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# ON STRONG QUASI-CONTINUITY OF FUNCTIONS OF TWO VARIABLES

**Abstract**

Some properties describing the strong quasicontinuity of functions of one and two variables are considered.

## Preliminaries

Let $\mathbb{R}$ be the set of all reals and let $E$ denote $\mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$. For $x \in E$ and for $r > 0$ let $K(x,r) = \{t \in X : |t-x| < r\}$. Moreover, let $\mu_e$ ($\mu$) be outer Lebesgue measure (Lebesgue measure) in $E$.

Denote by

$$d_u(A,x) = \limsup_{h \to 0} \mu_e(A \cap K(x,h))/\mu(K(x,h)),$$
$$(d_l(A,x) = \liminf_{h \to 0} \mu_e(A \cap K(x,h))/\mu(K(x,h)))$$

the upper (lower) outer density of $A \subset E$ at $x$. A $x \in E$ is called a density point of $A \subset E$ if there exists a measurable (in the sense of Lebesgue) set $B \subset A$ such that $d_l(B,x) = 1$. The family $\mathcal{T}_d = \{A \subset E$; $A$ is measurable and every $x \in A$ is a density point of $A\}$ is a topology called the density topology [2, 1, 7]. Moreover, let $\mathcal{T}_e$ denote the Euclidean topology in $E$.

## 1 Definitions and General Properties

A function $f : E \to \mathbb{R}$ has property $A(x)$ at a $x \in E$ (abbreviated $f \in A(x)$) if there is an open set $U$ such that $d_u(U,x) > 0$ and the restriction $f|(U \cup \{x\})$ is continuous at $x$. A function $f$ has property $B(x)$ at $x$ (abbreviated $f \in B(x)$)

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if for every $\eta > 0$ we have $d_u(\mathrm{int}(\{t : |f(t) - f(x)| < \eta\}), x) > 0$, where $\mathrm{int}(X)$ denotes the Euclidean interior of $X$.

A function $f$ is strongly quasicontinuous at $x$ (abbreviated $f$ is s.q.c. at $x$) (is strongly cliquish at $x$ (abbreviated $f$ is s.c.q. at $x$)) if for every $\eta > 0$ and for every $U \in \mathcal{T}_d$ such that $x \in U$ there is a nonempty open set $V$ such that $V \cap U \neq \emptyset$ and $|f(t) - f(x)| < \eta$ for all $t \in U \cap V$ ($\operatorname{osc} f < \eta$ on the set $U \cap V$) [3].

A function $f$ has the Denjoy-Clarkson property (abbreviated $f \in DCP$) if it is measurable and for all open sets $I \subset \mathcal{R}$, $J \subset E$ such that $J \cap f^{-1}(I) \neq \emptyset$ we have $\mu_e(J \cap f^{-1}(I)) > 0$.

Moreover, denote by $C(f)$ the set of all continuity points of $f$, by $Q_s(f)$ the set of all $x \in E$, at which $f$ is s.q.c., by $A(f)$ the set $\{x \in E; f \in A(x)\}$ and by $B(f)$ the set $\{x \in E; f \in B(x)\}$. Obviously, $C(f) \subset A(f) \subset B(f) \subset Q_s(f)$.

**Example 1** *Let $C \subset E$ be a closed, nowhere dense set with $\mu(C) > 0$. There is an isolated set $B \subset E \setminus C$ such that the closure* $\mathrm{cl}(B) \supset C$*. If $f$ is the characteristic function of the set $B$, then $f$ is s.q.c. at every point $x \in E \setminus B$, but $f$ doesn't have property $B(x)$ at any $x \in C$ which is a density point of the set $C$.*

**Remark 1** *There is an everywhere s.q.c. function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at every $x \neq 0$, and such that $f \notin A(0)$.*

PROOF. Let $\{I_{k,n} : k, n \in N\}$ ($N$ denotes the set of all positive integers) be a family of pairwise disjoint closed intervals such that

- $0 \notin I_{k,n}$ for $k, n \in N$,
- $d_l\big(\bigcup_{k\in N} I_{k,n}, 0\big) = 2^{-n}$ for $n \in N$.
- if $x_i \in I_{k_i,n_i}$ for $i \in N$, $(k_i, n_i) \neq (k_j, n_j)$ for $i \neq j$, $i, j \in N$, and $\lim_{i\to\infty} x_i = x$, then $x = 0$.

Such intervals $I_{k,n}$ exist, since in every interval $(1/(k+1), 1/k)$, $k \in N$, we can find disjoint closed intervals $J_{k,i}$, $i \leq n$, such that $\mu(J_{k,i}) = 2^{-i}/k(k+1)$ for $i \leq n$. Then every sequence $(I_{k,n})_{n\in\mathbb{N}}$ of all intervals $J_{k,n}$ and all intervals $-J_{k,n}$, $n \leq k$ and $k \in \mathbb{N}$, satisfies all required conditions.

Let

$$f(x) = \begin{cases} 1/n & \text{for } x \in I_{k,n},\ k, n \in \mathbb{N} \\ 0 & \text{for } x = 0 \\ \text{linear} & \text{otherwise.} \end{cases}$$

Then $f$ is continuous at every $x \neq 0$. Fix $\eta > 0$ and $A \in \mathcal{T}_d$ such that $0 \in A$. Let $n \in \mathbb{N}$ be such that $1/n < \eta$. Then $A \cap \bigcup_{k\in\mathbb{N}} I_{k,n} \neq \emptyset$ and there is $k$ such that $A \cap \operatorname{int}(I_{k,n}) \neq \emptyset$ and $|f(t) - f(0)| = f(t) = 1/n < \eta$ for $t \in A \cap \operatorname{int}(I_{k,n})$. So, $f$ is s.q.c. at 0.

Assume, to the contrary, that $f \in A(0)$. Then there is an open set $U$ such that $a = d_u(U, 0) > 0$ and $f|(U \cup \{0\})$ is continuous at 0. Fix $n_0$ such that $2^{-n_0} < a/2$. Then $d_u\big(\bigcup_{n\le n_0; k\in\mathbb{N}} I_{k,n}, 0\big) \geq 1 - a/2$ and consequently, for every open set $V$ with $0 \in V$ we have $V \cap U \cap \bigcup_{k\in\mathbb{N}; n\le n_0} I_{k,n} \neq \emptyset$. Since $f(t) \geq 1/n_0$ for each $t \in \bigcup_{k\in\mathbb{N}; n\le n_0} I_{k,n}$ and $f(0) = 0$, the restricted function $f|(U \cup 0)$ is not continuous at 0. So $f \notin A(0)$. □

**Remark 2** *Observe that for the function $f$ from the proof of Remark 1 we have $A(f) \neq B(f)$, since $f \in B(0)$.*

**Theorem 1** *Let $f \in DCP$. If $f$ is s.q.c. at $x \in E$, then $f \in B(x)$.*

PROOF. Assume, to the contrary, that $f \notin B(x)$. Then there is a $\eta > 0$ such that $d_u(\operatorname{int}(\{t : |f(t) - f(x)| < \eta\}), x) = 0$. Consequently, $d_l(\operatorname{cl}(\{t : |f(t) - f(x)| \geq \eta\}), x) = 1$, where $\operatorname{cl}(X)$ denotes the closure of $X$. Let $A \subset \operatorname{cl}(\{t : |f(t) - f(x)| \geq \eta\})$ belong to $\mathcal{T}_d$ with $d_l(A, x) = 1$. There is a countable set $B \subset \{t : |f(t) - f(x)| \geq \eta\}$ such that $A \subset \operatorname{cl}(B)$. Since $f \in DCP$, there is $H \subset \{t : |f(t) - f(x)| \geq \eta/2\}$ belonging to $\mathcal{T}_d$ such that $B \subset \operatorname{cl}(H)$. Then $A \subset \operatorname{cl}(B) \subset \operatorname{cl}(\operatorname{cl}(H)) = \operatorname{cl}(H)$ and $F = A \cup H \cup \{x\} \in \mathcal{T}_d$. Since $f$ is s.q.c. at $x$, there is an open set $U$ such that $U \cap F \neq \emptyset$ and $|f(t) - f(x)| < \eta/2$ for every $t \in U \cap F$, contrary to $U \cap H \neq \emptyset$ and $H \subset \{t : |f(t) - f(x)| \geq \eta/2\}$. □

**Corollary 1** *If $Q_s(f) = E$, then $B(f) = E$.*

**Remark 3** *The property DCP is well known in differentiation theory* [7] *and it can be considered also for nonmeasurable functions. Theorem* 1 *is true if measurability of $f$ is omitted.*

**Theorem 2** *Let $f : E \to \mathbb{R}$ and let $A \subset E$ satisfy $\mu(A \setminus B(f)) = 0$. Then $\mu(A \setminus C(f)) = 0$.*

PROOF. Assume, to the contrary, that $\mu_e(A \setminus C(f)) > 0$. Then there is an $\eta > 0$ such that $G = \{t \in A \cap B(f) : \operatorname{osc} f(t) \geq \eta\}$ is of positive outer measure. By the Lebesgue Density Theorem, $H = \{t : d_u(G, t) = 1\}$ is measurable and $H \in \mathcal{T}_d$. Fix $x \in H \cap G$. Since $f \in B(x)$, we have

$$d_u(\operatorname{int}(\{t : |f(t) - f(x)| < \eta/3\}), x) > 0.$$

So $G \cap \operatorname{int}(\{t : |f(t) - f(x)| < \eta/3\}) \neq \emptyset$. Let $u \in G \cap \operatorname{int}(\{t : |f(t) - f(x)| < \eta/3\})$. Since $u \in \operatorname{int}(\{t : |f(t) - f(x)| < \eta/3\}$, we obtain that $\operatorname{osc} f(u) \leq 2\eta/3$, contrary to $u \in G$ and $\operatorname{osc} f(u) \geq \eta$. □

**Corollary 2** *If $\mu(E \setminus B(f)) = 0$, then $\mu(E \setminus C(f)) = 0$.*

**Corollary 3** *If $Q_s(f) = E$, then $\mu(E \setminus C(f)) = 0$.*

**Remark 4** *Observe that $Q_s(f) \setminus C(f)$ need not have measure zero (e.g. for the function $f$ from Example 1).*

**Remark 5** *It is obvious that if the functions $f_n : E \to \mathbb{R}$, $n \in \mathbb{N}$, are s.q.c. at a point $x$ and if the sequence $(f_n)_n$ converges uniformly to $f$, then $f$ is also s.q.c. at $x$.*

**Theorem 3** *Let $f : E \to \mathbb{R}$ be a function such that $Q_s(f) = E$. Then there is a sequence of functions $f_n$, $n \in \mathbb{N}$, which converges uniformly to $f$ and for which $A(f_n) = E$ for $n \in \mathbb{N}$.*

PROOF. We prove that for every $\eta > 0$ there is $g : E \to \mathbb{R}$ such that $A(g) = E$ and $|f(x) - g(x)| < \eta$ for all $x \in \mathbb{R}$. Fix $\eta > 0$. By Corollary 3 $f$ is almost everywhere continuous. So, $V = \{y \in \mathbb{R} : \mu(\operatorname{cl}(f^{-1}(y))) > 0\}$ is countable. Consequently, the linear space $E_Q(V)$ over the field $\mathbb{Q}$ of all rationals generated by $V$ is also countable and there is a $c > 0$ which is not in $E_Q(V)$. Fix $n \in \mathbb{N}$ with $c < n\eta/6$. Observe that $\mu(\operatorname{cl}(f^{-1}((2k-1)c/n))) = 0$ for all integers $k$ and $h(x) = (2k-1)c/n$ if $(2k-1)c/n \leq f(x) < (2k+1)c/n$ is almost everywhere continuous and $h(x) \leq f(x) < h(x) + 2c/n < h(x) + \eta/3$ for every $x \in E$. If $d_u(\operatorname{int}(h^{-1}(h(x))), x) > 0$, set $g(x) = h(x)$. If $d_u(\operatorname{int}(h^{-1}(h(x))), x) = 0$, then set $g(x) = h(x) - 2c/n$.

Evidently, $|f - g| \leq |f - h| + |h - g| \leq 2c/n + 2c/n < \eta/3 + \eta/3 < \eta$. We will prove that $g \in A(x)$ for every $x \in E$. If $(2k-1)c/n < f(x) < /2k+1)c/n$ for some integer $k$, then there is a $r > 0$ such that

$$(f(x) - r, f(x) + r) \subset ((2k-1)c/n, (2k+1)c/n)$$

and, by Corollary 1, $d_u(\operatorname{int}(f^{-1}((f(x) - r, f(x) + r))), x) > 0$. Since $g(t) = h(t) = (2k-1)c/n$ for all $t \in \operatorname{int}(f^{-1}((f(x) - r, f(x) + r)))$, we obtain that $g \in A(x)$.

Now, let $f(x) = (2k+1)c/n$ for some integer $k$. If $d_u(\operatorname{int}(h^{-1}(h(x))), x) > 0$ then $g \in A(x)$, because $h$ is almost everywhere continuous. Assume that

$d_u(\text{int}\,(h^{-1}(h(x))), x) = 0$. From the definition of $h$, because $h$ is almost everywhere continuous and since $f \in B(x)$, we get

$$d_u(\text{int}\,(f^{-1}((f(x) - 2c/n, f(x)))), x) > 0$$

. Since $g(t) = (2k-1)c/n$ for all $t \in \text{int}\,(f^{-1}((f(x) - 2c/n, f(x))))$ and for $t = x$, we get $g \in A(x)$. □

Now for functions $f, g : E \to \mathbb{R}$ let $\varrho(f, g) = \min(1, \sup_{x \in E} |f(x) - g(x)|)$. Moreover, denote by $\mathcal{A}$ ($\mathcal{B}$) ($Q_s$) the family of all functions $f : E \to \mathbb{R}$ with $A(f) = E$ ($B(f) = E$) ($Q_s(f) = E$).

Observe that $\mathcal{B} = Q_s$ is a closed subset of the complete metric space $(DCP, \varrho)$. Moreover, by Theorem 4, the closure $\text{cl}_{\varrho}(\mathcal{A})$ of the set $\mathcal{A}$ in the metric $\varrho$ is the same as $\mathcal{B}$.

**Remark 6** *The set $Q_s = \mathcal{B}$ is nowhere dense in the space $(DCP, \varrho)$.*

Proof. Since $Q_s$ is closed, it suffices to prove that for every $\eta > 0$ and for every $f \in Q_s$ there is a $g \in DCP \setminus Q_s$ such that $\varrho(f, g) < \eta$. Fix $f \in Q_s$ and $\eta > 0$. Let $F$ be a nowhere dense nonempty set belonging to $\mathcal{T}_d$ such that $\text{cl}\,(F) \subset C(f)$ and let $h$ be the characteristic function of the set $F$. Then $g = f + \eta h/2 \in DCP \setminus Q_s$ and $\varrho(f, g) = \eta/2 < \eta$. □

## 2 Functions of Two Variables

Now let $E = \mathbb{R}^2$. There are functions $f : E \to \mathbb{R}$ such that all sections $f_x(t) = f(x, t)$, $f^y(t) = f(t, y)$, $t, x, y \in \mathbb{R}$, are continuous and $\mu(E \setminus C(f)) > 0$ [4]. Observe that such functions $f$ are not in $Q_s$. However, such functions have the following property $H(x, y)$ at every $(x, y) \in E$.

A function $f : E \to \mathbb{R}$ has property $H(x, y)$ ($K(x, y)$) at $(x, y)$ if for every $\eta > 0$ and for all $U$, $V \in \mathcal{T}_d$ such that $x \in U$ and $y \in V$ there is an open set $W$ such that $W \cap (U \times V) \neq \emptyset$ and $|f(u, v) - f(x, y)| < \eta$ for all $(u, v) \in W \cap (U \times V)$ ($\text{osc}\, f < \eta$ on the set $W \cap (U \times V)$).

**Theorem 4** *If all sections $f_x$ and $f^y$, $x, y \in \mathbb{R}$, of $f : E \to \mathbb{R}$ belong to $Q_s$, then $f$ has property $H(x, y)$ at every $(x, y) \in E$.*

Proof. Fix $(x, y) \in E$, a real $\eta > 0$ and $U, V \in \mathcal{T}_d$ such that $x \in U$ and $y \in V$. Since $f^y \in B(x)$, there is an open interval $I$ such that $I \cap U \neq \emptyset$ and $|f(t, y) - f(x, y)| < \eta/4$ for all $t \in I$. Let $F = \text{cl}\,(I \cap U)$. Since $f_t \in B(y)$ for all $t \in F$, for each $t \in F$ there is an open interval $J(t)$ with rational endpoints such that $J(t) \cap V \neq \emptyset$ and $|f(t, v) - f(t, y)| < \eta/4$ for all $v \in J(t)$. There is an open interval $J$ such that $G = \{t \in F : J(t) = J\}$ is of the second category in $F$. Consequently, there is an open interval $I_1 \subset I$ such that $I_1 \cap F \neq \emptyset$ and $I_1 \cap G$ is dense in $I_1 \cap F$. Evidently, $K = (I_1 \cap U) \times (J \cap V) \neq \emptyset$. Fix $(u, v) \in K$ and assume that $|f(u, v) - f(x, y)| > \eta/2$. Since $f^v \in B(u)$, there is an open interval $I_2 \subset I_1$ such that $I_2 \cap F \neq \emptyset$ and $|f(t, v) - f(x, y)| > \eta/2$ for all $t \in I_2$. Let $s \in I_2 \cap G$. Then $|f(s, v) - f(x, y)| > \eta/2$. But

$$|f(s, v) - f(x, y)| \leq |f(s, v) - f(s, y)| + |f(s, y) - f(x, y)| < \eta/4 + \eta/4 = \eta/2$$

This contradiction finishes the proof. □

Now, denote by $P_s$ the family of all functions $f : E \to \mathbb{R}$ which are strongly cliquish at every $x \in E$.

**Theorem 5** *If all sections $f^y$ of $f : E \to \mathbb{R}$ belong to $Q_s$ and all sections $f_x$ belong to $P_s$, then $f$ has property $K(x, y)$ at every $(x, y) \in E$.*

PROOF. Fix $(x, y) \in E$, and $U, V \in \mathcal{T}_d$ such that $x \in U$, $y \in V$ and $\eta > 0$. For every $t \in W = \mathrm{cl}\,(U)$ there are an open interval $I(t)$ with rational endpoints and a closed interval $J(t)$ with rational endpoints such that $\mu(J(t)) < \eta/2$, $I(t) \cap V \neq \emptyset$ and $f(t, v) \in J(t)$ for every $v \in V \cap J(t)$. Since the family of all pairs of intervals with rational endpoints is countable, there are open intervals $I$, $L$ and a closed interval $J$ such that $I \cap U \neq \emptyset$ and

$$A = \{t \in W : I(t) = L, J(t) = J\}$$

is dense in $I \cap U$. Fix $(u, v) \in (I \times L) \cap (U \times V)$. If $f(u, v) \notin J$, then since $f^v \in B(u)$, we obtain that there is a $w \in A \cap U$ such that $f(w, v) \notin J$, contrary to the definition of $A$ and the choice of $I(t)$ and $J(t)$. So, $f(u, v) \in J$ for every $(u, v) \in (I \times L) \cap (U \times V)$ and $\mathrm{osc}\, f \leq \eta/2 < \eta$ on $(I \times L) \cap (U \times V)$. □

**Problem 1** *Suppose that $f : E \to \mathbb{R}$ has all sections $f^y \in Q_s$ and all sections $f_x \in P_s$. Is $f$ in $P_s$?*

Now, denote by $\Phi$ the family of all $f : \mathbb{R} \to \mathbb{R}$ such that for every nonempty closed set $P$ of positive measure and for every $\eta > 0$ there is an open interval $I$ such that $I \cap P \neq \emptyset$ and $\mathrm{osc}\, f < \eta$ on $I \cap P$.

Observe that all Baire 1 functions and all almost everywhere continuous functions are in $\Phi$.

**Problem 2** *Let $f : E \to \mathbb{R}$ be such that all sections $f_x$ are in $\Phi$ and all sections $f^y$ are in $Q_s$. Is $f$ in $P_s$?*

Now we say that the functions $f_s : \mathbb{R} \to \mathbb{R}$, where $s \in S$ and $S$ is a set of indices, are strongly quasi-equicontinuous (abbreviated s.q.ec.) at $x \in \mathbb{R}$ if for every $\eta > 0$

$$d_u\big(\mathrm{int}\,\big(\bigcap_{s\in S}(f_s)^{-1}((f_s(x)-\eta, f_s(x)+\eta))\big), x\big) > 0.$$

**Theorem 6** *If all sections $f^y$ of $f : E \to \mathbb{R}$ are s.q.c. at every $x$ and if the sections $f_x$, $x \in \mathbb{R}$, are s.q.ec. at every $y$, then $f$ is s.q.c..*

PROOF. Fix $(x, y) \in E$, $\eta > 0$ and $U \subset E$ belonging to $\mathcal{T}_d$ and such that $(x, y) \in U$. Since $f^y$ is s.q.c., we get $f^y \in B(x)$. Consequently, for the interior $\mathrm{int}\,((f^y)^{-1}((f(x, y) - \eta/2, f(x, y) + \eta/2))) = G$ we have $d_u(G, x) > 0$. Let

$$H = \mathrm{int}\,\big(\bigcap_{t\in\mathcal{R}}(f_t)^{-1}((f(t, y) - \eta/2, f(t, y) + \eta/2))\big).$$

Since the sections $f_x$ are s.q.ec. at $y$, we obtain $d_u(H, y) > 0$. So $G \times H$ is open, $d_u((G \times H), (x, y)) > 0$ and $(G \times H) \cap U \neq \emptyset$. Let $(u, v) \in G \times H$. Then

$$|f(u, v) - f(x, y)| \leq |f(u, v) - f(u, y)| + |f(u, y) - f(x, y)| < \eta/2 + \eta/2 = \eta$$

and the proof is complete. □

**Theorem 7** *There is a function $f : E \to \mathbb{R}$ having continuous sections $f_x$ and $f^y$, $x, y \in \mathbb{R}$, such that $\mu(E \setminus C(f)) > 0$ and for every $\eta > 0$, for every $y \in \mathbb{R}$ and for every $U \in \mathcal{T}_d$ containing $y$ there is an open interval $I$ such that $I \cap U \neq \emptyset$ and $|f(x, t) - f(x, y)| < \eta$ for all $t \in U \cap I$ and for all $x \in \mathbb{R}$.*

PROOF. Let $C \subset [0, 1]$ be a Cantor set of positive measure. There are pairwise disjoint closed intervals $I_n \subset \mathbb{R} \setminus C$ such that

- if $x_i \in I_{n_i}$ for $i \in \mathbb{N}$, $I_{n_i} \neq I_{n_j}$ for $i \neq j$ and $\lim_{i\to\infty} = x$, then $x \in C$,
- for all $x \in C$ we have $d_u\big(\bigcup_{n\in\mathbb{N}} I_n, x\big) = 0$,
- $C \subset \mathrm{cl}\,\big(\bigcup_{n\in\mathbb{N}} I_{2n-1}\big) \cap \mathrm{cl}\,\big(\bigcup_{n\in\mathbb{N}} I_{2n}\big)$.

Let $f : E \to \mathbb{R}$ be a function such that $f(x, y) = 0$ if $(x, y) \notin I_{2n-1} \times I_{2n}$, $n \in \mathbb{N}$, $f$ is continuous at every $(x, y) \notin C \times C$ and $f(I_{2n-1} \times I_{2n}) = [0, 1]$ for $n \in \mathbb{N}$. Then $f$ satisfies all required conditions. □

**Remark 7** *Observe that Theorem* 7 *shows that in Theorem* 6 *the definition of strong quasi-equicontinuity of sections* $f_x$, $x \in \mathbb{R}$, *can't be the following:* $f_x$, $x \in \mathbb{R}$, *are s.q.ec. at a point* $y$ *if for every* $\eta > 0$ *and for every* $U \in \mathcal{T}_d$ *with* $y \in U$ *there is an open set* $V$ *such that* $V \cap U \neq \emptyset$ *and* $|f_x(v) - f_x(y)| < \eta$ *for all* $v \in U \cap V$ *and* $x \in \mathbb{R}$. *The function* $f$ *from Theorem* 7 *is not in* $Q_s$, *since* $\mu(E \setminus C(f)) > 0$.

**Theorem 8** *Let* $f : E \to \mathbb{R}$ *be a function such that all sections* $f_x$ *are s.q.ec. at every* $y$ *and all sections* $f^y$ *are almost everywhere continuous. Then* $f \in P_s$.

Proof. We proceed as in the proof of Theorem 6, but for each $U \in \mathcal{T}_d$ we find a $(x, y) \in U$ such that $x$ is a density point of $\{t : (t, y) \in U\}$ and $f^y$ is continuous at $x$.

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