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SPECIAL MAXIMAL OPERATOR AND A_p^+ WEIGHTS

Abstract

We introduce a special maximal operator associated to a special variant of Muckenhoupt's weights. By using this special maximal operator, we can construct the special weights. We also prove a weighted weak-type estimate of the special maximal operator.

1 Introduction

1.1 Setup

In this note, we study the special maximal operator on the upper half plane and the A_p^+ weights. To begin, we first introduce the A_p^+ class.

Definition 1.1. A measurable function μ is a *weight* on the upper half plane $\mathbb{R}^2_+$ if $\mu > 0$ almost everywhere and is locally integrable on $\mathbb{R}^2_+$.

Definition 1.2. A disk in $\mathbb{R}^2_+$ is said to be a *special disk* if it is centered at some $x \in \mathbb{R}$, where $\mathbb{R}$ denotes the topological boundary of $\mathbb{R}^2_+$.

Definition 1.3. For $p > 1$, let p' denote the conjugate exponent of p. We say the two weights μ_1 and μ_2 are in the $A_p^+(\mathbb{R}^2_+)$ class, denoted by $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}^2_+)$, if there is a positive constant c such that

$$\frac{1}{|D \cap \mathbb{R}^2_+|} \int_{D\cap\mathbb{R}^2_+} \mu_1(z)\, dA(z) \left(\frac{1}{|D \cap \mathbb{R}^2_+|} \int_{D\cap\mathbb{R}^2_+} \mu_2(z)^{-\frac{p'}{p}}\, dA(z) \right)^{\frac{p}{p'}} \le c$$

for all special disks D. For some weight μ, if $(\mu, \mu) \in A_p^+(\mathbb{R}^2_+)$, we may simply adopt the notation $\mu \in A_p^+(\mathbb{R}^2_+)$.

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We have defined the $A_p^+(\mathbb{R}_+^2)$ class for $p > 1$. Now we extend this class to the case $p = 1$.

Definition 1.4. Two weights μ_1 and μ_2 on $\mathbb{R}_+^2$ are in the class $A_1^+(\mathbb{R}_+^2)$, denoted by the ordered pair $(\mu_1, \mu_2) \in A_1^+(\mathbb{R}_+^2)$, if there is a $c > 0$ such that, for all special disks D, we have

$$\frac{1}{\left|D \cap \mathbb{R}_+^2\right|} \int_{D \cap \mathbb{R}_+^2} \mu_1(\zeta)\, dA(\zeta) \leq c\mu_2(z)$$

for any $z \in D \cap \mathbb{R}_+^2$. Again, for some weight μ, if it satisfies $(\mu, \mu) \in A_1^+(\mathbb{R}_+^2)$, we simply adopt the notation $\mu \in A_1^+(\mathbb{R}_+^2)$.

Now we introduce a suitable maximal operator associated to the $A_p^+(\mathbb{R}_+^2)$ class.

Definition 1.5. For any measurable function f on $\mathbb{R}_+^2$, we define the *special maximal operator* $\widetilde{M}^+$ by

$$\widetilde{M}^+(f)(z) = \sup_{z \in D} \frac{1}{\left|D \cap \mathbb{R}_+^2\right|} \int_{D \cap \mathbb{R}_+^2} |f(\zeta)|\, dA(\zeta)$$

for $z \in \mathbb{R}_+^2$, where the supremum is taken over all special disks D containing z.

Remark 1.6. It is clear that the special maximal function $\widetilde{M}^+(f)$ is lower semi-continuous.

Remark 1.7. It is easy to see that $(\mu_1, \mu_2) \in A_1^+(\mathbb{R}_+^2)$ if and only if $\widetilde{M}^+(\mu_1) \leq c\mu_2$. In particular, a weight μ belonging to the class $A_1^+(\mathbb{R}_+^2)$ is equivalent to $\widetilde{M}^+(\mu) \leq c\mu$.

1.2 Results

After introducing the basic definitions, we can state our main results.

Theorem 1.8. *Let f be a measurable funciton on $\mathbb{R}_+^2$. Then, for any $0 < q < 1$, the function $\left(\widetilde{M}^+(f)\right)^q$ is in $A_1^+(\mathbb{R}_+^2)$.*

Remark 1.9. This theorem tells us how to construct an A_1^+ weight. Together with Proposition 2.1, we introduce a way to construct an A_p^+ weight; for Proposition 2.1, see section 2.

Theorem 1.10. *Assume that $p \geq 1$. Suppose μ_1 and μ_2 are two weights on $\mathbb{R}_+^2$. Then we have a weak-type (p,p) inequality: namely, there is a constant $c > 0$ such that*

$$\mu_1\Big(\big\{z \in \mathbb{R}_+^2 : \ \widetilde{M}^+(f)(z) > \alpha\big\}\Big) \leq \frac{c}{\alpha^p} \int_{\mathbb{R}_+^2} |f(z)|^p \, \mu_2(z) \, dA(z) \tag{1.1}$$

for all $\alpha > 0$, if and only if $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$.

Remark 1.11. This theorem is an analogue of the classical result for the Hardy–Littlewood maximal operator; see [6, Theorem 1 and Theorem 8].

1.3 Background

In potential theory and classical harmonic analysis, it is of particular interest to obtain the weighted estimate of a singular integral and the weighted mean convergence of orthogonal series.

In the 1970s, Muckenhoupt introduced the A_p class to show that a necessary and sufficient condition for the Hardy–Littlewood maximal operator to be weighted L^p-bounded is the weight belonging to the A_p class; see [6]. This result is shown to be very important in proving the weigthed L^p and the weighted BMO estimates of the Hilbert transform by Hunt, Muckenhoupt and Wheeden; see [3, 7]. By considering Muckenhoupt's results, Coifman and Fefferman proved the weighted L^p boundedness of a general singular integral in [2].

In contrast to the classical results, we are interested in the weighted L^p estimate of the 2-dimensional analogue of the so-called Hilbert integral. The Hilbert integral, which was introduced by Phong and Stein in [9, 10], is a sibling of the Hilbert transform. It is defined by

$$\mathcal{H}(f)(x) = \int_0^\infty \frac{f(y)\,dy}{x+y}$$

for any $f \in C_c^\infty(\mathbb{R}^+)$, where $x > 0$. If we consider the Bergman projection on the upper half plane, then we have a similar expression

$$\mathcal{B}(f)(z) = \frac{1}{\pi} \int_{\mathbb{R}_+^2} - \frac{f(w)\,dA(w)}{(z-\overline{w})^2}$$

for any $f \in C_c^\infty(\mathbb{R}_+^2)$, where $z \in \mathbb{R}_+^2$. This type of operator, although not singular at the diagonal line of the product domain, has a very close relation with the general Calderón-Zygmund singular integrals; see [4, 5]. In particular, the follwoing two-weight estimate

$$\int_{\mathbb{R}^2_+} |\mathcal{B}(f)(z)|^p \, \mu_1(z)\, dA(z) \le C \int_{\mathbb{R}^2_+} |f(z)|^p \, \mu_2(z)\, dA(z), \tag{1.2}$$

where μ_1 and μ_2 are two weights on $\mathbb{R}^2_+$, has many interesting applications; see [1] for some special cases.

In [4], Lanzani and Stein proved, for $1 < p < \infty$, that the estimate (1.2) holds for a single weight $\mu = \mu_1 = \mu_2$ if and only if $\mu \in A_p^+(\mathbb{R}^2_+)$. By slight modifications of the proof in [4], the author in [1] improved the previous result: if μ_1 and μ_2 are two weights such that $c\mu_1 \ge \mu_2$ for some $c > 0$, then (1.2) holds if and only if $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}^2_+)$. Based on these facts, it is natural to consider the more general situation.

Conjecture 1.12. *For $1 < p < \infty$, if the two weights μ_1 and μ_2 satisfy $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}^2_+)$, then (1.2) holds for some $C > 0$.*

By considering the result in [8], we are also interested in the following variant.

Conjecture 1.13. *For $1 < p < \infty$, if the two weights μ_1 and μ_2 satisfy $(\mu_1^r, \mu_2^r) \in A_p^+(\mathbb{R}^2_+)$ for some $r > 1$, then (1.2) holds for some $C > 0$.*

Note that the A_p^+ class is a variant of the A_p class. So we suspect that the special maximal operator $\widetilde{M}^+$ is the corresponding variant of the Hardy-Littlewood maximal operator associated to the A_p^+ class. According to the classical results, the weighted L^p estimate of the special maximal operator $\widetilde{M}^+$ may play an important role in proving Conjecture 1.12 and Conjecture 1.13.

Moreover, if we consider the "absolute value" of the Bergman projection on the upper half plane

$$\widetilde{\mathcal{B}}(f)(z) = \frac{1}{\pi} \int_{\mathbb{R}^2_+} \frac{f(w)\, dA(w)}{|z - \overline{w}|^2},$$

then we see, for $f \ge 0$ and $z, z' \in \mathbb{R}^2_+$, that

$$\widetilde{\mathcal{B}}(f)(z') \ge \widetilde{\mathcal{B}}(f)(z)$$

whenever $\Re(z') = \Re(z)$ and $\Im(z') \le \Im(z)$. It is not difficult to see that the special maximal operator $\widetilde{M}^+$ also enjoys the same property above as $\widetilde{\mathcal{B}}$. In view of this, we hope that Theorem 1.8 and Theorem 1.10 could provide some clues to prove Conjecture 1.12 and Conjecture 1.13.

1.4 Organization and outline

In section 2, following the ideas in the classical results, we introduce a way to construct an A_p^+ weight from two A_1^+ weights. In section 3, we prove a useful covering lemma and an interesting mapping property of the special maximal operator $\widetilde{M}^+$. In section 4, we give the proof of Theorem 1.8. In section 5, we prove Theorem 1.10.

2 Construction of an A_p^+ weight

In this section, we give a simple observation for a special product of two A_1^+ weights.

Proposition 2.1. *Suppose that μ_1 and μ_2 are two weights and $\mu_j \in A_1^+(\mathbb{R}_+^2)$, $j = 1, 2$. Then, for $1 \le p < \infty$, we have $\mu_1\mu_2^{1-p} \in A_p^+(\mathbb{R}_+^2)$.*

PROOF. By definition, for $j = 1, 2$, we have

$$\frac{1}{\left|D \cap \mathbb{R}_+^2\right|} \int_{D\cap\mathbb{R}_+^2} \mu_j(z)\, dA(z) \le c \inf_{z\in D\cap\mathbb{R}_+^2} \mu_j(z)$$

for all special disks D. Then we see that

$$\frac{1}{\left|D \cap \mathbb{R}_+^2\right|} \int_{D\cap\mathbb{R}_+^2} \mu_1(z)\mu_2(z)^{1-p}\, dA(z) \le c \inf_{z\in D\cap\mathbb{R}_+^2} \mu_1(z) \Big(\inf_{z\in D\cap\mathbb{R}_+^2} \mu_2(z) \Big)^{1-p}$$

and

$$\begin{aligned} &\left(\frac{1}{\left|D \cap \mathbb{R}_+^2\right|} \int_{D\cap\mathbb{R}_+^2} \big(\mu_1(z)\mu_2(z)^{1-p}\big)^{-\frac{p'}{p}}\, dA(z) \right)^{\frac{p}{p'}} \\ &\le c \Big(\inf_{z\in D\cap\mathbb{R}_+^2} \mu_1(z) \Big)^{-1} \Big(\inf_{z\in D\cap\mathbb{R}_+^2} \mu_2(z) \Big)^{p-1}. \end{aligned}$$

Combining these two inequalities above, we obtain $\mu_1\mu_2^{1-p} \in A_p^+(\mathbb{R}_+^2)$. □

Remark 2.2. From Proposition 2.1, we see that as long as we have two $A_1^+(\mathbb{R}_+^2)$ weights, we can construct an $A_p^+(\mathbb{R}_+^2)$ weight by taking a special product of the two $A_1^+(\mathbb{R}_+^2)$ weights.

3 Two useful lemmas

3.1 Special squares

We first define the special squares associated to the special maximal operator $\widetilde{M}^+$ and the corresponding special disks, .

Definition 3.1. A square is said to be a *special square* if it is of the form

$$\widetilde{S}_{j,k} = \{x + iy \in \mathbb{R}^2_+ : \ j \cdot 2^k \le x \le (j+1) \cdot 2^k \ \text{ and } \ 0 \le y \le 2^k\},$$

where $j, k \in \mathbb{Z}$. Given a special square $\widetilde{S}_{j,k}$, we define

$$\widetilde{S}^*_{j,k} = \{x + iy \in \mathbb{R}^2_+ : \ (j-2) \cdot 2^k \le x \le (j+3) \cdot 2^k \ \text{ and } \ 0 \le y \le 5 \cdot 2^k\}.$$

3.2 A covering lemma

With the definition of the special squares, we can state a covering lemma, which is an analogue of [6, Lemma 7].

Lemma 3.2. *Let $f \ge 0$ be an integrable function on $\mathbb{R}^2_+$, and suppose $\alpha > 0$. Then there is a sequence of measurable sets $\{W_l\}$ and a sequence of special squares $\{\widetilde{S}_l\}$ such that*

(a) The intersection of different W_l's has measure 0,

(b) $\widetilde{S}_l \subset W_l \subset \widetilde{S}^*_l$,

(c) $\frac{\alpha}{16} \left|\widetilde{S}_l\right| \le \int_{W_l} f(\zeta)\, dA(\zeta)$,

(d) If $\widetilde{M}^+(f)(z) > \alpha$, then $z \in \bigcup W_l$.

Proof. Following the idea in [6], we argue as in the classical Calderón-Zygmund lemma. Since $\int_{\mathbb{R}^2_+} f(\zeta)\, dA(\zeta) < \infty$, there is a $k_0 \in \mathbb{Z}^+$ such that, for all $k \ge k_0$ and all $j \in \mathbb{Z}$, we have

$$\frac{1}{|\widetilde{S}_{j,k}|} \int_{\widetilde{S}_{j,k}} f(\zeta)\, dA(\zeta) \le \frac{\alpha}{16}. \tag{3.1}$$

For the $k = k_0 - 1$ level, to each $j \in \mathbb{Z}$, we have either (3.1) still true or

$$\frac{\alpha}{16} < \frac{1}{|\widetilde{S}_{j,k}|} \int_{\widetilde{S}_{j,k}} f(\zeta)\, dA(\zeta) \le \frac{\alpha}{4}. \tag{3.2}$$

The right-hand side of (3.2) follows from (3.1) in the $k+1$ level. If (3.2) holds for this j, we collect this special square $\widetilde{S}_{j,k}$ into the sequence $\{\widetilde{S}_l\}$; otherwise we continue this process to the $k-1$ level in this $\widetilde{S}_{j,k}$. Therefore, we obtain a sequence of almost disjoint special squares $\{\widetilde{S}_l\}$ satisfying (3.2).

Define $W_1 = \widetilde{S}_1^* \backslash \left(\bigcup_{m\neq 1} \widetilde{S}_m\right)$, and successively let

$$W_l = \widetilde{S}_l^* \backslash \left(\bigcup_{m\neq l} \widetilde{S}_m \cup \bigcup_{l'<l} W_{l'} \right),$$

for $l > 1$. Properties (a) and (b) are easy to check from this definition. Property (c) follows from $\widetilde{S}_l \subset W_l$ and $\widetilde{S}_l$ satisfies (3.2). If $\widetilde{M}^+(f)(z) > \alpha$, then there is a special disk $D_z = D_r(x_0)$ centered at $x_0 \in \mathbb{R}$ with radius $r > 0$ such that $z \in D_z$ and

$$\frac{1}{\left|D_z \cap \mathbb{R}_+^2\right|} \int_{D_z \cap \mathbb{R}_+^2} f(\zeta)\, dA(\zeta) > \alpha.$$

If $2^{k_1-1} \leq r < 2^{k_1}$ for some $k_1 \in \mathbb{Z}$, then D_z intersects at most three special squares $\widetilde{S}_{j,k_1}$'s and it is contained in the union of these squares. Moreover, we have

$$\left|D_z \cap \mathbb{R}_+^2\right| \leq \frac{\pi}{2} \left|\widetilde{S}_{j,k_1}\right| < 4 \left|D_z \cap \mathbb{R}_+^2\right|.$$

Therefore, at least one such special square, say $\widetilde{S}_{j_1,k_1}$, satisfies

$$\int_{D_z \cap \mathbb{R}_+^2 \cap \widetilde{S}_{j_1,k_1}} f(\zeta)\, dA(\zeta) > \frac{1}{3}\alpha \left|D_z \cap \mathbb{R}_+^2\right|.$$

So we obtain

$$\int_{\widetilde{S}_{j_1,k_1}} f(\zeta)\, dA(\zeta) > \frac{\pi}{24}\alpha \left|\widetilde{S}_{j_1,k_1}\right| > \frac{\alpha}{16} \left|\widetilde{S}_{j_1,k_1}\right|.$$

From our construction of the sequence $\{\widetilde{S}_l\}$, $\widetilde{S}_{j_1,k_1}$ cannot be any of those satisfying (3.1), so $\widetilde{S}_{j_1,k_1}$ is contained in one of the special squares $\{\widetilde{S}_l\}$. Since $\widetilde{S}_{j_1,k_1}$ intersects D_z, we must have $z \in D_z \subset \widetilde{S}_{j_1,k_1}^* \subset \widetilde{S}_{l_1}^*$ for some l_1. By the definition of $\{W_l\}$, if z is not in $W_1, \ldots, W_{l_1}$, then z must be in $\widetilde{S}_{m_1}$ for some m_1. Hence $z \in W_{m_1}$, which implies (d). $\square$

3.3 A mapping property of $\widetilde{M}^+$

Now we introduce an interesting mapping property of the special maximal operator $\widetilde{M}^+$.

Lemma 3.3. *Let f be a measurable funciton on $\mathbb{R}^2_+$. Then either $\widetilde{M}^+(f)(z) = \infty$ for all $z \in \mathbb{R}^2_+$, or $\widetilde{M}^+(f)(z) < \infty$ for all $z \in \mathbb{R}^2_+$.*

Proof. Assuming that $\widetilde{M}^+(f)(z) = \infty$ for some $z \in \mathbb{R}^2_+$, we show that $\widetilde{M}^+(f)(z') = \infty$ for any $z' \in \mathbb{R}^2_+$. By definition, there is a sequence of special disks $\{D_n\}$ with $z \in D_n$ and

$$\frac{1}{\left|D_n \cap \mathbb{R}^2_+\right|} \int_{D_n \cap \mathbb{R}^2_+} |f(\zeta)|\; dA(\zeta) > n \tag{3.3}$$

for all $n \in \mathbb{Z}^+$. Let r_n be the radius of D_n, and let x_n be the center. Then $D_n = D_{r_n}(x_n)$. Since $z \in D_n$, we see $r_n > \Im(z) > 0$.

If $\{r_n\}$ is not bounded above, then by selecting a subsequence, we may assume that $\lim r_n = \infty$. Then, given any $z' \in \mathbb{R}^2_+$, we have $r_n \geq |z' - z|$ for n sufficiently large. In this case, it is easy to see that $z' \in D_{2r_n}(x_n)$, the special disk centered at x_n with radius $2r_n$. From (3.3), we see that

$$\frac{1}{\left|D_{2r_n}(x_n) \cap \mathbb{R}^2_+\right|} \int_{D_{2r_n}(x_n) \cap \mathbb{R}^2_+} |f(\zeta)|\; dA(\zeta) > \frac{1}{4} n$$

for n sufficiently large. This implies $\widetilde{M}^+(f)(z') = \infty$.

If $\{r_n\}$ is bounded above, then by selecting a subsequence, we may assume that $\lim r_n = r$ for some r with $\Im(z) \leq r < \infty$. Note that, since $z \in D_n$, we have $\Re(z) \in D_n$, so $D_n \subset D_{3r}(\Re(z))$ for n sufficiently large, where $D_{3r}(\Re(z))$ is a special disk centered at $\Re(z)$ with radius $3r$. Therefore, from (3.3), we see that

$$\int_{D_{3r}(\Re(z)) \cap \mathbb{R}^2_+} |f(\zeta)|\; dA(\zeta) > \frac{1}{2} n \pi r_n^2 \geq cn,$$

where $c = \frac{1}{2}\pi \Im(z) > 0$, for n sufficiently large. So we obtain

$$\int_{D_{3r}(\Re(z)) \cap \mathbb{R}^2_+} |f(\zeta)|\; dA(\zeta) = \infty.$$

Now, for any $z' \in \mathbb{R}^2_+$, it is easy to see that $z' \in D_{3r+|z'-z|}(\Re(z))$, the special disk centered at $\Re(z)$ with radius $3r + |z' - z|$. From the equality above, we have

$$\frac{1}{\left|D_{3r+|z'-z|}(\Re(z)) \cap \mathbb{R}^2_+\right|} \int_{D_{3r+|z'-z|}(\Re(z)) \cap \mathbb{R}^2_+} |f(\zeta)|\; dA(\zeta) = \infty,$$

which implies $\widetilde{M}^+(f)(z') = \infty$. This completes the proof. $\square$

4 Construction of A_1^+ weights

Now we are ready to apply $\widetilde{M}^+$ to construct $A_1^+(\mathbb{R}_+^2)$ weights.

PROOF OF THEOREM 1.8. It suffices to show the conclusion for $f \geq 0$. By Lemma 3.3, we can assume $\widetilde{M}^+(f)(z) < \infty$ for all $z \in \mathbb{R}_+^2$; otherwise, the conclusion is trivial. If $\widetilde{M}^+(f)(z) = 0$ for some $z \in \mathbb{R}_+^2$, then it is easy to see that $f = 0$ on $\mathbb{R}_+^2$. In this case, the conclusiong is trivial again. So we may assume that $0 < \widetilde{M}^+(f) < \infty$ on $\mathbb{R}_+^2$.

We use an analogue of the argument in [11, Chapter 5.2]. Let $\mu(z) = \big(\widetilde{M}^+(f)(z)\big)^q$; we show that $\widetilde{M}^+(\mu)(z) \leq c\mu(z)$. That implies $\mu \in A_1^+(\mathbb{R}_+^2)$.

Fixing $z \in \mathbb{R}_+^2$, we normalize f by dividing by $\widetilde{M}^+(f)(z)$, so we may assume that $\widetilde{M}^+(f)(z) = 1$ and $\mu(z) = 1$. Hence it suffices to show that there is a $c > 0$ such that

$$\frac{1}{\left|D \cap \mathbb{R}_+^2\right|} \int_{D \cap \mathbb{R}_+^2} \mu(\zeta)\, dA(\zeta) \leq c \tag{4.1}$$

for any special D containing z.

Given a special disk $D = D_R(x_0)$ containing z, let $f_1 = \chi_{D_{2R}(x_0) \cap \mathbb{R}_+^2} f$ and $f_2 = f - f_1$. We first deal with f_1. Let $V_\alpha = \{\zeta \in D \cap \mathbb{R}_+^2 : \widetilde{M}^+(f_1)(\zeta) > \alpha\}$. We have

$$\begin{aligned} \int_{D \cap \mathbb{R}_+^2} \left(\widetilde{M}^+(f_1)(\zeta)\right)^q dA(\zeta) &= \int_0^\infty q\alpha^{q-1} \left|V_\alpha\right| d\alpha \\ &= \int_0^1 + \int_1^\infty q\alpha^{q-1} \left|V_\alpha\right| d\alpha. \end{aligned} \tag{4.2}$$

Since $|V_\alpha| \leq |D|$, we see that the first integral of (4.2) is bounded by cR^2. For the second integral, since $\widetilde{M}^+(f)(z) = 1$ and $z \in D_{2R}(x_0) \cap \mathbb{R}_+^2$, we see that

f_1 is integrable on $\mathbb{R}^2_+$. By Lemma 3.2, we have

$$\begin{aligned}
|V_\alpha| &\le \sum_l \left|\widetilde{S}_l^*\right| = \sum_l 25 \left|\widetilde{S}_l\right| \\
&\le c \sum_l \frac{1}{\alpha} \int_{W_l} f_1(\eta)\, dA(\eta) \\
&\le \frac{c}{\alpha} \int_{\mathbb{R}^2_+} f_1(\eta)\, dA(\eta) \\
&= \frac{c}{\alpha} \int_{D_{2R}(x_0)\cap\mathbb{R}^2_+} f(\eta)\, dA(\eta) \\
&\le \frac{cR^2}{\alpha} \widetilde{M}^+(f)(z) \\
&= \frac{cR^2}{\alpha}.
\end{aligned}$$

So the second integral of (4.2) is bounded by cR^2. Hence, so is (4.2).

Next we deal with f_2. For any $\zeta \in D \cap \mathbb{R}^2_+$, we consider an arbitrary special disk D'_r that contains ζ and whose radius is r. It is easy to see that $D'_r \subset D_{2r+R}(x_0)$. When $2r < R$, we have $D'_r \subset D_{2r+R}(x_0) \subset D_{2R}(x_0)$. Since f_2 vanishes on $D_{2R}(x_0) \cap \mathbb{R}^2_+$, we see that

$$\frac{1}{\left|D'_r \cap \mathbb{R}^2_+\right|} \int_{D'_r\cap\mathbb{R}^2_+} f_2(\eta)\, dA(\eta) = 0 < c,$$

for any $c > 0$. When $2r \ge R$, then $(2r+R)^2 \le 16r^2$, so we have

$$\begin{aligned}
\int_{D'_r\cap\mathbb{R}^2_+} f_2(\eta)\, dA(\eta) &\le \int_{D_{2r+R}(x_0)\cap\mathbb{R}^2_+} f(\eta)\, dA(\eta) \\
&\le \left|D_{2r+R}(x_0) \cap \mathbb{R}^2_+\right| \widetilde{M}^+(f)(z) \\
&= c(2r+R)^2 \\
&\le cr^2,
\end{aligned}$$

since $z \in D \subset D_{2r+R}(x_0)$. In either case, we obtain

$$\frac{1}{\left|D'_r \cap \mathbb{R}^2_+\right|} \int_{D'_r\cap\mathbb{R}^2_+} f_2(\eta)\, dA(\eta) < c.$$

Since D'_r is arbitrary, we see $\widetilde{M}^+(f_2)(\zeta) \le c$ for any $\zeta \in D \cap \mathbb{R}^2_+$. Therefore, we obtain

$$\int_{D\cap\mathbb{R}^2_+} \left(\widetilde{M}^+(f_2)(\zeta)\right)^q dA(\zeta) \le cR^2. \tag{4.3}$$

Combining (4.3) with the fact that (4.2) is bounded by cR^2, we see that

$$\int_{D\cap\mathbb{R}^2_+} \left(\widetilde{M}^+(f)(\zeta)\right)^q \, dA(\zeta) \le cR^2,$$

which implies (4.1). This completes the proof. □

5 The weak-type (p, p) estimate

Following the idea in [6], we now investigate the weak-type (p, p) mapping property of the special maximal operator $\widetilde{M}^+$.

Proof of Theorem 1.10. We use an analogue of the argument in [6, Theorem 8]. For the sufficient part, we only need to prove (1.1) for integrable $f \ge 0$. To see this, note that any measurable function $f \ge 0$ on $\mathbb{R}^2_+$ can be approximated by the increasing sequence $\{f\chi_{D_R}\}_{R>0}$, where $D_R = D_R(x_0)$ is a sequence of special disks centered at $x_0 \in \mathbb{R}$ with radius R. Note that the set $\{z \in \mathbb{R}^2_+ : \widetilde{M}^+(f)(z) > \alpha\}$ is the increasing union of the same sets formed with the $f\chi_{D_R}$'s. If we prove (1.1) for $f\chi_{D_R}$, then the monotonic convergent theorem will imply (1.1) for f. Note that any $f\chi_{D_R}$ can be approximated by an increasing sequence of simple functions. Since the support of $f\chi_{D_R}$ is bounded, these simple functions are integrable. By the same limiting argument, (1.1) for integrable functions will imply (1.1) for $f\chi_{D_R}$.

Let $V_\alpha = \{z \in \mathbb{R}^2_+ : \widetilde{M}^+(f)(z) > \alpha\}$. By Lemma 3.2, for $p > 1$, we have

$$\begin{aligned}
\mu_1(V_\alpha) &\le \sum_l \mu_1(W_l) \\
&\le \sum_l \mu_1(W_l) \left(\frac{16}{\alpha|\widetilde{S}_l|}\int_{W_l} f(z)\, dA(z)\right)^p \\
&\le \sum_l \frac{c}{\alpha^p}\frac{\mu_1(W_l)}{|\widetilde{S}_l|}\int_{W_l} f(z)^p \mu_2(z)\, dA(z) \left(\frac{1}{|\widetilde{S}_l|}\int_{W_l} \mu_2(z)^{-\frac{p'}{p}}\, dA(z)\right)^{p/p'} \\
&\le \sum_l \frac{c}{\alpha^p}\int_{W_l} f(z)^p \mu_2(z)\, dA(z) \frac{\mu_1(\widetilde{S}_l^*)}{|\widetilde{S}_l^*|}\left(\frac{1}{|\widetilde{S}_l^*|}\int_{\widetilde{S}_l^*} \mu_2(z)^{-\frac{p'}{p}}\, dA(z)\right)^{p/p'} \\
&\le \sum_l \frac{c}{\alpha^p}\int_{W_l} f(z)^p \mu_2(z)\, dA(z) \\
&\le \frac{c}{\alpha^p}\int_{\mathbb{R}^2_+} f(z)^p \mu_2(z)\, dA(z).
\end{aligned}$$

For $p = 1$, similarly, we have

$$\begin{aligned}\mu_1(V_\alpha) &\le \sum_l \frac{16\mu_1(W_l)}{\alpha|\widetilde{S}_l|}\int_{W_l} f(z)\,dA(z)\\ &\le \sum_l \frac{c}{\alpha}\frac{\mu_1(W_l)}{|\widetilde{S}_l|}\int_{W_l} f(z)\mu_2(z)\,dA(z)\left(\inf_{z\in W_l}\mu_2(z)\right)^{-1}\\ &\le \sum_l \frac{c}{\alpha}\int_{W_l} f(z)\mu_2(z)\,dA(z)\frac{\mu_1(\widetilde{S}_l^*)}{|\widetilde{S}_l^*|}\left(\inf_{z\in \widetilde{S}_l^*}\mu_2(z)\right)^{-1}\\ &\le \frac{c}{\alpha}\int_{\mathbb{R}^2_+} f(z)\mu_2(z)\,dA(z).\end{aligned}$$

For the necessary part, we first consider $p > 1$. Given any special disk D, we assume that $\int_{D\cap\mathbb{R}^2_+}\mu_2(z)^{-\frac{p'}{p}}\,dA(z) = \infty$. Then, by duality of the space $L^p(D\cap\mathbb{R}^2_+)$, there is a $g \in L^p(D\cap\mathbb{R}^2_+)$ such that $\int_{D\cap\mathbb{R}^2_+} g(z)\mu_2(z)^{-\frac{1}{p}}\,dA(z) = \infty$. Let $f = g\mu_2^{-\frac{1}{p}}\chi_{D\cap\mathbb{R}^2_+}$ on $\mathbb{R}^2_+$. Then $\widetilde{M}^+(f)(z) = \infty$ for all $z \in D\cap\mathbb{R}^2_+$. So (1.1) gives $\mu_1(D\cap\mathbb{R}^2_+) = 0$, which contradicts the assumption $\mu_1 > 0$ almost everywhere. We also exclude the trivial case $\mu_2 = \infty$ on $D\cap\mathbb{R}^2_+$ to see that indeed we have $0 < \int_{D\cap\mathbb{R}^2_+}\mu_2(z)^{-\frac{p'}{p}}\,dA(z) < \infty$.

Take $f = \mu_2^{-\frac{p'}{p}}\chi_{D\cap\mathbb{R}^2_+}$ and $\alpha = \frac{1}{|D\cap\mathbb{R}^2_+|}\int_{D\cap\mathbb{R}^2_+}\mu_2(z)^{-\frac{p'}{p}}\,dA(z)$ in (1.1). We see that

$$\begin{aligned}\mu_1(D\cap\mathbb{R}^2_+) &\le \frac{c}{\alpha^p}\int_{D\cap\mathbb{R}^2_+}\mu_2(z)^{-p'}\mu_2(z)\,dA(z)\\ &= \frac{c\left|D\cap\mathbb{R}^2_+\right|}{\alpha^{p-1}},\end{aligned}$$

which is equivalent to $(\mu_1,\mu_2) \in A_p^+(\mathbb{R}^2_+)$.

When $p = 1$, given any special disk D, we exclude the trivial case $\inf\mu_2 = \infty$, where the infimum is taken over all $z \in D\cap\mathbb{R}^2_+$. Then, for any $\epsilon > 0$, there must be a measurable set $U \subset D\cap\mathbb{R}^2_+$ with $|U| > 0$, so that $\mu_2(z) < \epsilon + \inf\mu_2$ on U.

Taking $f = \chi_U$ and $\alpha = \frac{|U|}{|D \cap \mathbb{R}^2_+|}$ in (1.1), we see that

$$\begin{aligned}\mu_1(D \cap \mathbb{R}^2_+) &\leq \frac{c\,\big|D \cap \mathbb{R}^2_+\big|}{|U|} \int_U \mu_2(z)\, dA(z) \\ &\leq c\,\big|D \cap \mathbb{R}^2_+\big| \left(\epsilon + \inf_{z \in D \cap \mathbb{R}^2_+} \mu_2(z)\right).\end{aligned}$$

Letting $\epsilon \to 0^+$, we see that the inequality above is equivalent to $(\mu_1, \mu_2) \in A_1^+(\mathbb{R}^2_+)$. This completes the proof. $\square$

6 Concluding remarks

1. For any measurable $f \geq 0$, we do not have the basic inequality $f \leq \widetilde{M}^+(f)$. Due to this special feature of the special maximal operator, some classical results, such as the reverse Hölder inequality or the factorization of weights, may fail in the A_p^+ setting. Hence, in order to obtain the weighted L^p and BMO estimates, or even to prove Conjecture 1.12 and Conjecture 1.13, we need to develop additional analysis on the behavior of this special maximal operator along the boundary.

2. It is natural to ask: what is the n-dimensional analogue of the Hilbert integral? This type of operator behaves like the Bergman projection on a general domain; see also [5]. It will be very interesting if we can apply the higher dimensional analogue to the function theory of several complex variables in the future.

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