] \arrow[d] \\
\hat{\Phi} \textit{ weakly mixing} \arrow[ddd] \arrow[dr] & \hat{\Phi} \textit{ elastic} \arrow[d] \\
 & \hat{\Phi} \textit{ totally transitive} \arrow[d] \\
 & \hat{\Phi} \textit{ transitive} \arrow[d] \\
\Phi \textit{ weakly mixing} \arrow[dr] & \Phi \textit{ elastic} \arrow[d] \\
 & \Phi \textit{ totally transitive} \arrow[d] \\
 & \Phi \textit{ transitive}
\end{tikzcd}
$$

Proof. It is obvious that the properties of $\hat{\Phi}$ are passed onto Φ. So in view of Theorem 1, it remains to establish 3 things:

Φ *strongly mixing* $\Rightarrow$ Φ *strongly mixing.* Consider a pair of open balls $B_\varepsilon(A), B_\varepsilon(B)$ of $(\mathcal{K}(M), h)$. Cover A by n balls $B_{\varepsilon/2}(x_i)$ of (M, d) with $S = \{x_1, \dots, x_n\} \subseteq A$ and B by n balls $B_{\varepsilon/2}(y_i)$ with $T = \{y_1, \dots, y_n\} \subseteq B$. Since Φ is strongly mixing, for each i the set of g such that $gB_{\varepsilon/2}(x_i) \cap B_{\varepsilon/2}(y_i) = \varnothing$ is finite. Therefore, all but finitely many $g \in G$ satisfy $gB_{\varepsilon/2}(x_i) \cap B_{(\varepsilon/2)}(y_i) \neq \varnothing$, $i \in \{1, \dots, n\}$. Consider one such g. For $i \in \{1, \dots, n\}$ let $x'_i \in B_{\varepsilon/2}(x_i)$ and $y'_i = gx'_i \in B_{\varepsilon/2}(y_i)$, and put $S' = \{x'_1, \dots, x'_n\}$, $T' = gS' = \{y'_1, \dots, y'_n\}$. Then

$$h(A, S') \leq h(A, S) + h(S, S') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and, likewise, $h(B, T') < \varepsilon$. Since $T' = gS'$, we have $gB_\varepsilon(A) \cap B_\varepsilon(B) \neq \varnothing$.

Φ *k-transitive for all k $\Rightarrow$ $\hat{\Phi}$ k-transitive for all k.* Consider $2k$ balls of $(\mathcal{K}(M), h)$: $B_\varepsilon(A_i), B_\varepsilon(B_i)$, $i \in \{1, \dots, k\}$. Cover each A_i and B_i by n balls of (M, d), all of radius $\varepsilon/2$ and with centers $x_{ij} \in A_i$, $y_{ij} \in B_i$. Since Φ is kn-transitive, some $g \in G$ satisfies $gB_{\varepsilon/2}(x_{ij}) \cap B_{\varepsilon/2}(y_{ij}) \neq \varnothing$, for all $i \in \{1, \dots, k\}$, $j \in \{1, \dots, n\}$. Arguing as in the previous paragraph, this g also satisfies $gB_\varepsilon(A_i) \cap B_\varepsilon(B_i) \neq \varnothing$, for all $i \in \{1, \dots, k\}$.

$\hat{\Phi}$ transitive $\Rightarrow$ Φ elastic. Suppose $B_\varepsilon(x), B_\varepsilon(x_1), \dots, B_\varepsilon(x_n)$ are open balls of (M, d). We must find a g such that $gB_(x) \cap B_\varepsilon(x_i) \neq \varnothing$, for all $i \in \{1, \dots, n\}$. Since $\hat{\Phi}$ is transitive there is a g such that $gB_\varepsilon(\{x\}) \cap B_\varepsilon(\{x_1, \dots, x_n\}) \neq \varnothing$ in $\mathcal{K}(M)$. That is, from the discussion immediately before the proposition, there is $A \in \mathcal{K}(M)$ with $A \subseteq B_\varepsilon(x)$, $gA \subseteq \cup_i B_\varepsilon(x_i)$ and $gA \cap B_\varepsilon(x_i) \neq \varnothing$, for all $i \in \{1, \dots, n\}$. But this also means $gB_\varepsilon(x) \cap B_\varepsilon(x_i) \neq \varnothing$, for all $i \in \{1, \dots, n\}$, as required. □

5. Actions of abelian groups.

Furstenberg showed that for flows, weakly mixing implies k-transitive for all k [**19**], see also [**16**, Proposition II.4.12]. Petersen showed that for actions of abelian groups, elastic implies weakly mixing [**35**]. Combining these ideas, one obtains:

Theorem 2. *For actions of abelian groups, one has the following implications*:

$$\begin{matrix}\textit{strongly}\\ \textit{mixing}\end{matrix} \Longrightarrow \left\{\begin{matrix}k\textit{-transitive for all } k\\ \textit{weakly mixing}\\ \textit{elastic}\end{matrix}\right\} \Longrightarrow \begin{matrix}\textit{totally}\\ \textit{transitive}\end{matrix} \Longrightarrow \textit{transitive},$$

where the conditions inside the braces are equivalent.

Remark 3. Notice that we do not impose the "second countable Baire" hypothesis, as we did for Theorem 1. Indeed, the "second countable Baire" hypothesis was only used in the proof of Theorem 1 to show that weak mixing implies total transitivity, and we don't require this here.

Proof. Given Theorem 1, we need only establish two things:

Weakly mixing $\Rightarrow$ k-transitive for all k. Consider k pairs of nonempty open sets U_i, V_i, $i \in \{1, \ldots, k\}$. If the action is weak mixing, there is some $g_2 \in G$ such that

$$g_2 U_2 \cap U_1 \neq \varnothing \quad \text{and} \quad g_2 V_2 \cap V_1 \neq \varnothing.$$

Next, there is some g_3 such that

$$g_3 U_3 \cap (g_2 U_2 \cap U_1) \neq \varnothing \quad \text{and} \quad g_3 V_3 \cap (g_2 V_2 \cap V_1) \neq \varnothing,$$

and so on, up to k. Let

$$U = g_k U_k \cap \cdots \cap g_2 U_2 \cap U_1 \neq \varnothing,$$

and define V similarly. There is some $g \in G$ such that $gU \cap V \neq \varnothing$. Setting $g_1 = \mathrm{id}$, we have for all $i \in \{1, \ldots, k\}$,

$$g g_i U_i \cap g_i V_i = g_i g U_i \cap g_i V_i = g_i (g U_i \cap V_i) \neq \varnothing.$$

Thus $gU_i \cap V_i \neq \varnothing$, for all i. So the action is k-transitive for all k.

Elastic $\Rightarrow$ weakly mixing. Consider nonempty open sets U_1, U_2, V_1, V_2. By elasticity, there is some g_1 such that

$$g_1 U_1 \cap U_2 \neq \varnothing \quad \text{and} \quad g_1 U_1 \cap V_2 \neq \varnothing,$$

and next, there is some g such that

$$g(U_1 \cap g_1^{-1} U_2) \cap V_1 \neq \varnothing \;\; \text{and} \;\; g(U_1 \cap g_1^{-1} U_2) \cap (U_1 \cap g_1^{-1} V_2) \neq \varnothing.$$

Then $gU_1 \cap V_1 \neq \varnothing$ and

$$g g_1^{-1} U_2 \cap g_1^{-1} V_2 = g_1^{-1} g U_2 \cap g_1^{-1} V_2 = g_1^{-1} (g U_2 \cap V_2) \neq \varnothing,$$

which gives $gU_2 \cap V_2 \neq \varnothing$. Therefore Φ is weakly mixing.

This completes the proof of Theorem 2. □

As discussed at the beginning of subsection 4.3, if G is strongly mixing on M, then the action of $G \times G$ on M^2 is k-transitive for all k, but not strongly mixing; in particular, there are such actions of abelian groups.

An action of an abelian group which is totally transitive but not weakly mixing was given in Example 2. The group in Example 1 (whose action is transitive but not totally transitive) is abelian.

We now turn to the inheritance properties. In Proposition 5 we saw that if H is a finite index subgroup of G, then G weakly mixing doesn't necessarily imply H weakly mixing. However, this does hold if G is abelian, as was observed in [**24**]. This can be expressed in a somewhat more useful way: if G has an abelian subgroup A whose induced action is weakly mixing, then every finite index subgroup H of G is weakly mixing. Indeed, if A is abelian and weakly mixing, then by Theorem 2, A is elastic, so G is elastic by Proposition 4, and thus H is elastic, by Proposition 5, and H is weakly mixing, again by Theorem 2.

For products, notice that from Theorem 2 and Remark 2, if the action of an abelian group G is weakly mixing on M, then the action of G on M^2 is also weakly mixing; this was proved in [**24**].

For the induced action on hyperspace, the next result follows immediately from Theorem 2 and Proposition 7:

Proposition 8. *For actions of abelian groups on metric spaces, one has the following implications*:

$$\left\{\Phi \text{ strongly mixing, } \hat{\Phi} \text{ strongly mixing}\right\}$$

$$\Downarrow$$

$$\left\{\begin{array}{cccc} \hat{\Phi}k\text{-transitive } \forall\, k, & \hat{\Phi} \text{ elastic}, & \hat{\Phi} \text{ weakly mixing}, & \hat{\Phi} \text{ totally transitive} \\ \Phi k\text{-transitive } \forall\, k, & \Phi \text{ elastic}, & \Phi \text{ weakly mixing}, & \hat{\Phi} \text{ transitive} \end{array}\right\}$$

$$\Downarrow$$

$$\Phi \text{ totally transitive}$$

$$\Downarrow$$

$$\Phi \text{ transitive},$$

where the conditions inside the braces are equivalent.

6. Chaotic actions.

Definition 5. The action of G on M is *chaotic* if it is transitive and the set of points in M whose orbit under G is finite is a dense subset of M.

A group G has a faithful chaotic action on some Hausdorff topological space M if and only if G is residually finite [**11**]. (Curiously, it is also true that a finitely generated group is residually finite if and only if it is the group of isometries of a metric compactum [**18**]). Chaotic actions enjoy the usual "sensitivity to initial conditions" property [**11**]. Every compact triangulable manifold of dimension greater than 1 admits a weakly mixing chaotic action of every countably generated free group [**12**].

The following result is a generalization of a result by Peter Stacey for single maps, see [**5**].

Theorem 3. *For chaotic actions on second countable Baires, one has*:

$$\textit{strongly mixing} \Longrightarrow \left\{\begin{array}{c} k\textit{-transitive for all } k \\ \textit{weakly mixing} \\ \textit{elastic} \\ \textit{totally transitive} \end{array}\right\} \Longrightarrow \textit{transitive,}$$

where the conditions inside the braces are equivalent.

Proof. In view of Theorem 1, it suffices to show that if the action of G is totally transitive and chaotic, then G is k-transitive for all k. We argue by induction: we assume that G is totally transitive, chaotic and k-transitive for some $k \geq 1$, and we will show that G is $(k+1)$-transitive. Let $U_1, \ldots, U_{k+1}$ and $V_1, \ldots, V_{k+1}$ be nonempty open sets of M. By the inductive hypothesis, there exists $g \in G$ such that $gU_i \cap V_i \neq \varnothing$ for all $1 \leq i \leq k$. As G is chaotic, for each $1 \leq i \leq k$ there exists $x_i \in U_i \cap g^{-1}V_i$ such that Gx_i is finite. Consider the intersection H of the stabilizer subgroups of the points x_i:

$$H = \{h \in G;\ h(x_i) = x_i,\ \text{for all } 1 \leq i \leq k\}.$$

H has finite index and so, as G is totally transitive, there exists $h \in H$ such that $hU_{k+1} \cap g^{-1}V_{k+1} \neq \varnothing$. Then $ghU_{k+1} \cap V_{k+1} \neq \varnothing$, and $ghU_i \cap V_i \neq \varnothing$ for all $1 \leq i \leq k$ since $ghx_i = gx_i \in V_i$. □

Example 8. Consider the linear action of $SL(n, \mathbf{Z})$ on the torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$, for $n \geq 2$. Recall that a subgroup generated by a hyperbolic matrix is weakly mixing on $\mathbf{T}^n$, see [**19**]. Moreover, for each $m \in \mathbf{N}$, the image in $\mathbf{T}^n$ of the points of the form $(1/m)x$, where $x \in \mathbf{Z}^n$, is a finite $SL(n, \mathbf{Z})$-invariant set. It follows that the action of $SL(n, \mathbf{Z})$ on $\mathbf{T}^n$ is chaotic. Thus, by Theorem 3, the action is k-transitive for all k. However, the action is clearly not strongly mixing. (A totally transitive chaotic function which is not strongly mixing was given in [**15**]).

Now let M denote the disjoint union of two copies of $\mathbf{T}^n$, and let G be the direct product of $SL(n, \mathbf{Z})$ and the two element group $\{\mathrm{id}, \tau\}$. There is an obvious action of G on M; $SL(n, \mathbf{Z})$ acts linearly on each connected component of M and τ is the homeomorphism that interchanges the two components. This action is clearly transitive (and chaotic), but it is not totally transitive.

Turning now to the inheritance properties, it is immediate from Proposition 5 and Theorem 3 that finite index subgroups of chaotic weakly mixing groups are also chaotic weakly mixing. In particular, by Theorem 3, if G is chaotic and weakly mixing on M, then for every point $x \in M$ with finite orbit, the action on M of the stabilizer subgroup $G_x = \{g \in G;\ gx = x\}$ is k-transitive for all k.

For products, notice that, from Theorem 3 and Remark 2, if G is totally transitive and chaotic on M, then the action of G on M^2 is also totally transitive and chaotic. Notice however, that for a chaotic action of G on M, the action of G on M^2 may fail to be chaotic; indeed, if the action of G on M is chaotic but not totally transitive, as in Example 8, then G is not weakly mixing on M by Theorem 3, and so G is not transitive on M^2.

For the induced action on hyperspace, the next result follows immediately from Theorem 3 and Proposition 7:

Proposition 9. *For chaotic actions on second countable complete metric spaces, one has*:

$$\Big\{\Phi \textit{ strongly mixing, } \hat{\Phi} \textit{ strongly mixing}\Big\}$$

$$\Downarrow$$

$$\left\{\begin{array}{ccc} \hat{\Phi}k\textit{-transitive } \forall\, k, & \hat{\Phi} \textit{ elastic}, & \hat{\Phi} \textit{ weakly mixing} \\ \Phi k\textit{-transitive } \forall\, k, & \Phi \textit{ elastic}, & \Phi \textit{ weakly mixing} \\ \hat{\Phi} \textit{ totally transitive}, & \hat{\Phi} \textit{ transitive}, & \Phi \textit{ totally transitive} \end{array}\right\}$$

$$\Downarrow$$

$$\Phi \textit{ transitive}$$

where the conditions inside the braces are equivalent.

Remark 4. We do not know whether the second countable Baire hypothesis is necessary in Theorems 1 and 3 and in Proposition 9.

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