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# A SUFFICIENT CONDITION FOR A BOUNDED SET OF POSITIVE LEBESGUE MEASURE IN $\mathbb{R}^2$ OR $\mathbb{R}^3$ TO CONTAIN ITS CENTROID

**Abstract**

In this paper, we give a sufficient condition for a domain in either two- or three-dimensional Euclidean space to contain its centroid. We show that the condition is sharp. The condition is not, however, necessary.

## 1 Introduction

It is a matter of some interest to understand when the centroid of a region in space actually lies in that region. Experts in robotics need this type of information, just because the *balance* of a robot is an essential consideration. The question is also of some considerable interest from a purely mathematical point of view.

These ideas were first explored in [1]. We continue that research here. In particular, we find a geometrically elegant sufficient condition for a region to contain its centroid. This condition is, unfortunately, not necessary. But, the question that we begin to answer here has been open for a number of years, and what we present is a notable first step. And, we are able to show that our result is sharp.

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## 2 Some preliminaries

We begin with some definitions and notational conventions.

**Definition 1.** Let $\lambda = \lambda_N$ be the Lebesgue measure on $\mathbb{R}^N$.

**Definition 2.** Let $\pi_i : \mathbb{R}^N \to \mathbb{R}$ be the $i$th natural coordinate function, given by $\pi_i((p_1, \ldots, p_N)) = p_i$.

**Definition 3.** Let $U$ be a bounded Lebesgue measurable subset of $\mathbb{R}^N$ with $\lambda(U) > 0$, and for each $i \in \{1, 2, \ldots, N\}$, define

$$\overline{x_i} = \frac{\int_U \pi_i d\lambda}{\lambda(U)}.$$

Then the *centroid* of $U$, denoted $C(U)$, is given by $C(U) = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_N})$.

**Definition 4.** Let $d = d_N$ be the Euclidean distance in $\mathbb{R}^N$. That is, for any $p, q \in \mathbb{R}^N$, we have

$$d(p,q) = \sqrt{\sum_{i=1}^{N} (\pi_i(q) - \pi_i(p))^2}.$$

Additionally, let $||\cdot||$ be the usual norm on $\mathbb{R}^N$, so that $d(p,q) = ||p - q||$.

**Definition 5.** If $p \in \mathbb{R}^N$ and $r > 0$, then let $B(p, r)$ be the open (Euclidean) ball with center $p$ and radius $r$.

**Definition 6.** For each positive integer $N$, let $\Sigma_N$ denote the volume of the unit ball in $\mathbb{R}^N$; i.e., $\Sigma_N = \lambda(B(0, 1))$.

Roughly speaking, our goal in this paper is to show that, if $U$ is a bounded subset of $\mathbb{R}^N$ with positive Lebesgue measure, and if the ratio of the radius of the smallest ball containing $U$ to the radius of the largest ball contained in $U$ is sufficiently small, then $U$ must contain its centroid. By "sufficiently small," we mean that this ratio is less than a value $a_N$, which is dependent only on the dimension $N$ of the space in which $U$ is embedded. We will present a technique for calculating these values, and we will see that they are sharp, in the sense that the result would not hold if we replaced $a_N$ with $a_N + \epsilon$ for any $\epsilon > 0$. However, we will only calculate $a_N$ here in the particular cases where $N = 2$ or 3, since the general case is unwieldy. In fixed higher dimensions, we believe that our technique can still be used, though unfortunately the calculations become prohibitive.

Let us conclude this section by stating our main result. Technical artifacts in the statement of the theorem will be explained in the next section.

**Theorem 7.** *Let $U$ be a bounded, Lebesgue-measurable subset of $\mathbb{R}^2$ (respectively, of $\mathbb{R}^3$) with $\lambda(U) > 0$. Suppose that that the largest open ball contained in $U$ has radius $r_1$ and the smallest closed ball containing $U$ has radius $r_2$. Then, $C(U) \in U$ if $r_2/r_1 < a_2$, where $a_2 \approx 1.82001$ is the unique positive zero of the function*

$$\begin{aligned} f_2(x) =& \frac{2}{3}(4(x-1))^{\frac{3}{2}} - \pi \\ & - (2-x)\left(\frac{\pi}{2}x^2 - (2-x)\sqrt{4(x-1)} - x^2 \sin^{-1}(\frac{2}{x} - 1)\right) \end{aligned}$$

*(respectively, if $r_2/r_1 < a_3$, where $a_3 \approx 1.71667$ is the unique positive real root of the polynomial $f_3(x) = x^4 - 2x^3 + 2x - 2$).*

## 3 Some technical matters

Before we get to the heart of the matter, though, we will pause to investigate a somewhat subtle detail. In the previous paragraphs, we mentioned the "largest ball" (call it $B$) that is contained in $U$. Here, we are of course referring to an open ball $B$ whose radius is greater than or equal to the radius of any other ball that is contained in $U$. But one may ask whether such a largest ball must even exist. Certainly, if $U$ is allowed to be unbounded, there may be no such thing—consider $U = \mathbb{R}^2 \setminus \{(0,0)\}$, or $U = \{(x,y) \in \mathbb{R}^2 : x \geq 1, y \geq 0, \text{ and } y \leq 1 - \frac{1}{x}\}$. But our paper deals with centroids and it is therefore natural to consider only bounded sets, since the centroid of an unbounded set may not even exist. As it turns out, the requirement that $U$ be bounded is enough to ensure that such a largest open ball $B$ does indeed exist. The following lemma formalizes this notion.[1]

**Lemma 8.** *Let $U$ be a bounded subset of $\mathbb{R}^N$. Then there exist $p \in U$ and $r \geq 0$ such that:*

- $B(p, r) \subseteq U$
- *if* $B(p', r') \subseteq U$, *then* $r \geq r'$.

*We will refer to $B(p, r)$ as "the largest open ball contained in $U$."*[2]

[1] It is worth noting that we could ease the need for these first two lemmas by modifying the main argument of this paper to assume merely that $U$ has a containing ball and a contained ball of sufficiently similar radii, thereby avoiding the use of the terms "smallest" and "largest" altogether. But we find these matters to be interesting in their own right, and we believe that they ultimately lead to a more satisfying theorem.

[2] It may be more accurate to use the article "a" instead of "the" here, since the largest ball may not be unique. However, for ease of exposition, we do not do so in this paper.

Proof. The case $U = \emptyset$ is trivial, so we assume that $U$ is nonempty. Let $f : U \to \mathbb{R}$ be given by $f(p) = \sup\{r \in \mathbb{R} : B(p,r) \subseteq U\}$, and define $r = \sup\{f(p) : p \in U\}$. Certainly $r < \infty$ since $U$ is bounded.

First, observe that for any $u \in U$, we actually do have $B(u, f(u)) \subseteq U$, since for each $v \in B(u, f(u))$, we can set $t = \frac{f(u)+d(u,v)}{2}$ so that $v \in B(u,t)$, and then, since $t < f(u)$, we know from the definition of $f(u)$ that there must be some $t' > t$ for which $B(u,t') \subseteq U$; hence, $v \in U$. We will make use of this fact toward the end of the proof.

Now let $(r_i)_{i\geq 0}$ be a sequence of values from $\{f(p) : p \in U\}$ such that $r_i \to r$ as $i \to \infty$, and choose $(p_i)_{i\geq 0}$ so that $f(p_i) = r_i$ for each $i \geq 0$. Since $U$ is bounded, we know from the Bolzano-Weierstrass theorem that there is some subsequence $(p_{i_j})_{j\geq 0}$ and some $p \in \overline{U}$ such that $p_{i_j} \to p$ as $j \to \infty$. Note also that the corresponding sequence $(r_{i_j})_{j\geq 0}$ converges to $r$, since $(r_i)_{i\geq 0}$ does.

Now choose any point $q \in B(p,r)$. Then $d(p,q) = r - \epsilon$ for some $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $j \geq N$, we have $d(p, p_{i_j}) < \frac{\epsilon}{2}$, and choose $M \in \mathbb{N}$ such that for all $k \geq M$, we have $r - r_{i_k} < \frac{\epsilon}{2}$. Let $L = \max\{N, M\}$. Then $d(p_{i_L}, q) \leq d(p_{i_L}, p) + d(p,q) < \frac{\epsilon}{2} + r - \epsilon = r - \frac{\epsilon}{2} < r_{i_L}$. Therefore, $q \in B(p_{i_L}, r_{i_L})$.

Since $r_{i_L} = f(p_{i_L})$, we know from our observation that $B(p_{i_L}, r_{i_L}) \subseteq U$, and hence $q \in U$. Since $q$ was chosen arbitrarily from $B(p,r)$, this shows that $B(p,r) \subseteq U$. Finally, it is clear that if $B(p',r') \subseteq U$, then $r' \leq f(p') \leq r$. So, the proof is complete. $\square$

We are also concerned with the parallel notion of the "smallest ball" (call it $B'$) containing $U$. Here, we can quickly see that even when $U$ is bounded, there may be no smallest containing open ball—consider $U = \{(0,0)\}$. However, if we make $B'$ a *closed* ball, the problem is solved. Throughout the following lemma—and indeed, the rest of this paper—we will denote the closure of any set $S$ by $\overline{S}$. The appropriate topology should be clear from context.

**Lemma 9.** *Let $U$ be a bounded subset of $\mathbb{R}^N$. Then there exist $p \in U$ and $r \geq 0$ such that:*

- $U \subseteq \overline{B(p,r)}$
- *if $U \subseteq \overline{B(p',r')}$, then $r \leq r'$, with equality only if $U = \emptyset$ or $p' = p$.*

*We will refer to $\overline{B(p,r)}$ as "the smallest closed ball containing $U$."*

Proof. The idea here is very similar to that of the previous proof.

Again, we assume for the duration of the proof that $U \neq \emptyset$. Since $U$ is bounded, we can choose some $w \in \mathbb{R}^N$ and some $s \in \mathbb{R}^+$ such that $U \subseteq B(w,s)$.

Let $f : B(w, 2s) \to \mathbb{R}$ be given by $f(p) = \inf\{r \in \mathbb{R} : U \subseteq \overline{B(p, r)}\}$, and define $r = \inf\{f(p) : p \in B(w, 2s)\}$.

First, we make a helpful observation. Suppose that $u \in B(w, 2s)$. Choose $v \in \mathbb{R}^N \setminus \overline{B(u, f(u))}$, and set $t = \frac{f(u)+d(u,v)}{2}$ so that $v \notin \overline{B(u, t)}$ despite the fact that $t > f(u)$. Then, from the definition of $f(u)$, we know that $t$ is not a lower bound of the set $\{r \in \mathbb{R} : U \subseteq \overline{B(u, r)}\}$, so there must be some $t' < t$ such that $U \subseteq \overline{B(u, t')}$. But of course $\overline{B(u, t')} \subseteq \overline{B(u, t)}$, so $U \subseteq \overline{B(u, t)}$ and hence $v \notin U$. Therefore, it follows that $U \subseteq \overline{B(u, f(u))}$, which is the observation we were seeking.

Now, let $(r_i)_{i \geq 0}$ be a sequence of values from $\{f(p) : p \in B(w, 2s)\}$ such that $r_i \to r$ as $i \to \infty$, and choose $(p_i)_{i \geq 0}$ so that $f(p_i) = r_i$ for each $i \geq 0$. By the Bolzano-Weierstrass theorem, there is some subsequence $(p_{i_j})_{j \geq 0}$ and some $p \in \overline{B(w, 2s)}$ such that $p_{i_j} \to p$ as $j \to \infty$. Note also that the corresponding sequence $(r_{i_j})_{j \geq 0}$ converges to $r$, since $(r_i)_{i \geq 0}$ does.

Next, choose any point $q \in \mathbb{R}^N \setminus \overline{B(p, r)}$. Then $d(p, q) = r + \epsilon$ for some $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $j \geq N$, we have $d(p, p_{i_j}) < \epsilon/2$, and choose $M \in \mathbb{N}$ such that for all $k \geq M$, we have $r_{i_k} - r < \epsilon/2$. Let $L = \max\{N, M\}$. Then, from the triangle inequality, we have $d(p_{i_L}, q) \geq d(p, q) - d(p, p_{i_L}) > r + \epsilon - \frac{\epsilon}{2} = r + \frac{\epsilon}{2} > r_{i_L}$. Therefore, $q \notin \overline{B(p_{i_L}, r_{i_L})}$. Finally, since $r_{i_L} = f(p_{i_L})$, we know from our first observation that $U \subseteq \overline{B(p_{i_L}, r_{i_L})}$, and hence $q \notin U$. It follows that $U \subseteq \overline{B(p, r)}$.

Suppose that $U \subseteq \overline{B(p', r')}$. Then $r' \geq f(p')$. If $p' \in B(w, 2s)$, then it is clear that $f(p') \geq r$, and hence $r' \geq r$. Otherwise, if $p' \notin B(w, 2s)$, then $\overline{B(p', s)} \cap U = \emptyset$, so $f(p') > s$. Since $s \geq f(w) \geq r$, we see that in this case, too, $r' \geq r$.

Now, we want to show that this inequality is in fact strict in all non-trivial cases. Suppose that, for some $p' \neq p$, we have $U \subseteq \overline{B(p', r)}$, and write $p'' = \frac{p+p'}{2}$ and $r'' = \sqrt{r^2 - \frac{d(p,p')^2}{4}}$. Choose $q \in \mathbb{R}^N \setminus \overline{B(p'', r'')}$. Suppose that the angle $\alpha = \angle pp''q$ is between $\frac{\pi}{2}$ and $\pi$, inclusive. Then from the law of cosines, we know that $d(p, q)^2 = d(p, p'')^2 + d(p'', q)^2 - 2d(p, p'')d(p'', q)\cos\alpha \geq d(p, p'')^2 + d(p'', q)^2 > d(p, p'')^2 + r''^2 = d(p, p'')^2 + r^2 - \frac{d(p,p'')^2}{4} = r^2 + \frac{3}{4}d(p, p'')^2 > r^2$. Hence $d(p, q) > r$, so $q \notin \overline{B(p, r)}$, and therefore, $q \notin \overline{B(p, r)} \cap \overline{B(p', r)}$. If $\alpha \in [0, \pi/2)$, then $\angle p'p''q \in [\pi/2, \pi]$, and so the same argument works if we swap the roles of $p$ and $p'$. Thus, $q \notin \overline{B(p, r)} \cap \overline{B(p', r)}$, so it follows that $\overline{B(p, r)} \cap \overline{B(p', r')} \subseteq \overline{B(p'', r'')}$. And of course, since $U \subseteq \overline{B(p, r)} \cap \overline{B(p', r)}$, it follows that $U \subseteq \overline{B(p'', r'')}$. But this is a contradiction, since $r'' < r$. $\square$

Now that we have laid these matters to rest, we must take one more preliminary step before we are ready to move forward with our main argument.

The following little lemma will help us to construct the critical shape that we seek.

**Lemma 10.** *Let $B$ be an open $N$-ball with radius $r > 0$. Choose $p \in B$ and select a coordinate system so that $B$ has center $\mathbf{0}$ and $p$ has $x_1$-coordinate in the interval $[0, r)$ and $x_j$-coordinate $0$ for $j = 2, 3, \ldots, N$. Then there exists $L \in [-r, r)$ such that $V_L = \{q \in B : \pi_1(q) > L\}$ has centroid $p$. What is more, if $U \subset B$ also has centroid $p$, then $\lambda(V_L) \geq \lambda(U)$.*

Proof. For any $L$, it is clear that $\pi_j(C(V_L)) = 0$ for $j = 2, 3, \ldots, N$. We will use the Intermediate Value Theorem to show that $\pi_1(C(V_L)) = \pi_1(p)$.

Say $L_0 = -r$ and $L_1 = \pi_1(p)$. Then $\pi_1(C(V_{L_0})) = 0$ since $V_{L_0} = B$, and clearly $\pi_1(C(V_{L_1})) \geq \pi_1(p)$. We have therefore found two values $L_0$ and $L_1$ so that $\pi_1(C(V_{L_0})) \leq \pi_1(p) \leq \pi_1(C(V_{L_1}))$. For arbitrary $L \in [-r, r)$, we have

$$\pi_1(C(V_L)) = \frac{\int_L^r x \Sigma_{N-1} \left[\sqrt{r^2 - x^2}\right]^{N-1} dx}{\lambda(V_L)} .$$

This is the ratio of two continuous functions of $L$, so it is continuous at all points where the denominator is non-zero; i.e., on all of $[-r, r)$. So, by the Intermediate Value Theorem, there exists an $L$ such that $\pi_1(C(V_L)) = \pi_1(p)$, and hence, $C(V_L) = p$.

For the second part of the statement, suppose that $U$ and $V_L$ are as described. Translate the whole system so that $L = 0$. By Lemma 5.3 in [1], we can write

$$\begin{aligned} & \frac{\lambda(U \cap V_L)}{\lambda(U)} \pi_j(C(U \cap V_L)) + \frac{\lambda(U \setminus V_L)}{\lambda(U)} \pi_j(C(U \setminus V_L)) \\ = \quad & \frac{\lambda(U \cap V_L)}{\lambda(V_L)} \pi_j(C(U \cap V_L)) + \frac{\lambda(V_L \setminus U)}{\lambda(V_L)} \pi_j(C(V_L \setminus U)) . \end{aligned}$$

So, after rearranging, we have

$$\begin{aligned} & \lambda(V_L) - \lambda(U) = \\ & \qquad \frac{\lambda(U)\lambda(V_L \setminus U)\pi_j(C(V_L \setminus U)) - \lambda(V_L)\lambda(U \setminus V_L)\pi_j(C(U \setminus V_L))}{\lambda(U \cap V_L)\pi_j(C(U \cap V_L))} . \end{aligned} \tag{1}$$

But if $q \in V_L \setminus U$, then $\pi_1(q) > 0$, so $\pi_1(C(V_L \setminus U)) > 0$. Similarly, if $q \in U \cap V_L$, then $\pi_1(q) > 0$, so $\pi_1(C(U \cap V_L)) > 0$. Finally, if $q \in U \setminus V_L$, then $\pi_1(q) \leq 0$, so $\pi_1(C(U \setminus V_L)) \leq 0$. Thus, the righthand side of (1) is not less than zero. So $\lambda(V_L) \geq \lambda(U)$. □

Finally, we are ready to prove the theorem.

## 4 Proof of the main theorem

PROOF. We prove the equivalent contrapositive statement. We work at first in $\mathbb{R}^N$ in order to suggest a general technique for approaching this problem in any given dimension $N \in \mathbb{N}$, but toward the end of the proof we will specialize down to the physically-relevant cases $N = 2$ and $N = 3$ in order to facilitate computation.

Let $U$ be a bounded, Lebesgue-measurable subset of $\mathbb{R}^N$ with $\lambda(U) > 0$. Denote by $B_1$ the largest open ball contained in $U$, and by $B_2$ the smallest closed ball containing $U$. Let $r_1$ be the radius of $B_1$, let $r_2$ be the radius of $B_2$, and let $c_1$ and $c_2$ denote the centers of the balls $B_1$ and $B_2$, respectively. Let $U_1 = U \cap B_1 = B_1$ and let $U_2 = U \setminus U_1$. Say that $p = C(U)$, $p_1 = C(U_1) = c_1$, and $p_2 = C(U_2)$. Without loss of generality, suppose that $c_2 = \mathbf{0}$ and that $p_2$ lies on the nonnegative $x_1$-axis.

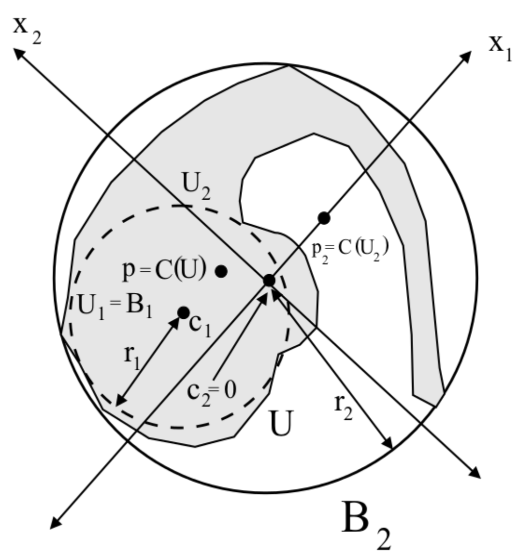


Figure 1: An example of a two-dimensional set $U$ and the various objects associated with it.

*Suppose that $U$ does not contain its centroid.* It follows that $p \notin B_1$. That is, $r_1 \le d(p_1, p)$. Since $U_1$ and $U_2$ are disjoint, Lemma 5.3 in [1] shows that

$$p = \frac{\lambda(U_1)}{\lambda(U_1) + \lambda(U_2)} p_1 + \frac{\lambda(U_2)}{\lambda(U_1) + \lambda(U_2)} p_2 \,.$$

So we have

$$\begin{aligned} r_1 &\le d\left(p_1, \frac{\lambda(U_1)}{\lambda(U_1) + \lambda(U_2)} p_1 + \frac{\lambda(U_2)}{\lambda(U_1) + \lambda(U_2)} p_2\right) \\ &= d\left(p_1, p_1 + \frac{\lambda(U_2)}{\lambda(U_1) + \lambda(U_2)}(p_2 - p_1)\right) \\ &= \left\| \frac{\lambda(U_2)}{\lambda(U_1) + \lambda(U_2)}(p_2 - p_1) \right\| \\ &= \frac{\lambda(U_2)}{\lambda(U_1) + \lambda(U_2)} \|p_2 - p_1\| \\ &= \frac{\lambda(U_2)}{\lambda(U_1) + \lambda(U_2)} d(p_1, p_2) \,. \end{aligned} \tag{2}$$

By Lemma 10, there exists an $L \in [-r_2, r_2)$ such that $V_L = \{q \in B_2 : \pi_1(q) > L\}$ has centroid $p_2$ and $\lambda(V_L) \ge \lambda(U_2)$. So, we have

$$\begin{aligned} \lambda(U_2) &\le \lambda(V_L) \\ \lambda(U_2)\lambda(U_1) &\le \lambda(V_L)\lambda(U_1) \\ \lambda(U_2)\lambda(U_1) + \lambda(U_2)\lambda(V_L) &\le \lambda(V_L)\lambda(U_1) + \lambda(U_2)\lambda(V_L) \\ \lambda(U_2)[\lambda(U_1) + \lambda(V_L)] &\le \lambda(V_L)[\lambda(U_1) + \lambda(U_2)] \\ \frac{\lambda(U_2)}{\lambda(U_1) + \lambda(U_2)} &\le \frac{\lambda(V_L)}{\lambda(U_1) + \lambda(V_L)} \,. \end{aligned}$$

So, from (2), we have

$$r_1 \le \frac{\lambda(V_L)}{\lambda(U_1) + \lambda(V_L)} d(p_1, p_2) \,. \tag{3}$$

Now we will put an upper bound on $d(p_1, p_2)$. Let $q_\epsilon = p_1 + \frac{p_1}{||p_1||}(r_1 - \epsilon)$. Then $d(p_1, q_\epsilon) = |r_1 - \epsilon|$, so when $\epsilon \in (0, 2r_1)$, we know that $q_\epsilon \in B_1$. Then, since $B_1 \subseteq B_2$, we have $d(\mathbf{0}, q_\epsilon) < r_2$. But $d(\mathbf{0}, q_\epsilon) = ||p_1|| + |r_1 - \epsilon| = d(\mathbf{0}, p_1) + |r_1 - \epsilon|$. Hence, we have $d(p_1, \mathbf{0}) < r_2 - |r_1 - \epsilon|$ for all $\epsilon \in (0, 2r_1)$, so $d(p_1, \mathbf{0}) \le r_2 - r_1$. Next, recall that $d(\mathbf{0}, p_2) = \pi_1(p_2)$. Applying the triangle inequality, we see that $d(p_1, p_2) \le d(p_1, \mathbf{0}) + d(\mathbf{0}, p_2) \le r_2 - r_1 + \pi_1(p_2)$. This is the upper bound we were seeking.

So, from (3), we have

$$r_1 \le \frac{\lambda(V_L)}{\lambda(U_1)+\lambda(V_L)}(r_2 - r_1 + \pi_1(p_2))\,. \tag{4}$$

After doing some easy manipulations, we arrive at the inequality

$$0 \le \pi_1(p_2)\lambda(V_L) + (r_2 - 2r_1)\lambda(V_L) - r_1\lambda(U_1)\,. \tag{5}$$

Since

$$\begin{aligned}\pi_1(p_2)\lambda(V_L) &= \int_{V_L} \pi_1 d\lambda \\ &= \int_L^{r_2} \Sigma_{N-1}\left[\sqrt{r_2^2 - x^2}\right]^{N-1} x\,dx \\ &= \frac{\Sigma_{N-1}}{N+1}(r_2^2 - L^2)^{(N+1)/2}\,,\end{aligned}$$

and from [2] we know that

$$\lambda(U_1) = r_1^N \frac{2\pi^{\frac{N}{2}}}{\Gamma(N/2)\cdot N}$$

(where $\Gamma$ is the usual gamma function), we can rewrite this inequality as

$$0 \le \frac{\Sigma_{N-1}}{N+1}(r_2^2 - L^2)^{(N+1)/2} + (r_2 - 2r_1)\lambda(V_L) - r_1^{N+1}\frac{2\pi^{\frac{N}{2}}}{\Gamma(N/2)\cdot N}\,. \tag{6}$$

Let the function $F : [-r_2, r_2] \to \mathbb{R}$, a function of $L$, be defined by

$$F(L) = \frac{\Sigma_{N-1}}{N+1}(r_2^2 - L^2)^{(N+1)/2} + (r_2 - 2r_1)\lambda(V_L) - r_1^{N+1}\frac{2\pi^{\frac{N}{2}}}{\Gamma(N/2)\cdot N}\,. \tag{7}$$

We would like to use our knowledge that $F$ is nonnegative to show that $r_2/r_1$ must be large. To do this, we first need to evaluate $\lambda(V_L)$ in terms of $L$. This is certainly doable, since

$$\begin{aligned}\lambda(V_L) &= \int_L^{r_2} \Sigma_{N-1}\left[\sqrt{r_2^2 - x^2}\right]^{N-1} dx \\ &= \Sigma_{N-1}\cdot \int_0^{\cos^{-1}(L/r_2)} r_2^N \sin^N\theta\, d\theta\end{aligned}$$

and

$$\int \sin^N \theta\, d\theta = \begin{cases} -\cos\theta \sum_{r=0}^{m-1} \frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2} \sin^{2r+1}\theta + \frac{(2m)!}{2^{2m}(m!)^2}\theta \\ \qquad \text{if } N = 2m \text{ is even} \\ -\cos\theta \sum_{r=0}^{m-1} \frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)!(r!)^2} \sin^{2r}\theta \\ \qquad \text{if } N = 2m+1 \text{ is odd}. \end{cases}$$

See [3] for formulas of this type.

However, proceeding in the general case would get extremely tedious, largely due to the cumbersome nature of this equation for $\int \sin^N \theta\, d\theta$. So, we instead specialize down to dimensions $N = 2$ and $N = 3$, and trust that the reader can extract the technique for approaching the problem when $N$ is an arbitrary fixed natural number.

### 4.1 The case $N = 2$

Let $\lambda$ denote two-dimensional Lebesgue measure here. After some simplification, we have

$$\lambda(V_L) = \frac{\pi}{2} r_2^2 - L\sqrt{r_2^2 - L^2} - r_2^2 \sin^{-1}\left(\frac{L}{r_2}\right).$$

Of course, we also know that

$$\lambda(U_1) = \pi r_1^2 .$$

Now let us expand $F(L)$ so that it is written in terms of $L$:

$$\begin{aligned} F(L) = {} & \frac{2}{3}(r_2^2 - L^2)^{\frac{3}{2}} \\ & + (r_2 - 2r_1)\left(\frac{\pi}{2} r_2^2 - L\sqrt{r_2^2 - L^2} - r_2^2 \sin^{-1}\frac{L}{r_2}\right) - \pi r_1^3 . \end{aligned}$$

We see that

$$\begin{aligned} F(2r_1 - r_2) = {} & \frac{2}{3}(4r_1(r_2 - r_1))^{\frac{3}{2}} - \pi r_1^3 \\ & - (2r_1 - r_2)\left(\frac{\pi}{2} r_2^2 - (2r_1 - r_2)\sqrt{4r_1(r_2 - r_1)} - r_2^2 \sin^{-1}(\frac{2r_1 - r_2}{r_2})\right). \end{aligned}$$

Now write $d = r_2/r_1$. So

$$\frac{1}{r_1^3}F(2r_1 - r_2) = \frac{2}{3}\left(\frac{1}{r_1}\sqrt{4r_1(r_2 - r_1)}\right)^3 - \pi$$
$$-(2-d)\left(\frac{\pi}{2}d^2 - (2-d)(\frac{1}{r_1})\sqrt{4r_1(r_2-r_1)} - d^2\sin^{-1}(\frac{2}{d} - 1)\right)$$
$$= \frac{2}{3}(4(d-1))^{\frac{3}{2}} - \pi$$
$$-(2-d)\left(\frac{\pi}{2}d^2 - (2-d)\sqrt{4(d-1)} - d^2\sin^{-1}(\frac{2}{d} - 1)\right).$$

We know that the lefthand side of the equation is nonnegative, so of course the righthand side must be as well. It follows that $d \geq a_2$, where $a_2$ is the unique positive zero of the righthand side, taken as a function of $d$. Using a calculator or computer algebra system, we can determine that $a_2 = 1.82001\ldots$.

### 4.2 The case $N = 3$

We now let $\lambda$ denote three-dimensional Lebesgue measure and set $N = 3$. Here we have

$$\lambda(V_L) = \int_L^{r_2} \Sigma_2\left(\sqrt{r_2^2 - x^2}\right)^2 dx = \pi\left(\frac{2r_2^3}{3} - Lr_2^2 + \frac{L^3}{3}\right),$$

and

$$\lambda(U_1) = \frac{4}{3}\pi r_1^3.$$

Now, tedious calculation shows that

$$F(L) = \frac{\pi}{4}(r_2^2 - L^2)^2 +$$
$$(r_1 - r_2)\frac{4\pi}{3}r_1^3 + (2r_1 - r_2)\left[\frac{4\pi}{3}r_1^3 + \pi\left(\frac{2r_2^3}{3} - Lr_2^2 + \frac{L^3}{3}\right)\right].$$

As a result,

$$\frac{1}{r_1^4}\cdot F(2r_1 - r_2) = \frac{4\pi}{3}\left(-2 + (r_2/r_1)^4 - 2(r_2/r_1)^3 + 2(r_2/r_1)\right).$$

Setting $d = r_2/r_1$, we can rewrite this last line as

$$\frac{1}{r_1^4}\cdot F(2r_1 - r_2) = \frac{4\pi}{3}\left[d^4 - 2d^3 + 2d - 2\right].$$

Since the lefthand side is positive, the righthand side must be as well, and hence $d \geq a_3$, where $a_3$ is the unique positive real root of the polynomial $d^4 - 2d^3 + 2d - 2$. We can use a calculator or computer algebra system to see that $a_3 = 1.71667\ldots$. □

## 5 Sharpness

The following corollaries show that our result is sharp in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

**Corollary 11.** *There exists a bounded Lebesgue-measurable $U \subset \mathbb{R}^2$ such that $C(U) \notin U$ even though the ratio of the radii of the smallest closed ball containing $U$ and the largest open ball contained in $U$ is equal to $a_2$.*

PROOF. Let $r_1 = 1$ and let $r_2 = a_2$. Let $p_1 = (r_1 - r_2, 0)$ and let $p_2 = (0, 0)$. Let $B_1 = B(p_1, r_1)$ and let $B_2 = \overline{B(p_2, r_2)}$. Let $U_1 = B_1$ and let $V = \{p \in B_2 : \pi_1(p) > 2r_1 - r_2\}$. Let $U = U_1 \cup V$. It is clear that $B_1$ is the largest open ball contained in $U$ and that $B_2$ is the smallest closed ball containing $U$. Hence, the relevant ratio is indeed equal to $a_2$.

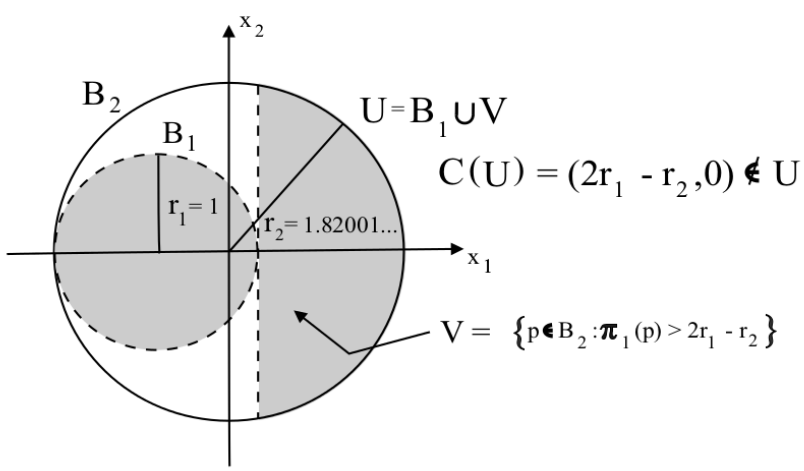


Figure 2: The "critical shape" in $\mathbb{R}^2$.

Now we will show that $U$ does not contain its centroid. Since $\frac{r_2}{r_1} = a_2$, we know from the end of Section 3 that $F(2r_1 - r_2) = 0$, where the function

$F(L)$ is as defined in Section 3. Using the fact that $V = V_{2r_1-r_2}$ and working backward through the proof, we arrive at equation (7) where we see that

$$0 = \frac{\Sigma_{N-1}}{N+1}(r_2^2 - (2r_1 - r_2)^2)^{(N+1)/2} + (r_2 - 2r_1)\lambda(V_{2r_1-r_2}) - r_1^{N+1}\frac{2\pi^{\frac{N}{2}}}{\Gamma(N/2)\cdot N}.$$

Continuing backward, the inequalities in (6), (5), and (4) become equalities, so that once we reach (4) we have

$$r_1 = \frac{\lambda(V)}{\lambda(U_1)+\lambda(V)}(r_2 - r_1 + \pi_1(p_2)).$$

A bit of manipulation yields

$$\begin{aligned} r_1 + \frac{\lambda(U_1)}{\lambda(U_1)+\lambda(V)}(r_1 - r_2) - \frac{\lambda(V)}{\lambda(U_1)+\lambda(V)}(r_2 - r_1) \\ = \frac{\lambda(V)}{\lambda(U_1)+\lambda(V)}\pi_1(p_2) + \frac{\lambda(U_1)}{\lambda(U_1)+\lambda(V)}(r_1 - r_2)\,, \end{aligned}$$

which we can simplify to

$$2r_1 - r_2 = \pi_1(C(U)),$$

by Lemma 5.3 in [1]. It follows $C(U) \notin U$. □

**Corollary 12.** *There exists a bounded Lebesgue-measurable $U \subset \mathbb{R}^3$ such that $C(U) \notin U$ even though the ratio of the radii of the smallest closed ball containing $U$ and the largest open ball contained in $U$ is equal to $a_3$.*

Proof. The proof is essentially identical to the above. □

## 6 Concluding remarks

It is clear that our sufficient condition is not necessary. In the plane, a long, thin rectangle certainly contains its centroid, yet the ratio of radii of minimal outer disc with maximal inner disc can be arbitarily large.

The condition that we do present here has intuitive appeal, and we suspect that an invariant version of this condition may be closer to necessary and sufficient.

We hope to explore necessary conditions in future work.

## References

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