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PRODUCTS OF BAIRE ONE DOUBLE STAR FUNCTIONS

Abstract

We characterize products of Baire one double star functions.

1 Preliminaries

The letters $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{Z}$ denote the set of nonnegative integers, the real line, and the set of all integers, respectively. The symbols ω_0 and ω_1 denote the first infinite ordinal and the first uncountable ordinal, respectively. The word *function* denotes a mapping from a subset of $\mathbb{R}$ into $\mathbb{R}$. The symbol $\mathcal{C}(f)$ stands for the set of points of continuity of a function f.

Let $A \subset \mathbb{R}$. We use the symbols $\operatorname{cl} A$, $\operatorname{bd} A$, and A' to denote the closure, the boundary, and the set of all accumulation points of A, respectively. For each $x \in \mathbb{R}$, we define $\varrho(x, A)$ to be the distance between x and A; i.e., $\varrho(x, A) \overset{\mathrm{df}}{=} \inf\{|x - a| \,;\, a \in A\}$.

If $f\colon \mathbb{R} \to \mathbb{R}$, then for each $a \in \mathbb{R}$, we define $[f = a] \overset{\mathrm{df}}{=} \{x \in \mathbb{R} \,;\, f(x) = a\}$; the symbols $[f \neq a]$, $[f > a]$, etc. are defined analogously.

If $I \subset \mathbb{R}$ is an interval and $\psi\colon I \to \mathbb{R}$, then for every ordinal α, we define [2]

$$\mathcal{U}_\alpha(\psi) \overset{\mathrm{df}}{=} \operatorname{int}\bigl(\bigcup_{\beta<\alpha} \mathcal{U}_\beta(\psi) \cup \mathcal{C}\bigl(\psi{\restriction}I \setminus \bigcup_{\beta<\alpha} \mathcal{U}_\beta(\psi)\bigr)\bigr)$$

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(int stands for the interior operator in I) and $\widetilde{\mathcal{U}}_\alpha(\psi) \stackrel{\mathrm{df}}{=} \mathcal{U}_\alpha(\psi) \setminus \bigcup_{\beta<\alpha} \mathcal{U}_\beta(\psi)$.

For each $\alpha < \omega_1$, we denote

$$\mathcal{S}_\alpha \stackrel{\mathrm{df}}{=} \bigl\{f \colon \mathbb{R} \to \mathbb{R}\,;\, \mathcal{U}_\alpha(f) = \mathbb{R}\bigr\}.$$

(Notice that the domain of each $f \in \mathcal{S}_\alpha$ is $\mathbb{R}$.) In particular, $\mathcal{S}_0$ is the class of all continuous functions and $\mathcal{S}_1$ is the class $\mathcal{B}_1^{**}$ defined by R.J. Pawlak [4].

We say that $f \colon \mathbb{R} \to \mathbb{R}$ is a *Baire one star function* [3], if for each nonempty closed set $P \subset \mathbb{R}$, there is a nonempty portion $Q \stackrel{\mathrm{df}}{=} P \cap (a,b)$ of P such that $f{\restriction}Q$ is continuous. We denote the family of all Baire one star functions by $\mathcal{B}_1^*$.

In the article [2], the first author proved that $\langle \mathcal{S}_\alpha\,;\, \alpha < \omega_1\rangle$ is a classification of Baire one star functions, and characterized sums of functions from these classes. The goal of this paper is to make the first step toward the characterization of the products of functions from these classes.

2 The theorem

We will prove the following theorem.

Theorem 2.1. *Let $f \colon \mathbb{R} \to \mathbb{R}$. Denote by $\mathcal{I}$ the family of all bounded connected components $I = (a,b)$ of $\mathcal{U}_0(f)$ with the property that $f(a)f(b) < 0$ and $I \cap [f = 0] = \emptyset$. The following are equivalent*:

i) *there exist $g, h \in \mathcal{S}_1$ such that $f = gh$ on $\mathbb{R}$;*

ii) *$f \in \mathcal{S}_2$ and for each $x \in [f \neq 0] \setminus \mathcal{U}_1(f)$:*

$$(\exists \delta > 0)\ (x-\delta, x+\delta) \cap \bigl(\textstyle\bigcup_{I\in\mathcal{I}} \operatorname{bd} I\bigr)' \cap \mathcal{U}_1(f) = \emptyset, \tag{2.1}$$

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall I \in \mathcal{I}) \\ \bigl(I \subset [f \cdot f(x) > 0] \cap (x-\delta, x+\delta) \Rightarrow \varrho(f(x), f[\operatorname{cl} I]) < \varepsilon\bigr). \tag{2.2}$$

It seems surprising that despite the simplicity of the definition of the class $\mathcal{S}_1$ and the not-very-complicated characterization of the family of the products of two such functions, the proof of this characterization is very long. Therefore we adjourn the proof of this theorem to another section. First we will prove several auxiliary lemmas used in the proof of the implication ii) ⇒ i).

Remark. If $f \in \mathcal{S}_2$, then there exist $g, h \in \mathcal{S}_1$ such that $f = g + h$ on $\mathbb{R}$, and we can require that $\mathcal{U}_1(f) \subset \mathcal{U}_0(g)$ and $\mathcal{U}_0(f) \subset \mathcal{U}_0(h)$. (Cf. [2, Proposition 8] and its proof.) The main difficulty in the proof of our main theorem is that in general, if f can be expressed as the product of two $\mathcal{S}_1$ functions, then we cannot require that either factor is continuous on $\mathcal{U}_0(f)$.

Example. Define $f\colon \mathbb{R} \to \mathbb{R}$ by the formula:

$$f(x) \overset{\text{df}}{=} \begin{cases} (-2)^n & \text{if } x = n^{-1} \text{ for some } n \in \mathbb{N} \setminus \{0\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then f can be written as the product of two $\mathcal{S}_1$ functions. However, if $f = g_0 g_1$ and $g_0, g_1 \in \mathcal{S}_1$, then $\mathcal{U}_0(f) \not\subset \mathcal{U}_0(g_0)$.

PROOF. Indeed, 0 is the only element of $\mathbb{R} \setminus \mathcal{U}_1(f)$, and since $f = f(0)$ on $\mathcal{U}_0(f)$, condition (2.2) is fulfilled. To prove condition (2.1), notice that 0 is the only accumulation point of $\mathbb{R} \setminus \mathcal{U}_0(f)$ and $0 \notin \mathcal{U}_1(f)$. So by Theorem 2.1, there exist $g, h \in \mathcal{S}_1$ such that $f = gh$ on $\mathbb{R}$.

Now let $f = g_0 g_1$, where $g_0, g_1 \in \mathcal{S}_1$. Observe that $0 \notin \mathcal{U}_0(g_0) \cup \mathcal{U}_0(g_1)$, since otherwise we would obtain $0 \in \mathcal{U}_1(f)$, which is not the case.

Since $f(0) = 1$, we may assume that $g_0(0) = g_1(0) = 1$ as well. There is a $\delta > 0$ such that

$$0 < g_0 < 2 \quad \text{on } (0,\delta) \setminus \mathcal{U}_0(g_0), \qquad 0 < g_1 < 2 \quad \text{on } (0,\delta) \setminus \mathcal{U}_0(g_1). \tag{2.3}$$

Fix an odd integer $n > 1/\delta$. Put $x \overset{\text{df}}{=} n^{-1}$ and $z \overset{\text{df}}{=} (n+1)^{-1}$. Then $f(x) < 0$, so $g_i(x) < 0$ for some $i \in \{0, 1\}$. From (2.3) we conclude that $x \in \mathcal{U}_0(g_i)$. Let C be the connected component of $\mathcal{U}_0(g_i)$ to which x belongs, and let $t \overset{\text{df}}{=} \inf C$. Notice that $g_i < 0$ on $(t, x]$ since $g_i(x) < 0$ and $g_i \neq 0$ on $\mathbb{R}$.

If $t < z$, then $g_i(z) < 0$, so $g_{1-i}(z) = f(z)/g_i(z) < 0$. By (2.3), we obtain $z \in \mathcal{U}_0(g_{1-i})$. Hence $z \in \mathcal{U}_0(g_i) \cap \mathcal{U}_0(g_{1-i}) \subset \mathcal{U}_0(f)$, a contradiction.

If $t = z$, then by (2.3), $0 < g_i(z) < 2$. (Recall that $t \notin \mathcal{U}_0(g_i)$.) Hence $g_{1-i}(z) > 0$. Since $g_{1-i} = f/g_i < 0$ on $(t, x]$, we obtain $z \notin \mathcal{U}_0(g_{1-i})$. Using (2.3) once more, we get $g_{1-i}(z) < 2$. Consequently,

$$2^{n+1} = f(z) = g_i(z) g_{1-i}(z) < 2 \cdot 2 = 4,$$

a contradiction.

So, $t > z$. Then $t \in \mathcal{U}_0(f) \setminus \mathcal{U}_0(g_i)$. Since f is continuous in a neighborhood of t, we conclude that $t \notin \mathcal{U}_0(g_{1-i})$. $\square$

We would like to state several open problems.

- Given an integer $n > 2$, characterize products of n functions from $\mathcal{S}_1$.
- Given nonzero ordinals $\alpha, \beta < \omega_1$, characterize products of functions $g \in \mathcal{S}_\alpha$ and $h \in \mathcal{S}_\beta$.
- Characterize products of real Baire one double star functions defined on some topological space different from $\mathbb{R}$. (See [1] for the definition.)

3 The lemmas

Throughout this section we assume that the assumptions listed in Theorem 2.1.ii) are fulfilled. For brevity, for each interval J and each $x \in \mathbb{R}$, we define

$$\mathrm{w}(x, J) \stackrel{\mathrm{df}}{=} \sup\big\{\varrho\big(f(x), f\big[[f \cdot f(x) > 0] \cap \operatorname{cl} I\big]\big)\,;\, I \in \mathcal{I}, I \subset [f \cdot f(x) > 0] \cap J\big\}.$$

(If J contains no interval $I \in \mathcal{I}$ such that $I \subset [f \cdot f(x) > 0]$, then we define $\mathrm{w}(x, J) \stackrel{\mathrm{df}}{=} 0$.) With this notation, (2.2) becomes

$$(\forall \varepsilon > 0)(\exists \delta > 0) \;\; \mathrm{w}(x, (x - \delta, x + \delta)) < \varepsilon.$$

We start with the construction on an interval disjoint from $\bigcup \mathcal{I}$.

Lemma 3.1. *Assume that $S = (a, b) \subset \mathcal{U}_1(f) \setminus \bigcup \mathcal{I}$ and $a, b \in \mathbb{R} \setminus \mathcal{U}_0(f)$. For each $z \neq 0$, there are functions $\varphi, \psi \colon \operatorname{cl} S \to \mathbb{R}$ such that $f = \varphi\psi$ on $\operatorname{cl} S$, φ is continuous on $\operatorname{cl} S \cap \mathcal{U}_1(f)$ and $\psi = z$ on $(\operatorname{cl} S \setminus \mathcal{U}_0(\psi)) \cup \operatorname{bd} S$.*

Proof. Arrange all connected components of the set $S \cap \mathcal{U}_0(f)$ in a sequence $\langle S_n\,;\, n < \xi\rangle$, where $\xi \leq \omega_0$. For each $n < \xi$, we have $S_n \notin \mathcal{I}$. So, we can construct a piecewise linear continuous function $\varphi_n \colon \operatorname{cl} S_n \to \mathbb{R}$ such that $\varphi_n = f/z$ on $\operatorname{bd} S_n$, $[\varphi_n = 0] \subset [f = 0]$, and

$$\left|\varphi_n - \frac{f(\inf S_n)}{z}\right| < \left|\frac{f(\sup S_n) - f(\inf S_n)}{z}\right| + \frac{1}{n+1} \qquad \text{on } S_n.$$

Define

$$\varphi(x) \stackrel{\mathrm{df}}{=} \begin{cases} \varphi_n(x) & \text{if } x \in S_n,\, n \in \mathbb{N}, \\ f(x)/z & \text{if } x \in \operatorname{cl} S \setminus \mathcal{U}_0(f), \end{cases}$$

$$\psi(x) \stackrel{\mathrm{df}}{=} \begin{cases} f(x)/\varphi(x) & \text{if } x \in \mathcal{U}_0(f) \setminus [\varphi = 0], \\ z & \text{if } x \in (\operatorname{cl} S \setminus \mathcal{U}_0(f)) \cup [\varphi = 0]. \end{cases}$$

Take any sequence $(x_k) \subset \operatorname{cl} S$ convergent to some $x \in \operatorname{cl} S \cap \mathcal{U}_1(f)$ such that $\varphi(x_k) \to y \in [-\infty, \infty]$. Since $\mathcal{U}_1(f)$ is open, we may assume $(x_k) \subset \mathcal{U}_1(f)$. If there is a subsequence $(x_{k_s}) \subset \widetilde{\mathcal{U}}_1(f)$, then $x \in \widetilde{\mathcal{U}}_1(f)$ as well, and using the continuity of $f \restriction \widetilde{\mathcal{U}}_1(f)$ we conclude that

$$y = \lim_{s \to \infty} \varphi(x_{k_s}) = \lim_{s \to \infty} f(x_{k_s})/z = f(x)/z = \varphi(x).$$

So, assume that $(x_k) \subset \operatorname{cl} S \cap \mathcal{U}_0(f)$. For each k, choose an $n_k < \xi$ such that $x_k \in S_{n_k}$. (We can find such n_k because $\operatorname{bd} S \cap \mathcal{U}_0(f) = \emptyset$.) If there is an

$n < \xi$ such that $x_k \in S_n$ for infinitely many k, then $y = \varphi(x)$ by the continuity of $\varphi_n = \varphi{\restriction}S_n$ on the interval $\operatorname{cl} S_n$. In the opposite case $n_k \to \infty$, whence $x = \lim_{k\to\infty} \inf S_{n_k} = \lim_{k\to\infty} \sup S_{n_k}$. Since $\inf S_{n_k}, \sup S_{n_k} \in \widetilde{\mathcal{U}}_1(f)$ for sufficiently big k, we conclude that $x \in \widetilde{\mathcal{U}}_1(f)$ as well, and using the continuity of $f{\restriction}\widetilde{\mathcal{U}}_1(f)$, we obtain

$$\begin{aligned} |y - \varphi(x)| &\le \varlimsup_{k\to\infty} |\varphi(x_k) - \varphi(\inf S_{n_k})| + \varlimsup_{k\to\infty} |\varphi(\inf S_{n_k}) - \varphi(x)| \\ &\le \varlimsup_{k\to\infty} \Bigl(\Bigl| \frac{f(\sup S_{n_k}) - f(\inf S_{n_k})}{z} \Bigr| + \frac{1}{n_k+1} \Bigr) = 0. \end{aligned}$$

We have proved that φ is continuous on $\operatorname{cl} S \cap \mathcal{U}_1(f)$. Clearly the remaining requirements are also fulfilled. $\square$

Next we turn to intervals in which there is no accumulation point of the set $\bigcup_{I\in\mathcal{I}} \operatorname{bd} I$.

Lemma 3.2. *Assume that* $L = (a, b] \subset \mathcal{U}_1(f)$, $a \in [f \neq 0] \setminus \mathcal{U}_1(f)$, $b \in \widetilde{\mathcal{U}}_1(f)$, *and*

$$L \cap \bigl(\textstyle\bigcup_{I\in\mathcal{I}} \operatorname{bd} I\bigr)' = \emptyset. \tag{3.1}$$

For each $\varepsilon > 0$, *there are functions* $g, h\colon \operatorname{cl} L \to \mathbb{R}$ *such that*

- $f = gh$ *on* $\operatorname{cl} L$,
- $h(a) = h(b) = \sqrt{|f(a)|}$,
- $g{\restriction}\operatorname{cl} L \setminus \mathcal{U}_0(g)$ *and* $h{\restriction}\operatorname{cl} L \setminus \mathcal{U}_0(h)$ *are continuous,*
- $|g - g(a)| \le \mathrm{w}(a, L)/\sqrt{|f(a)|} + \varepsilon$ *on* $\operatorname{cl} L \setminus \mathcal{U}_0(g)$,
- $|h - h(a)| \le \mathrm{w}(a, L)/\sqrt{|f(a)|} + \varepsilon$ *on* $\operatorname{cl} L \setminus \mathcal{U}_0(h)$.

PROOF. Put $z \overset{\mathrm{df}}{=} \sqrt{|f(a)|}$ and $z' \overset{\mathrm{df}}{=} z \cdot \operatorname{sgn} f(a)$. Using (3.1) we conclude that we can arrange all elements of $\{b\} \cup \bigl(L \cap \bigcup_{I\in\mathcal{I}} \operatorname{bd} I\bigr)$ in a strictly decreasing sequence $\langle b_n\,;\, n < \xi\rangle$, where $\xi \le \omega_0$; if $\xi < \omega_0$, then put $b_\xi \overset{\mathrm{df}}{=} a$. Notice that

- if $\xi = \omega_0$, then $b_n \to a$ by (3.1),
- if $f(b_n) = 0$ for some $n < \xi + 1$, then $n = 0$ and $b_n = b$.

For each $n < \xi$, put $S_n \overset{\mathrm{df}}{=} (b_{n+1}, b_n)$. Observe that if $S_n \notin \mathcal{I}$, then

- either $b_{n+1} = a$ or $b_{n+1} = \sup I$ for some $I \in \mathcal{I}$,

- either $b_n = b$ or $b_n = \inf I$ for some $I \in \mathcal{I}$.

If $S_n \in \mathcal{I}$, then

- if $S_n \subset [f \cdot f(a) < 0]$, then choose arbitrary $d_n \in S_n$,
- if $S_n \subset [f \cdot f(a) > 0]$, then choose a $d_n \in \operatorname{cl} S_n$ such that $f(d_n)f(a) > 0$ and
$$|f(d_n) - f(a)| < \mathrm{w}(a, (a, b_n)) + \frac{z\varepsilon}{n+1}.$$

For each $n < \xi$, put $S_n \overset{\text{df}}{=} (b_{n+1}, b_n)$. Put $S_{-1} \overset{\text{df}}{=} \{b\}$, $g_{-1}(b) \overset{\text{df}}{=} f(b)/z$, and $h_{-1}(b) \overset{\text{df}}{=} z$. By induction on $n < \xi$ we will define functions $g_n, h_n \colon \operatorname{cl} S_n \to \mathbb{R}$ such that:

$$f = g_n h_n \qquad \text{on } \operatorname{cl} S_n, \tag{3.2}$$
$$g_n(b_{n+1}) = z' \qquad \text{or} \qquad h_n(b_{n+1}) = z, \tag{3.3}$$
$$h_n(b_n) = h_{n-1}(b_n) \qquad (\text{whence also } g_n(b_n) = g_{n-1}(b_n)), \tag{3.4}$$
$$g_n \restriction \operatorname{cl} S_n \setminus \mathfrak{U}_0(g_n) \text{ and } h_n \restriction \operatorname{cl} S_n \setminus \mathfrak{U}_0(h_n) \text{ are continuous}, \tag{3.5}$$
$$|g_n - z'| \le \mathrm{w}(a, (a, b_n))/z + \varepsilon/(n+1) \qquad \text{on } \operatorname{cl} S_n \setminus \mathfrak{U}_0(g_n), \tag{3.6}$$
$$|h_n - z| \le \mathrm{w}(a, (a, b_n))/z + \varepsilon/(n+1) \qquad \text{on } \operatorname{cl} S_n \setminus \mathfrak{U}_0(h_n). \tag{3.7}$$

Assume that for some $n < \xi$ we have already defined the functions g_{n-1} and h_{n-1} on $\operatorname{cl} S_{n-1}$ according to the induction hypothesis. We consider several cases.

Case 1. $h_{n-1}(b_n) = z$.

Case 1.a) $S_n \cap \bigcup \mathcal{I} = \emptyset$.

Recall that $\operatorname{bd} S_n \subset \operatorname{bd} L \cup \bigcup_{I \in \mathcal{I}} \operatorname{bd} I \subset \mathbb{R} \setminus \mathfrak{U}_0(f)$. Use Lemma 3.1 to construct functions $g_n, h_n \colon \operatorname{cl} S_n \to \mathbb{R}$ such that $f = g_n h_n$ on $\operatorname{cl} S_n$, g_n is continuous on $\operatorname{cl} S_n \cap \mathfrak{U}_1(f)$ and $h_n = z$ on $(\operatorname{cl} S_n \setminus \mathfrak{U}_0(h_n)) \cup \operatorname{bd} S_n$. Then clearly (3.2)–(3.4) and (3.7) are fulfilled. Notice that if $\operatorname{cl} S_n \setminus \mathfrak{U}_0(g_n) \neq \emptyset$, then the only element of this set is $b_{n+1} = a$ (so (3.5) holds) and $g_n(a) = f(a)/z = z'$, so (3.6) holds as well.

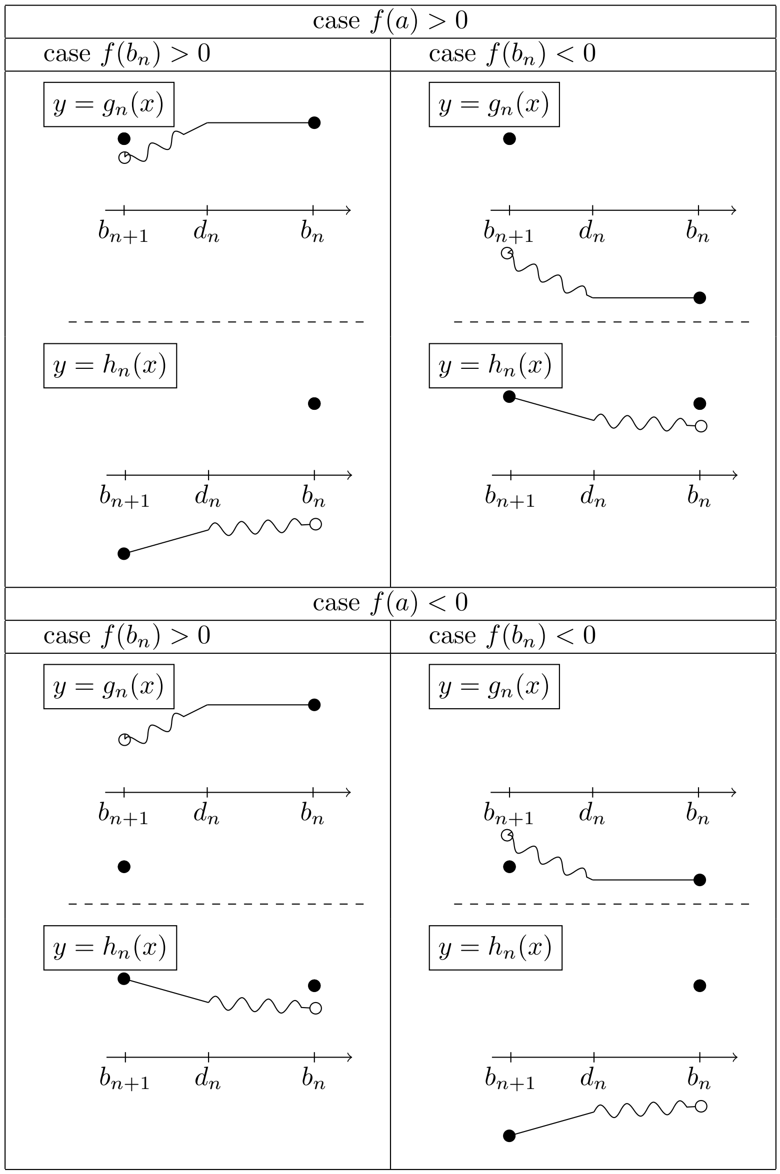
Case 1.b) $S_n \in \mathcal{I}$ and $S_n \subset [f \cdot f(a) < 0]$.

Define

$$h_n(x) \overset{\text{df}}{=} \begin{cases} f(b_{n+1})/z' & \text{if } x = b_{n+1}, \\ f(x) \cdot z/f(b_n) & \text{if } x \in [d_n, b_n], \\ \text{linearly} & \text{in the interval } [b_{n+1}, d_n]. \end{cases}$$

Then h_n is continuous on $\operatorname{cl} S_n \setminus \{b_n\}$. Observe that since $S_n \in \mathcal{I}$,

Figure 1: The graphs of g_n and h_n in Case 1.b):
$h_{n-1}(b_n) = z$, $S_n \in \mathcal{I}$, and $S_n \subset [f \cdot f(a) < 0]$.



$$\frac{f(b_{n+1})}{z'}\cdot\frac{f(d_n)z}{f(b_n)}=f(d_n)f(a)\cdot\frac{f(b_n)f(b_{n+1})}{z'^2(f(b_n))^2}>0.$$

So, $h_n\neq 0$ on $\operatorname{cl} S_n$, and we can define $g_n \overset{\text{df}}{=} f/h_n$. Notice that g_n is continuous on $\operatorname{cl} S_n\setminus\{b_{n+1}\}$ and $g_n(b_{n+1})=z'$.

Case 1.c) $S_n\in\mathcal{I}$ and $S_n\subset[f\cdot f(a)>0]$.

Define

$$g_n(x)\overset{\text{df}}{=}\begin{cases}f(b_{n+1})/z & \text{if } x\in[b_{n+1},d_n),\\ f(d_n)/z & \text{if } x=d_n,\\ f(b_n)/z & \text{if } x\in(d_n,b_n],\end{cases}\qquad h_n\overset{\text{df}}{=} f/g_n.$$

Then g_n is continuous except at d_n (which may happen to be an end point of S_n), and

$$|g_n(d_n)-z'|=\left|\frac{f(d_n)}{z}-z'\right|=\frac{|f(d_n)-f(a)|}{z}<\frac{\mathrm{w}(a,(a,b_n))}{z}+\frac{\varepsilon}{n+1}.$$

On the other hand, h_n is continuous except at b_{n+1}, d_n, and b_n, and h_n takes on the value z at these points; in particular, $h_n(b_{n+1})=z$.

Case 2. $h_{n-1}(b_n)\neq z$. Then $g_{n-1}(b_n)=z'$, so $h_{n-1}(b_n)=f(b_n)/z'$.

Case 2.a) $S_n\cap\bigcup\mathcal{I}=\emptyset$.

Recall that $\operatorname{bd} S_n\subset\operatorname{bd} L\cup\bigcup_{I\in\mathcal{I}}\operatorname{bd} I\subset\mathbb{R}\setminus\mathcal{U}_0(f)$. Use Lemma 3.1 to construct functions $g_n,h_n\colon \operatorname{cl} S_n\to\mathbb{R}$ such that $f=g_nh_n$ on $\operatorname{cl} S_n$, h_n is continuous on $\operatorname{cl} S_n\cap\mathcal{U}_1(f)$ and $g_n=z'$ on $(\operatorname{cl} S_n\setminus\mathcal{U}_0(g_n))\cup\operatorname{bd} S_n$. Then clearly (3.2)–(3.4) and (3.6) are fulfilled. Notice that if $\operatorname{cl} S_n\setminus\mathcal{U}_0(h_n)\neq\emptyset$, then the only element of this set is $b_{n+1}=a$ (so (3.5) holds) and $h_n(a)=f(a)/z'=z$, so (3.7) holds as well.

Case 2.b) $S_n\in\mathcal{I}$ and $S_n\subset[f\cdot f(a)<0]$.

Define

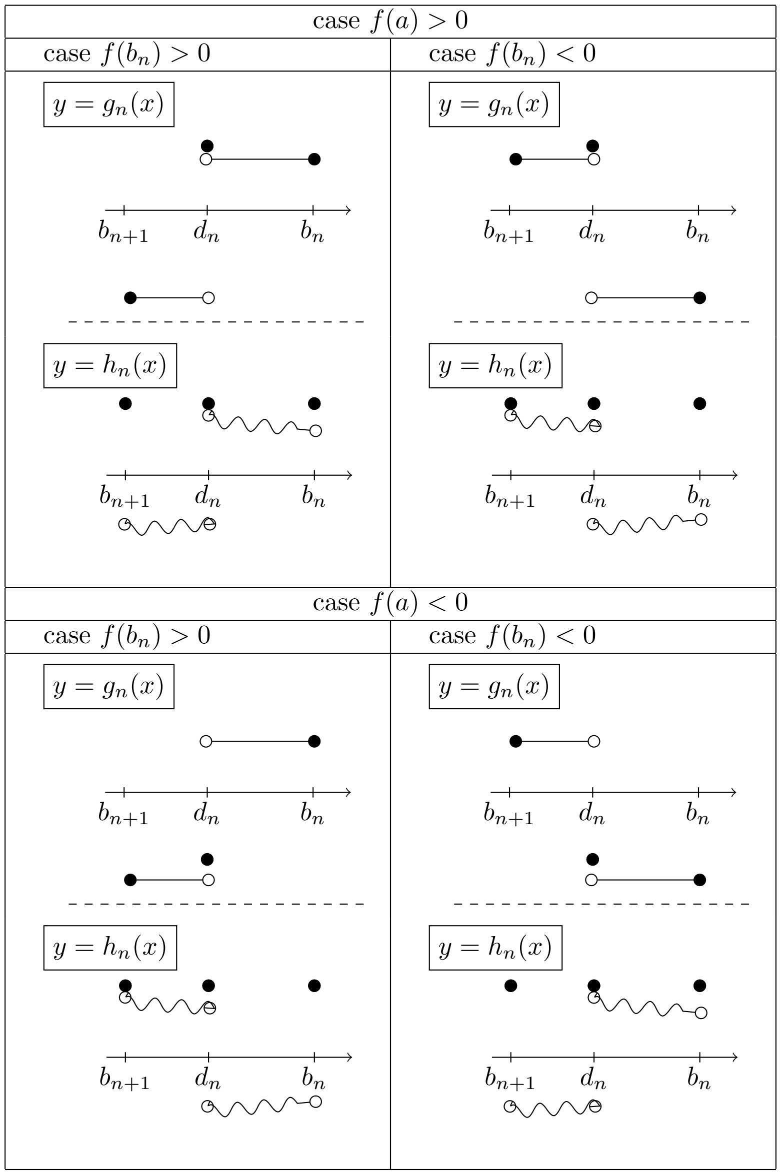
$$g_n(x)\overset{\text{df}}{=}\begin{cases}f(b_{n+1})/z & \text{if } x=b_{n+1},\\ f(x)\cdot z'/f(b_n) & \text{if } x\in[d_n,b_n],\\ \text{linearly} & \text{in the interval } [b_{n+1},d_n].\end{cases}$$

Then g_n is continuous on $\operatorname{cl} S_n\setminus\{b_n\}$. Observe that since $S_n\in\mathcal{I}$,

$$\frac{f(b_{n+1})}{z}\cdot\frac{f(d_n)z'}{f(b_n)}=f(d_n)f(a)\cdot\frac{f(b_n)f(b_{n+1})}{z^2(f(b_n))^2}>0.$$

So, $g_n\neq 0$ on $\operatorname{cl} S_n$, and we can define $h_n\overset{\text{df}}{=} f/g_n$. Notice that h_n is continuous on $\operatorname{cl} S_n\setminus\{b_{n+1}\}$ and $h_n(b_{n+1})=z$.

Figure 2: The graphs of g_n and h_n in Case 1.c):
$h_{n-1}(b_n) = z$, $S_n \in \mathcal{I}$, and $S_n \subset [f \cdot f(a) > 0]$.



Case 2.c) $S_n \in \mathcal{I}$ and $S_n \subset [f \cdot f(a) > 0]$.
Define

$$h_n(x) \stackrel{\text{df}}{=} \begin{cases} f(b_{n+1})/z' & \text{if } x \in [b_{n+1}, d_n), \\ f(d_n)/z' & \text{if } x = d_n, \\ f(b_n)/z' & \text{if } x \in (d_n, b_n], \end{cases} \qquad g_n \stackrel{\text{df}}{=} f/h_n.$$

Then h_n is continuous except at d_n (which may happen to be an end point of S_n), and

$$|h_n(d_n) - z| = \left| \frac{f(d_n)}{z'} - z \right| = \frac{|f(d_n) - f(a)|}{|z'|} < \frac{\mathrm{w}(a, (a, b_n))}{z} + \frac{\varepsilon}{n+1}.$$

On the other hand, g_n is continuous except at b_{n+1}, d_n, and b_n, and g_n takes on the value z' at these points; in particuar, $g_n(b_{n+1}) = z'$. This completes the induction procedure.

To complete the proof define

$$g(x) \stackrel{\text{df}}{=} \begin{cases} g_n(x) & \text{if } x \in \operatorname{cl} S_n,\ n < \xi, \\ z' & \text{if } x = a, \end{cases} \qquad h(x) \stackrel{\text{df}}{=} \begin{cases} h_n(x) & \text{if } x \in \operatorname{cl} S_n,\ n < \xi, \\ z & \text{if } x = a. \end{cases}$$

Notice that if $\xi < \omega_0$, then $a = b_\xi$, and by (3.3), we get $g_{\xi-1}(a) = z'$ and $h_{\xi-1}(a) = z$. So, the functions g and h are well-defined.

Evidently $f = gh$ on $\operatorname{cl} L$. We have $\operatorname{cl} L \setminus \mathcal{U}_0(g) \subset \bigcup_{n<\xi} (\operatorname{cl} S_n \setminus \mathcal{U}_0(g_n)) \cup \{a\}$. So by (3.5), we conclude that $g \restriction \operatorname{cl} L \setminus \mathcal{U}_0(g)$ is continuous except, maybe, the point a. However if $\xi = \omega_0$, then by (3.6) and (2.2), we obtain

$$\lim_{t \to a, t \in \operatorname{cl} L \setminus \mathcal{U}_0(g)} |g(t) - g(a)| \le \lim_{n \to \infty} \left(\frac{\mathrm{w}(a, (a, b_n))}{z} + \frac{\varepsilon}{n+1} \right) = 0.$$

Similarly we can prove that $h \restriction \operatorname{cl} L \setminus \mathcal{U}_0(h)$ is continuous. Since for each $n < \xi$, $\mathrm{w}(a, (a, b_n)) \le \mathrm{w}(a, L)$, the functions g and h fulfill also the other requirements. $\square$

Evidently a lemma analogous to Lemma 3.2, in which we assume that $a \in \widetilde{\mathcal{U}}_1(f)$ and $b \in [f \ne 0] \setminus \mathcal{U}_1(f)$, is also true. (Consider the function $x \mapsto f(-x)$ and the interval $[-b, -a]$.) Using this result in conjunction with Lemma 3.2, we obtain the next lemma.

Lemma 3.3. *Assume that $K = (a, b)$ is a bounded connected component of $\mathcal{U}_1(f)$ such that $K \cap \left(\bigcup_{I \in \mathcal{I}} \operatorname{bd} I\right)' = \emptyset$ and $f(a)f(b) > 0$. For each $\varepsilon > 0$, there are functions $g, h \colon \operatorname{cl} K \to \mathbb{R}$ such that*

- $f = gh$ *on* $\operatorname{cl} K$,
- $h = \sqrt{|f|}$ *on* $\operatorname{bd} K$,
- $g\restriction \operatorname{cl} K \setminus \mathcal{U}_0(g)$ *and* $h\restriction \operatorname{cl} K \setminus \mathcal{U}_0(h)$ *are continuous,*
- $|g - g(a)| \le \theta \mathrm{w}(a, K)/\sqrt{|f(a)|} + (\theta^2 + \theta - 2)\sqrt{|f(a)|} + \varepsilon$ *on* $\operatorname{cl} K \setminus \mathcal{U}_0(g)$, *where*
$$\theta \stackrel{\mathrm{df}}{=} \max\bigl\{\sqrt{f(a)/f(b)}, \sqrt{f(b)/f(a)}\,\bigr\} \ge 1,$$
- $|h - h(a)| \le \theta \mathrm{w}(a, K)/\sqrt{|f(a)|} + (\theta^2 + \theta - 2)\sqrt{|f(a)|} + \varepsilon$ *on* $\operatorname{cl} K \setminus \mathcal{U}_0(h)$.

PROOF. If $K \subset \mathcal{U}_0(f)$, then we can define $g \stackrel{\mathrm{df}}{=} \sqrt{|f|} \cdot (\operatorname{sgn} \circ f)$ and $h \stackrel{\mathrm{df}}{=} \sqrt{|f|}$.

In the opposite case take any $c \in K \cap \widetilde{\mathcal{U}}_1(f)$ and use Lemma 3.2 or its analogue to define functions $\bar{g}, \bar{h}\colon [a, c] \to \mathbb{R}$ and $\hat{g}, \hat{h}\colon [c, b] \to \mathbb{R}$ such that

- $f = \bar{g}\bar{h}$ on $[a, c]$, $f = \hat{g}\hat{h}$ on $[c, b]$,
- $\bar{h}(a) = \bar{h}(c) = \sqrt{|f(a)|}$, $\hat{h}(c) = \hat{h}(b) = \sqrt{|f(b)|}$,
- $\bar{g}\restriction[a, c] \setminus \mathcal{U}_0(\bar{g})$, $\bar{h}\restriction[a, c] \setminus \mathcal{U}_0(\bar{h})$, $\hat{g}\restriction[c, b] \setminus \mathcal{U}_0(\hat{g})$, and $\hat{h}\restriction[c, b] \setminus \mathcal{U}_0(\hat{h})$ are continuous,
- $|\bar{g} - \bar{g}(a)| \le \mathrm{w}(a, (a, c])/\sqrt{|f(a)|} + \varepsilon/\theta$ on $[a, c] \setminus \mathcal{U}_0(\bar{g})$,
- $|\bar{h} - \bar{h}(a)| \le \mathrm{w}(a, (a, c])/\sqrt{|f(a)|} + \varepsilon/\theta$ on $[a, c] \setminus \mathcal{U}_0(\bar{h})$,
- $|\hat{g} - \hat{g}(b)| \le \mathrm{w}(b, [c, b))/\sqrt{|f(b)|} + \varepsilon$ on $[c, b] \setminus \mathcal{U}_0(\hat{g})$,
- $|\hat{h} - \hat{h}(b)| \le \mathrm{w}(b, [c, b))/\sqrt{|f(b)|} + \varepsilon$ on $[c, b] \setminus \mathcal{U}_0(\hat{h})$.

Let $\varphi\colon [a, c] \to [1/\theta, \theta]$ be the linear function such that
$$\varphi(a) = 1, \qquad \varphi(c) = \hat{h}(c)/\bar{h}(c) = \sqrt{f(b)/f(a)}.$$
Observe that then for each $x \in [a, c]$,
$$\theta - 1 \ge \frac{1}{\varphi(x)} - 1 \ge \frac{1}{\theta} - 1 = \frac{1 - \theta}{\theta} \ge 1 - \theta,$$
so $|1/\varphi(x) - 1| \le \theta - 1$.

Define
$$g(x) \stackrel{\mathrm{df}}{=} \begin{cases} \bar{g}(x)/\varphi(x) & \text{if } x \in [a, c], \\ \hat{g}(x) & \text{if } x \in [c, b], \end{cases} \qquad h(x) \stackrel{\mathrm{df}}{=} \begin{cases} \bar{h}(x) \cdot \varphi(x) & \text{if } x \in [a, c], \\ \hat{h}(x) & \text{if } x \in [c, b]. \end{cases}$$

Then g and h are well-defined. Clearly $f = gh$ on $\operatorname{cl} K$, $h = \sqrt{|f|}$ on $\operatorname{bd} K$, and $g\restriction \operatorname{cl} K \setminus \mathcal{U}_0(g)$ and $h\restriction \operatorname{cl} K \setminus \mathcal{U}_0(h)$ are continuous.

Fix an $x \in \operatorname{cl} K \setminus \mathcal{U}_0(g)$. If $x \in [a, c] \setminus \mathcal{U}_0(g)$, then

$$\begin{aligned} |g(x) - g(a)| &\le \frac{|\bar g(x) - \bar g(a)|}{\varphi(x)} + \Big|\frac{1}{\varphi(x)} - 1\Big||g(a)| \\ &\le \frac{\theta \mathrm{w}(a, (a, c])}{\sqrt{|f(a)|}} + \varepsilon + (\theta - 1)\sqrt{|f(a)|} \\ &\le \frac{\theta \mathrm{w}(a, K)}{\sqrt{|f(a)|}} + (\theta^2 + \theta - 2)\sqrt{|f(a)|} + \varepsilon. \end{aligned}$$

(Notice that $\bar g(a) = g(a)$.) If $x \in [c, b] \setminus \mathcal{U}_0(g)$, then

$$\begin{aligned} |g(x) - g(a)| &\le |\hat g(x) - \hat g(b)| + |g(b) - g(a)| \\ &\le \frac{\mathrm{w}(b, [c, b))}{\sqrt{|f(b)|}} + \varepsilon + (\theta - 1)\sqrt{|f(a)|} \\ &\le \frac{\mathrm{w}(a, K) + |f(b) - f(a)|}{\sqrt{|f(b)|}} + (\theta - 1)\sqrt{|f(a)|} + \varepsilon \\ &\le \frac{\theta \mathrm{w}(a, K)}{\sqrt{|f(a)|}} + (\theta^2 - 1)\sqrt{|f(a)|} + (\theta - 1)\sqrt{|f(a)|} + \varepsilon \\ &= \frac{\theta \mathrm{w}(a, K)}{\sqrt{|f(a)|}} + (\theta^2 + \theta - 2)\sqrt{|f(a)|} + \varepsilon. \end{aligned}$$

Similarly we can show that for each $x \in \operatorname{cl} K \setminus \mathcal{U}_0(h)$,

$$|h(x) - h(a)| \le \frac{\theta \mathrm{w}(a, K)}{\sqrt{|f(a)|}} + (\theta^2 + \theta - 2)\sqrt{|f(a)|} + \varepsilon. \qquad \square$$

The next two lemmas allow us to change a bit the constructed functions. Lemma 3.4 enables joining constructed functions, while Lemma 3.5 helps us diminish the maximal values taken by each of the constructed functions on its set of points of discontinuity.

Lemma 3.4. *Assume $g, h\colon [c, d] \to \mathbb{R}$ are continuous. There are continuous functions $\bar g, \bar h, \hat g, \hat h\colon [c, d] \to \mathbb{R}$ such that $\bar g\bar h = \hat g\hat h = gh$ on $[c, d]$, $\bar g(c) = g(c)$, $\bar h(c) = h(c)$, $|\bar g(d)| = 1$, $|\hat g(c)| = 1$, $\hat g(d) = g(d)$, and $\hat h(d) = h(d)$.*

PROOF. If $g(e) \neq 0$ for some $e \in [c, d)$, then define

$$\bar g(x) \overset{\mathrm{df}}{=} \begin{cases} g(x) & \text{if } x \in [c, e], \\ \operatorname{sgn} g(e) & \text{if } x = d, \\ \text{linearly} & \text{in } [e, d], \end{cases} \qquad \bar h(x) \overset{\mathrm{df}}{=} \begin{cases} h(x) & \text{if } x \in [c, e], \\ \dfrac{g(x)h(x)}{\bar g(x)} & \text{if } x \in [e, d]. \end{cases}$$

In the opposite case notice that $g = 0$ on $[c, d]$. Take any $e \in (c, d)$ and define

$$\bar{g}(x) \stackrel{\text{df}}{=} \begin{cases} 0 & \text{if } x \in [c,e], \\ 1 & \text{if } x = d, \\ \text{linearly} & \text{in } [e,d], \end{cases} \qquad \bar{h}(x) \stackrel{\text{df}}{=} \begin{cases} h(c) & \text{if } x = c, \\ 0 & \text{if } x \in [e,d], \\ \text{linearly} & \text{in } [c,e]. \end{cases}$$

Clearly in both cases the functions $\bar{g}$ and $\bar{h}$ fulfill the claimed conditions.

Analogously we can construct the functions $\hat{g}$ and $\hat{h}$. □

Lemma 3.5. *Assume that $L = (a, b) \subset \mathcal{U}_1(f)$ is bounded and $\operatorname{bd} L \subset \mathcal{U}_0(f)$. For each $\varepsilon > 0$, there is a continuous function $g \colon \operatorname{cl} L \to (0, \infty)$ such that $g = 1$ on $\operatorname{bd} L$ and $|f/g| \le \varepsilon$ on $L \setminus \mathcal{U}_0(f)$.*

PROOF. Notice that $C \stackrel{\text{df}}{=} [a, b] \setminus \mathcal{U}_0(f) \subset L$ is compact. So, since $C \subset \widetilde{\mathcal{U}}_1(f)$, the restriction $f \restriction C$ is continuous and bounded. Put

$$a' \stackrel{\text{df}}{=} \min C > a, \qquad b' \stackrel{\text{df}}{=} \max C < b, \qquad T \stackrel{\text{df}}{=} \max |f|[C].$$

Define

$$g(x) \stackrel{\text{df}}{=} \begin{cases} 1 & \text{if } x \in \operatorname{bd} L, \\ T/\varepsilon & \text{if } x \in [a', b'], \\ \text{linearly} & \text{in the intervals } [a, a'] \text{ and } [b', b]. \end{cases}$$

Clearly g has all required properties. □

Now we turn to arbitrary connected components of $\mathcal{U}_1(f)$.

Lemma 3.6. *Assume that J is a connected component of $\mathcal{U}_1(f)$ (J need not be bounded). For each $\varepsilon > 0$, there are functions $g, h \colon \operatorname{cl} J \to \mathbb{R}$ such that*

- *$f = gh$ on $\operatorname{cl} J$,*
- *$|g| = |h| = \sqrt{|f|}$ on $\operatorname{bd} J$,*
- *$g \restriction \operatorname{cl} J \setminus \mathcal{U}_0(g)$ and $h \restriction \operatorname{cl} J \setminus \mathcal{U}_0(h)$ are continuous,*
- *$|g| \le \sqrt{\sup |f|[\operatorname{bd} J]} + \varepsilon$ on $J \setminus \mathcal{U}_0(g)$,*
- *$|h| \le \sqrt{\sup |f|[\operatorname{bd} J]} + \varepsilon$ on $J \setminus \mathcal{U}_0(h)$.*

PROOF. Put $a \stackrel{\text{df}}{=} \inf J$ and $b \stackrel{\text{df}}{=} \sup J$. We consider several cases.

Case 1. $(J \cap \widetilde{\mathcal{U}}_1(f))' \cap \operatorname{bd} J \subset [f = 0]$.

Choose a strictly increasing sequence $(a_z)_{z\in\mathbb{Z}} \subset J \cap \mathcal{U}_0(f)$ with limit points a and b. For each $z \in \mathbb{Z}$, use Lemma 3.5 to define a continuous function $g_z\colon [a_z, a_{z+1}] \to (0,\infty)$ such that

$$g_z = 1 \quad \text{on } \{a_z, a_{z+1}\}, \qquad |f/g_z| \le \frac{\varepsilon}{|z|+1} \quad \text{on } (a_z, a_{z+1}) \setminus \mathcal{U}_0(f).$$

Define

$$g(x) \overset{\text{df}}{=} \begin{cases} \sqrt{|f(x)|} \cdot \operatorname{sgn} f(x) & \text{if } x \in \operatorname{bd} J, \\ g_z(x), & \text{if } x \in [a_z, a_{z+1}],\ z \in \mathbb{Z}, \end{cases}$$

$$h(x) \overset{\text{df}}{=} \begin{cases} \sqrt{|f(x)|}, & \text{if } x \in \operatorname{bd} J. \\ f(x)/g(x) & \text{otherwise.} \end{cases}$$

Then evidently $f = gh$ on $\operatorname{cl} J$, $|g| = |h| = \sqrt{|f|}$ on $\operatorname{bd} J$, and $g{\restriction}J$ is continuous.

Clearly $J \cap \mathcal{U}_0(h) = J \cap \mathcal{U}_0(f)$. Since

$$J \setminus \mathcal{U}_0(h) = \bigcup_{n\in\mathbb{Z}} \big([a_z, a_{z+1}] \setminus \mathcal{U}_0(h)\big),$$

$h{\restriction}J \setminus \mathcal{U}_0(h)$ is continuous. If $x \in \operatorname{bd} J \cap (J \cap \widetilde{\mathcal{U}}_1(f))'$, then using the properties of g_z, we obtain

$$\lim_{t\to x,\, t\in J\setminus\mathcal{U}_0(h)} |h(t)| \le \lim_{z\to\pm\infty} \frac{\varepsilon}{|z|+1} = 0 = h(x).$$

Finally if $x \in \operatorname{bd} J \setminus (J \cap \widetilde{\mathcal{U}}_1(f))'$, then x is isolated in $\operatorname{cl} J \setminus \mathcal{U}_0(h)$. Clearly the other requirements of our lemma are also fulfilled.

Case 2. $a \in [f \ne 0] \cap (J \cap \widetilde{\mathcal{U}}_1(f))'$ and $b \notin (J \cap \widetilde{\mathcal{U}}_1(f))'$ or $b \in [f = 0]$.

Recall that $a \notin \mathcal{U}_1(f)$. Using (2.1) and (2.2) we can find a $\delta > 0$ such that

$$(a, a+\delta) \cap \big(\textstyle\bigcup_{I\in\mathcal{I}} \operatorname{bd} I\big)' = \emptyset, \qquad \mathrm{w}(a, (a, a+\delta)) < \varepsilon\sqrt{|f(a)|}/2.$$

By our assumption, there is a $b' \in (a, a+\delta) \setminus \mathcal{U}_0(f)$. Use Lemma 3.2 to construct functions $\bar{g}, \bar{h}\colon [a, b'] \to \mathbb{R}$ such that

- $f = \bar{g}\bar{h}$ on $[a, b']$,
- $\bar{h}(a) = \bar{h}(b') = \sqrt{|f(a)|}$,
- $\bar{g}{\restriction}[a, b'] \setminus \mathcal{U}_0(\bar{g})$ and $\bar{h}{\restriction}[a, b'] \setminus \mathcal{U}_0(\bar{h})$ are continuous,
- $|\bar{g} - \bar{g}(a)| \le \varepsilon$ on $[a, b'] \setminus \mathcal{U}_0(\bar{g})$,

- $|\bar{h} - \bar{h}(a)| \le \varepsilon$ on $[a, b'] \setminus \mathcal{U}_0(\bar{h})$.

Since $\bar{g}$ and $\bar{h}$ are Baire one star functions, we can find a closed interval

$$[c, d] \subset (a, b') \cap \mathcal{U}_0(\bar{g}) \cap \mathcal{U}_0(\bar{h}).$$

From Lemma 3.4 we conclude that we may assume $|\bar{g}(d)| = 1$.

Take a sequence $(b_n) \subset J \cap \mathcal{U}_0(f)$ such that $b_0 = d$ and $b_n \nearrow b$. For each n, use Lemma 3.5 to construct a continuous function $g_n\colon [b_n, b_{n+1}] \to (0, \infty)$ such that $g_n(b_n) = g_n(b_{n+1}) = 1$ and $|f/g_n| \le \varepsilon/(n+1)$ on $(b_n, b_{n+1}) \setminus \mathcal{U}_0(f)$.

Define

$$g(x) \stackrel{\text{df}}{=} \begin{cases} \bar{g}(d) \cdot \bar{g}(x) & \text{if } x \in [a, d], \\ g_n(x) & \text{if } x \in [b_n, b_{n+1}],\ n \in \mathbb{N}, \\ \sqrt{|f(x)|} \cdot \operatorname{sgn} f(x) & \text{if } x = b \in \mathbb{R}, \end{cases}$$

$$h(x) \stackrel{\text{df}}{=} \begin{cases} \bar{g}(d) \cdot \bar{h}(x) & \text{if } x \in [a, d], \\ f(x)/g_n(x) & \text{if } x \in [b_n, b_{n+1}],\ n \in \mathbb{N}, \\ \sqrt{|f(x)|} & \text{if } x = b \in \mathbb{R}. \end{cases}$$

Then clearly $f = gh$ on $\operatorname{cl} J$, $|g| = |h| = \sqrt{|f|}$ on $\operatorname{bd} J$, and $g\restriction[a, c] \setminus \mathcal{U}_0(g)$, $h\restriction[a, c] \setminus \mathcal{U}_0(h)$, $g\restriction[c, b)$, and $h\restriction[c, b) \setminus \mathcal{U}_0(h)$ are continuous. Similarly to Case 1 we can prove that if $b \in \mathbb{R}$, then $g\restriction \operatorname{cl} J \setminus \mathcal{U}_0(g)$ and $h\restriction \operatorname{cl} J \setminus \mathcal{U}_0(h)$ are continuous at b. The other two requirements of our lemma are evident.

Case 3. $b \in [f \ne 0] \cap (J \cap \widetilde{\mathcal{U}}_1(f))'$ and $a \notin (J \cap \widetilde{\mathcal{U}}_1(f))'$ or $a \in [f = 0]$.

We proceed analogously to Case 2, using an analog of Lemma 3.2.

Case 4. $a, b \in [f \ne 0] \cap (J \cap \widetilde{\mathcal{U}}_1(f))'$.

Choose $b', a' \in J \cap \widetilde{\mathcal{U}}_1(f)$ such that $b' < a'$ and

$$\begin{aligned} &(a, b') \cap \left(\textstyle\bigcup_{I \in \mathcal{I}} \operatorname{bd} I\right)' = \emptyset, && \mathrm{w}(a, (a, b')) < \varepsilon\sqrt{|f(a)|}/2, \\ &(a', b) \cap \left(\textstyle\bigcup_{I \in \mathcal{I}} \operatorname{bd} I\right)' = \emptyset, && \mathrm{w}(b, (a', b)) < \varepsilon\sqrt{|f(b)|}/2. \end{aligned}$$

Proceeding similarly to Case 2, we can construct functions $\bar{g}, \bar{h}\colon [a, b'] \to \mathbb{R}$ and $\hat{g}, \hat{h}\colon [a', b] \to \mathbb{R}$ such that

- $f = \bar{g}\bar{h}$ on $[a, b']$,
- $\bar{h}(a) = \bar{h}(b') = \sqrt{|f(a)|}$,
- $\bar{g}\restriction[a, b'] \setminus \mathcal{U}_0(\bar{g})$ and $\bar{h}\restriction[a, b'] \setminus \mathcal{U}_0(\bar{h})$ are continuous,
- $|\bar{g} - \bar{g}(a)| \le \varepsilon$ on $[a, b'] \setminus \mathcal{U}_0(\bar{g})$,

- $|\bar{h} - \bar{h}(a)| \le \varepsilon$ on $[a, b'] \setminus \mathcal{U}_0(\bar{h})$,
- $f = \hat{g}\hat{h}$ on $[a', b]$,
- $\hat{h}(a') = \hat{h}(b) = \sqrt{|f(b)|}$,
- $\hat{g}\restriction[a', b] \setminus \mathcal{U}_0(\hat{g})$ and $\hat{h}\restriction[a', b] \setminus \mathcal{U}_0(\hat{h})$ are continuous,
- $|\hat{g} - \hat{g}(b)| \le \varepsilon$ on $[a', b] \setminus \mathcal{U}_0(\hat{g})$,
- $|\hat{h} - \hat{h}(b)| \le \varepsilon$ on $[a', b] \setminus \mathcal{U}_0(\hat{h})$.

Take any closed intervals

$$[c', d] \subset (a, b') \cap \mathcal{U}_0(\bar{g}) \cap \mathcal{U}_0(\bar{h}), \qquad [c, d'] \subset (a', b) \cap \mathcal{U}_0(\hat{g}) \cap \mathcal{U}_0(\hat{h}).$$

From Lemma 3.4 we conclude that we may assume $|\bar{g}(d)| = |\hat{g}(c)| = 1$. Use Lemma 3.5 to construct a continuous function $\tilde{g}\colon [d, c] \to (0, \infty)$ such that $\tilde{g}(d) = \tilde{g}(c) = 1$ and $|f/\tilde{g}| \le \varepsilon$ on $(d, c) \setminus \mathcal{U}_0(f)$. Define

$$g(x) \stackrel{\text{df}}{=} \begin{cases} \bar{g}(d) \cdot \bar{g}(x) & \text{if } x \in [a, d], \\ \tilde{g}(x) & \text{if } x \in [d, c], \\ \hat{g}(c) \cdot \hat{g}(x) & \text{if } x \in [c, b], \end{cases} \qquad h(x) \stackrel{\text{df}}{=} \begin{cases} \bar{g}(d) \cdot \bar{h}(x) & \text{if } x \in [a, d], \\ f(x)/\tilde{g}(x) & \text{if } x \in [d, c], \\ \hat{g}(c) \cdot \hat{h}(x) & \text{if } x \in [c, b]. \end{cases}$$

It is easy to verify that the requirements of our lemma are fulfilled. □

Before we state the next lemma, notice that since $f \in \mathcal{S}_2$, the restriction of f to the closed set $\widetilde{\mathcal{U}}_2(f) = \mathbb{R} \setminus \mathcal{U}_1(f)$ is continuous. So, the set

$$F_0 \stackrel{\text{df}}{=} [f = 0] \cap \widetilde{\mathcal{U}}_2(f)$$

is closed. We will construct the functions g and h on connected components of its complement. Lemma 3.7 is the key to the construction of functions g and h on $\mathbb{R}$.

Lemma 3.7. *Assume that P is a connected component of $\mathbb{R} \setminus F_0$. For each $\varepsilon > 0$, there are $g, h\colon \operatorname{cl} P \to \mathbb{R}$ such that*

- $f = gh$ *on* P,
- $g = h = 0$ *on* $\operatorname{bd} P$,
- $g\restriction \operatorname{cl} P \setminus \mathcal{U}_0(g)$ *and* $h\restriction \operatorname{cl} P \setminus \mathcal{U}_0(h)$ *are continuous,*
- $|g| < 2\sqrt{2}\sqrt{\sup |f|[\operatorname{cl} P \cap \widetilde{\mathcal{U}}_2(f)]} + \varepsilon$ *on* $P \setminus \mathcal{U}_0(g)$,

- $|h| < 2\sqrt{2}\sqrt{\sup|f|[\operatorname{cl} P \cap \widetilde{\mathcal{U}}_2(f)]} + \varepsilon$ *on* $P \setminus \mathcal{U}_0(h)$.

Proof. Let $\mathcal{J}$ be the family of all connected components of $P \cap \mathcal{U}_1(f)$. Take any $M \in \mathcal{J}$ and let $\mathcal{J}'$ be the family of all intervals $J = (a, b) \in \mathcal{J}$ which satisfy at least one of the following conditions:

- J is unbounded,
- $J = M$,
- $J \cap \left(\bigcup_{I \in \mathcal{I}} \operatorname{bd} I\right)' \neq \emptyset$,
- $f(a) = 0$ or $f(b)/f(a) \notin (1/2, 2)$,
- $\mathrm{w}(a, J) \geq |f(a)|$.

(Notice that in particular, if $J \in \mathcal{J}$ and $\operatorname{bd} J \cap \operatorname{bd} P \neq \emptyset$, then $J \in \mathcal{J}'$, since $\operatorname{bd} P \subset [f = 0]$.) Define

$$F \stackrel{\mathrm{df}}{=} \bigcup_{J \in \mathcal{J}'} \operatorname{bd} J \cup \operatorname{bd} P \subset \widetilde{\mathcal{U}}_2(f).$$

In the rest of the proof, we will mark several parts as claims. We will use the symbol $\lhd$ to denote the end of the proof of such a claim.

Claim 3.1. $F' \subset \operatorname{bd} P$.

Indeed, take any one-to-one sequence $(x_k) \subset F$ convergent to some $x \in \mathbb{R}$. Then clearly $x \in \widetilde{\mathcal{U}}_2(f)$. Without loss of generality, we may assume that for each $k \in \mathbb{N}$, there is an interval $J_k \in \mathcal{J}'$ such that $x_k \in \operatorname{bd} J_k$, and that the sequence (J_k) is one-to-one.

Toward a contradiction, suppose that $x \notin \operatorname{bd} P$. Then $f(x) \neq 0$, since $x \notin F_0$. By (2.1) and (2.2), there is a $\delta > 0$ such that

$$(x - \delta, x + \delta) \cap \left(\bigcup_{I \in \mathcal{I}} \operatorname{bd} I\right)' \cap \mathcal{U}_1(f) = \emptyset, \tag{3.8}$$

$$\mathrm{w}(x, (x - \delta, x + \delta)) < \frac{|f(x)|}{3}. \tag{3.9}$$

Recall that $f \restriction \widetilde{\mathcal{U}}_2(f)$ is continuous. So, we may assume that

$$(\forall t \in \widetilde{\mathcal{U}}_2(f)) \left(|t - x| \leq \delta \Rightarrow |f(t) - f(x)| < \frac{|f(x)|}{3}\right). \tag{3.10}$$

Since the sequence (J_k) is one-to-one and $x_k \to x$, there is a $k \in \mathbb{N}$ such that $J_k \subset P \cap (x - \delta, x + \delta)$ and $J_k \neq M$. We will show that $J_k \notin \mathcal{J}'$, which is a contradiction.

Clearly J_k is bounded, and by (3.8), $J_k \cap \left(\bigcup_{I\in\mathcal{I}} \operatorname{bd} I\right)' = \emptyset$. Put $a \overset{\text{df}}{=} \inf J_k$ and $b \overset{\text{df}}{=} \sup J_k$. By (3.10), $f(a)f(x) > 0$ and $f(b)f(x) > 0$. Hence also $f(b)/f(a) > 0$ and

$$\frac{1}{2} = \frac{2|f(x)|}{3}\Big/\frac{4|f(x)|}{3} < f(b)/f(a) < \frac{4|f(x)|}{3}\Big/\frac{2|f(x)|}{3} = 2.$$

Moreover by (3.10) and (3.9),

$$\mathrm{w}(a, J_k) \le |f(a) - f(x)| + \mathrm{w}(x, J_k) < \frac{|f(x)|}{3} + \frac{|f(x)|}{3} = \frac{2|f(x)|}{3} < |f(a)|.$$

It follows that $J_k \notin \mathcal{J}'$, a contradiction which completes the proof of Claim 3.1. $\lhd$

Claim 3.2. If $L \subset P$ is an open interval disjoint from $\bigcup \mathcal{J}'$, then $f(t)f(u) > 0$ for all $t, u \in L \cap \widetilde{\mathfrak{U}}_2(f)$.

Toward a contradiction suppose there are $t, u \in L \cap \widetilde{\mathfrak{U}}_2(f)$ with $t < u$ such that $f(t)f(u) \le 0$. Then $t, u \notin F_0$, whence $f(t)f(u_n) < 0$. Let, e.g., $f(t) < 0 < f(u)$. (The other case is analogous.) Since $f{\restriction}\widetilde{\mathfrak{U}}_2(f)$ is continuous, the sets $\widetilde{\mathfrak{U}}_2(f) \cap [f \le 0]$ and $\widetilde{\mathfrak{U}}_2(f) \cap [f \ge 0]$ are closed. Put

$$t' \overset{\text{df}}{=} \max\bigl([t, u] \cap \widetilde{\mathfrak{U}}_2(f) \cap [f \le 0]\bigr),$$
$$u' \overset{\text{df}}{=} \min\bigl([t', u] \cap \widetilde{\mathfrak{U}}_2(f) \cap [f \ge 0]\bigr).$$

Observe that $f(t') < 0$ (because $t' \in \widetilde{\mathfrak{U}}_2(f) \setminus F_0$), whence $t' < u$, and similarly $f(u') > 0$ and $u' > t'$. Since $(t', u') \cap \widetilde{\mathfrak{U}}_2(f) = \emptyset$, we have $(t', u') \in \mathcal{J}$. The relation $f(t')f(u') \notin (1/2, 2)$ yields $(t', u') \in \mathcal{J}'$. But $(t', u') \subset (t, u)$, contrary to our assumption. $\lhd$

Claim 3.3. If $L \subset P$ is an open interval disjoint from $\bigcup \mathcal{J}'$ and $u \in L \cap \widetilde{\mathfrak{U}}_2(f)$, then $|f(t)| \operatorname{sgn} f(u) = f(t)$ for each $t \in \operatorname{cl} L \cap \widetilde{\mathfrak{U}}_2(f)$.

Toward a contradiction suppose there is a $t \in \operatorname{cl} L \cap \widetilde{\mathfrak{U}}_2(f)$ such that $|f(t)| \operatorname{sgn} f(u) \ne f(t)$. Then $\operatorname{sgn} f(t) = -\operatorname{sgn} f(u)$. (Notice that by Claim 3.2, $\operatorname{sgn} f(u) \ne 0$.) So by Claim 3.2, $t \notin L$. Let, e.g., $t = \inf L$. (The other case is analogous.) Put $z \overset{\text{df}}{=} \inf\bigl((t, u] \cap \widetilde{\mathfrak{U}}_2(f)\bigr)$. Since $(t, u] \cap \widetilde{\mathfrak{U}}_2(f) \subset [f \cdot f(u) > 0]$ (cf. Claim 3.2) and the restriction $f{\restriction}\widetilde{\mathfrak{U}}_2(f)$ is continuous, $f(z)f(u) \ge 0$, whence $z > t$. It follows that $(t, z) \in \mathcal{J}'$, contrary to our assumption. $\lhd$

Arrange all elements of $\mathcal{J}'$ in a sequence $\langle J_m\,;\, m < \xi_0\rangle$, where $\xi_0 \le \omega_0$. For each $m < \xi_0$, use Lemma 3.6 to construct functions $\bar{g}_m, \bar{h}_m \colon \operatorname{cl} J_m \to \mathbb{R}$ such that

- $f = \bar{g}_m \bar{h}_m$ on $\operatorname{cl} J_m$,
- $|\bar{g}_m| = |\bar{h}_m| = \sqrt{|f|}$ on $\operatorname{bd} J_m$,
- $\bar{g}_m \restriction \operatorname{cl} J_m \setminus \mathcal{U}_0(\bar{g}_m)$ and $\bar{h}_m \restriction \operatorname{cl} J_m \setminus \mathcal{U}_0(\bar{h}_m)$ are continuous,
- $|\bar{g}_m| \le \sqrt{\sup |f|[\operatorname{bd} J_m]} + \varepsilon/(m+2)$ on $J_m \setminus \mathcal{U}_0(\bar{g}_m)$,
- $|\bar{h}_m| \le \sqrt{\sup |f|[\operatorname{bd} J_m]} + \varepsilon/(m+2)$ on $J_m \setminus \mathcal{U}_0(\bar{h}_m)$.

Similarly arrange all elements of $\mathcal{J} \setminus \mathcal{J}'$ in a sequence $\langle K_p \,;\, p < \xi_1 \rangle$, where $\xi_1 \le \omega_0$. For each $p < \xi_1$, define $a_p \overset{\text{df}}{=} \inf K_p$ and $b_p \overset{\text{df}}{=} \sup K_p$, observe that $f(a_p) f(b_p) > 0$ (cf. Claim 3.2), and use Lemma 3.3 to construct functions $\hat{g}_p, \hat{h}_p \colon \operatorname{cl} K_p \to \mathbb{R}$ such that

- $f = \hat{g}_p \hat{h}_p$ on $\operatorname{cl} K_p$,
- $\hat{h}_p = \sqrt{|f|}$ on $\operatorname{bd} K_p$,
- $\hat{g}_p \restriction \operatorname{cl} K_p \setminus \mathcal{U}_0(\hat{g}_p)$ and $\hat{h}_p \restriction \operatorname{cl} K_p \setminus \mathcal{U}_0(\hat{h}_p)$ are continuous,
- $|\hat{g}_p - \hat{g}_p(a_p)| \le \theta_p \mathrm{w}(a_p, K_p)/\sqrt{|f(a_p)|} + (\theta_p^2 + \theta_p - 2)\sqrt{|f(a_p)|} + \varepsilon/(p+2)$ on $\operatorname{cl} K_p \setminus \mathcal{U}_0(\hat{g}_p)$, where
$$\theta_p \overset{\text{df}}{=} \max\bigl\{ \sqrt{f(a_p)/f(b_p)}, \sqrt{f(b_p)/f(a_p)} \,\bigr\} \in [1, \sqrt{2}),$$
- $|\hat{h}_p - \hat{h}_p(a_p)| \le \theta_p \mathrm{w}(a_p, K_p)/\sqrt{|f(a_p)|} + (\theta_p^2 + \theta_p - 2)\sqrt{|f(a_p)|} + \varepsilon/(p+2)$ on $\operatorname{cl} K_p \setminus \mathcal{U}_0(\hat{h}_p)$.

By induction on $n \in \mathbb{N}$ we will choose an interval $L_n = (u_n, v_n) \subset P$ and define functions $g_n, h_n \colon \operatorname{cl} L_n \to \mathbb{R}$ so that:

- $\operatorname{bd} L_n \subset F$,
- if $n > 0$ and $\inf P < u_{n-1}$, then $u_n < u_{n-1}$,
- if $n > 0$ and $\sup P > v_{n-1}$, then $v_n > v_{n-1}$,
- if $n > 0$, then $g_n \restriction \operatorname{cl} L_{n-1} = g_{n-1}$ and $h_n \restriction \operatorname{cl} L_{n-1} = h_{n-1}$,
- $f = g_n h_n$ on $\operatorname{cl} L_n$,
- $|g_n| = |h_n| = \sqrt{|f|}$ on $\operatorname{bd} L_n$,

- for each $m < \xi_0$, if $J_m \subset L_n$, then
$$|g_n(x)| = |\bar{g}_m(x)|, \qquad |h_n(x)| = |\bar{h}_m(x)|$$
for all $x \in \operatorname{cl} J_m$,
- for each $m < \xi_0$, if $K_p \subset L_n$, then
$$|g_n(x)| = |\hat{g}_p(x)|, \qquad |h_n(x)| = |\hat{h}_p(x)|$$
for all $x \in \operatorname{cl} K_p$,
- $g_n \restriction \operatorname{cl} L_n \setminus \mathcal{U}_0(g_n)$ and $h_n \restriction \operatorname{cl} L_n \setminus \mathcal{U}_0(h_n)$ are continuous.

First put $L_0 \stackrel{\text{df}}{=} M$. For all $x \in \operatorname{cl} L_0$, let $g_0(x) \stackrel{\text{df}}{=} \bar{g}_0(x)$ and $h_0(x) \stackrel{\text{df}}{=} \bar{h}_0(x)$. Then clearly the above conditions hold for $n = 0$.

Assume that for some $n \in \mathbb{N}$ we have already chosen the interval L_n and defined the functions g_n and h_n according to the induction hypothesis. Put
$$u_{n+1} \stackrel{\text{df}}{=} \begin{cases} \inf J_m & \text{if } u_n = \sup J_m \text{ for some } m < \xi_0, \\ \inf P & \text{if } (\inf P, u_n) \cap \bigcup \mathcal{J}' = \emptyset, \\ \sup\big((\inf P, u_n) \cap \bigcup \mathcal{J}'\big) & \text{otherwise,} \end{cases}$$
$$v_{n+1} \stackrel{\text{df}}{=} \begin{cases} \sup J_m & \text{if } v_n = \inf J_m \text{ for some } m < \xi_0, \\ \sup P & \text{if } (v_n, \sup P) \cap \bigcup \mathcal{J}' = \emptyset, \\ \inf\big((v_n, \sup P) \cap \bigcup \mathcal{J}'\big) & \text{otherwise,} \end{cases}$$
and let $L_{n+1} \stackrel{\text{df}}{=} (u_{n+1}, v_{n+1})$.

If $\inf P < u_n$, then

- either $u_n = \sup J_m$ for some $m < \xi_0$—then $u_{n+1} = \inf J_m < u_n$,
- or $(\inf P, u_n) \cap \bigcup \mathcal{J}' = \emptyset$—then $u_{n+1} = \inf P < u_n$,
- or neither of the above cases holds—then by Claim 3.1, there is an $m < \xi_0$ such that $u_{n+1} = \sup\big((\inf P, u_n) \cap \bigcup \mathcal{J}'\big) = \sup J_m < u_n$.

Notice that in all the above cases $u_{n+1} \in F$. Similarly we can show that if $\sup P > v_n$, then $v_{n+1} > v_n$ and that $v_{n+1} \in F$. So, $\operatorname{bd} L_{n+1} \subset F$.

For each $x \in \operatorname{cl} L_n$, define $g_{n+1}(x) \stackrel{\text{df}}{=} g_n(x)$ and $h_{n+1}(x) \stackrel{\text{df}}{=} h_n(x)$. Next we will define the functions g_{n+1} and h_{n+1} on $\operatorname{cl}(u_{n+1}, u_n)$. We may assume that

$\inf P < u_n$, since otherwise $\mathrm{cl}(u_{n+1}, u_n) = \emptyset$. If $u_n = \sup J_m$ for some $m < \xi_0$, then notice that since $u_n \notin F_0$, we have $f(u_n) \neq 0$ and

$$\tau \stackrel{\mathrm{df}}{=} \operatorname{sgn} h_n(u_n) \cdot \operatorname{sgn} \bar{h}_m(u_n) \in \{-1, 1\}.$$

Moreover:

$$\begin{aligned}\tau \bar{g}_m(u_n) &= \frac{|h_n(u_n)|}{h_n(u_n)} \cdot \frac{\bar{h}_m(u_n)\bar{g}_m(u_n)}{|\bar{h}_m(u_n)|} = \frac{f(u_n)}{h_n(u_n)} = g_n(u_n),\\ \tau \bar{h}_m(u_n) &= \operatorname{sgn} h_n(u_n) \cdot |\bar{h}_m(u_n)| = \operatorname{sgn} h_n(u_n) \cdot |h_n(u_n)| = h_n(u_n).\end{aligned}$$

(Recall that $|h_n(u_n)| = |\bar{h}_m(u_n)| = \sqrt{|f(u_n)|}$.) So, if we define

$$g_{n+1}(x) \stackrel{\mathrm{df}}{=} \tau \bar{g}_m(x), \qquad h_{n+1}(x) \stackrel{\mathrm{df}}{=} \tau \bar{h}_m(x)$$

for $x \in \mathrm{cl}(u_{n+1}, u_n) = \mathrm{cl}\, J_m$, then g_{n+1} and h_{n+1} are well-defined. Evidently

$$f = g_{n+1}h_{n+1}, \quad |g_{n+1}| = |\bar{g}_m|, \quad |h_{n+1}| = |\bar{h}_m| \qquad \text{on } \mathrm{cl}\, J_m,$$

and

$$|g_{n+1}(\inf L_{n+1})| = |h_{n+1}(\inf L_{n+1})| = \sqrt{|f(\inf L_{n+1})|}.$$

By definition, $g_{n+1} \restriction \mathrm{cl}(u_{n+1}, v_n) \setminus \mathcal{U}_0(g_{n+1})$ and $h_{n+1} \restriction \mathrm{cl}(u_{n+1}, v_n) \setminus \mathcal{U}_0(h_{n+1})$ are continuous.

Now assume that $u_n \notin \{\sup J_m\,;\, m < \xi_0\}$. Then either $u_{n+1} = \inf P$ or $u_{n+1} = \sup J_m$ for some $m < \xi_0$. For each $x \in \mathrm{cl}(u_{n+1}, u_n)$, define

$$g_{n+1}(x) \stackrel{\mathrm{df}}{=} \operatorname{sgn} h_n(u_n) \cdot \begin{cases} \hat{g}_p(x) & \text{if } x \in K_p \subset (u_{n+1}, u_n),\ p < \xi_1,\\ \sqrt{|f(x)|} \cdot \operatorname{sgn} f(u_n) & \text{if } x \in \mathrm{cl}(u_{n+1}, u_n) \setminus \bigcup \mathcal{J}, \end{cases}$$

$$h_{n+1}(x) \stackrel{\mathrm{df}}{=} \operatorname{sgn} h_n(u_n) \cdot \begin{cases} \hat{h}_p(x) & \text{if } x \in K_p \subset (u_{n+1}, u_n),\ p < \xi_1,\\ \sqrt{|f(x)|} & \text{if } x \in \mathrm{cl}(u_{n+1}, u_n) \setminus \bigcup \mathcal{J}. \end{cases}$$

Since $|g_n(u_n)| = |h_n(u_n)| = \sqrt{|f(u_n)|}$,

$$\begin{aligned}\operatorname{sgn} h_n(u_n) \cdot \sqrt{|f(u_n)|} \cdot \operatorname{sgn} f(u_n) &= \operatorname{sgn} g_n(u_n) \cdot |g_n(u_n)| = g_n(u_n),\\ \operatorname{sgn} h_n(u_n) \cdot \sqrt{|f(u_n)|} &= \operatorname{sgn} h_n(u_n) \cdot |h_n(u_n)| = h_n(u_n).\end{aligned}$$

It follows that the functions g_{n+1} and h_{n+1} are well-defined. By Claim 3.3, $f = g_{n+1}h_{n+1}$ on $\mathrm{cl}(u_{n+1}, u_n)$. It is evident that if $K_p \subset (u_{n+1}, u_n)$ for some $p < \xi_1$, then $|g_{n+1}| = |\hat{g}_p|$ and $|h_{n+1}| = |\hat{h}_p|$ on K_p, and that $|g_{n+1}(u_{n+1})| = |h_{n+1}(u_{n+1})| = \sqrt{|f(u_{n+1})|}$.

Claim 3.4. $g_{n+1}\restriction \operatorname{cl}(u_{n+1}, v_n) \setminus \mathcal{U}_0(g_{n+1})$ and $h_{n+1}\restriction \operatorname{cl}(u_{n+1}, v_n) \setminus \mathcal{U}_0(h_{n+1})$ are continuous.

Fix an $x \in \operatorname{cl}(u_{n+1}, v_n) \setminus \mathcal{U}_0(g_{n+1})$. Let $(x_k) \subset \operatorname{cl}(u_{n+1}, v_n) \setminus \mathcal{U}_0(g_{n+1})$ be any sequence convergent to x such that $g_{n+1}(x_k) \to y \in [-\infty, \infty]$. We consider several cases.

Case 1. If there is a subsequence $(x_{k_s}) \subset \widetilde{\mathcal{U}}_2(f) \setminus \operatorname{cl} L_n$, then $x \in \widetilde{\mathcal{U}}_2(f)$ as well. Using the continuity of $f\restriction\widetilde{\mathcal{U}}_2(f)$ we conclude that

$$\begin{aligned} y = \lim_{s\to\infty} g_{n+1}(x_{k_s}) &= \lim_{s\to\infty} \operatorname{sgn} h_n(u_n) \cdot \sqrt{|f(x_{k_s})|} \cdot \operatorname{sgn} f(u_n) \\ &= \operatorname{sgn} h_n(u_n) \cdot \sqrt{|f(x)|} \cdot \operatorname{sgn} f(u_n) = g_{n+1}(x). \end{aligned}$$

Case 2. If there is a subsequence $(x_{k_s}) \subset \operatorname{cl} L_n$, then by induction assumption, $y = \lim_{s\to\infty} g_n(x_{k_s}) = g_n(x) = g_{n+1}(x)$.

Case 3. If none of the above cases holds, then we may assume that for each $k \in \mathbb{N}$, there is a $p_k < \xi_1$ with $x_k \in K_{p_k}$.

Case 3.a) If $p_k \to \infty$, then $a_{p_k} \to x$ and $b_{p_k} \to x$, whence $x \in \widetilde{\mathcal{U}}_2(f)$. Using the continuity of $f\restriction\widetilde{\mathcal{U}}_2(f)$, Case 1, and the properties of $\hat{g}_p$, we conclude that

$$\begin{aligned} |y - g_{n+1}(x)| &= \lim_{k\to\infty} |g_{n+1}(x_k) - g_{n+1}(a_{p_k})| = \lim_{k\to\infty} |\hat{g}_{p_k}(x_k) - \hat{g}_{p_k}(a_{p_k})| \\ &\le \overline{\lim_{k\to\infty}} \Big(\frac{\theta_{p_k} \operatorname{w}(a_{p_k}, K_{p_k})}{\sqrt{|f(a_{p_k})|}} + (\theta_{p_k}^2 + \theta_{p_k} - 2)\sqrt{|f(a_{p_k})|} + \frac{\varepsilon}{p_k + 1} \Big). \end{aligned}$$

If $f(x) = 0$ (which is possible if $x = \inf P$), then since each $K_{p_k} \notin \mathcal{J}'$,

$$\begin{aligned} &\frac{\theta_{p_k} \operatorname{w}(a_{p_k}, K_{p_k})}{\sqrt{|f(a_{p_k})|}} + (\theta_{p_k}^2 + \theta_{p_k} - 2)\sqrt{|f(a_{p_k})|} \\ &\qquad \le \frac{\sqrt{2}\,|f(a_{p_k})|}{\sqrt{|f(a_{p_k})|}} + (\sqrt{2}^2 + \sqrt{2} - 2)\sqrt{|f(a_{p_k})|} \to 2\sqrt{2} \cdot \sqrt{|f(x)|} = 0. \end{aligned}$$

If $f(x) \neq 0$, then $\theta_{p_k} \to 1$ (because $f(a_{p_k}) \to f(x)$ and $f(b_{p_k}) \to f(x)$). For each k, let

$$\delta_k \stackrel{\text{df}}{=} \max\{|x - a_k|, |x - b_k|\}.$$

Since each $K_{p_k} \notin \mathcal{J}'$, by (2.2), we obtain

$$\begin{aligned}
&\varlimsup_{k\to\infty} \Big(\frac{\theta_{p_k} \mathrm{w}(a_{p_k}, K_{p_k})}{\sqrt{|f(a_{p_k})|}} + (\theta_{p_k}^2 + \theta_{p_k} - 2)\sqrt{|f(a_{p_k})|} \Big) \\
&\quad \le \varlimsup_{k\to\infty} \sqrt{\mathrm{w}(a_{p_k}, K_{p_k})} + (1^2 + 1 - 2)\sqrt{|f(x)|} \\
&\quad \le \varlimsup_{k\to\infty} \sqrt{\mathrm{w}(a_{p_k}, (x-\delta_k, x+\delta_k))} \\
&\quad \le \varlimsup_{k\to\infty} \sqrt{|f(a_{p_k}) - f(x)| + \mathrm{w}(x, (x-\delta_k, x+\delta_k))} = 0.
\end{aligned}$$

Case 3.b) Now assume that $p_k \not\to \infty$. There are a $p < \xi_1$ and a subsequence (x_{k_s}) such that $p_{k_s} = p$ for each s. Since $g_{n+1} \restriction \mathrm{cl}\, K_p = \operatorname{sgn} h_n(u_n) \cdot \hat{g}_p$,

$$y = \lim_{s\to\infty} g_{n+1}(x_{k_s}) = \operatorname{sgn} h_n(u_n) \lim_{s\to\infty} \hat{g}_p(x_{k_s}) = \operatorname{sgn} h_n(u_n) \cdot \hat{g}_p(x) = g_{n+1}(x).$$

In all cases we conclude that $y = g_{n+1}(x)$. So, $g_{n+1} \restriction \mathrm{cl}(u_{n+1}, v_n) \setminus \mathcal{U}_0(g_{n+1})$ is continuous. Similarly we can prove that $h_{n+1} \restriction \mathrm{cl}(u_{n+1}, v_n) \setminus \mathcal{U}_0(h_{n+1})$ is continuous. $\triangleleft$

Proceeding analogously we can define the functions g_{n+1} and h_{n+1} on $\mathrm{cl}(v_n, v_{n+1})$. This completes the induction procedure.

By Claim 3.1, $u_n \to \inf P$ and $v_n \to \sup P$. Define

$$g(x) \overset{\mathrm{df}}{=} \begin{cases} g_n(x) & \text{if } x \in \mathrm{cl}\, L_n,\ n \in \mathbb{N}, \\ 0 & \text{otherwise}, \end{cases} \qquad h(x) \overset{\mathrm{df}}{=} \begin{cases} h_n(x) & \text{if } x \in \mathrm{cl}\, L_n,\ n \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}$$

Then clearly $f = gh$ on P and $g = h = 0$ on $\mathrm{bd}\, P$.

Claim 3.5. The restrictions $g \restriction \mathrm{cl}\, P \setminus \mathcal{U}_0(g)$ and $h \restriction \mathrm{cl}\, P \setminus \mathcal{U}_0(h)$ are continuous.

By construction, the restrictions $g \restriction P \setminus \mathcal{U}_0(g)$ and $h \restriction P \setminus \mathcal{U}_0(h)$ are continuous. If $\inf P \in \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{cl}\, L_n$, then take any sequence $(x_k) \subset P \setminus \mathcal{U}_0(g)$ convergent to $\inf P$. Fix an $\eta > 0$ and choose an $n \in \mathbb{N}$ so that

$$\begin{gathered}
(\inf P, u_n) \cap \bigcup_{m < 2\varepsilon/\eta} J_m = (\inf P, u_n) \cap \bigcup_{p < 3\varepsilon/\eta} K_p = \emptyset, \\
|f| < \eta^2/18 \qquad \text{on } (\inf P, u_n) \cap \widetilde{\mathcal{U}}_2(f).
\end{gathered}$$

(The latter can be required since $f \restriction \widetilde{\mathcal{U}}_2(f)$ is continuous and $f(\inf P) = 0$.) Then for each $p < \xi_1$, if $K_p \subset (\inf P, u_n)$, then since $\mathrm{w}(a_p, K_p) < |f(a_p)|$,

$$\frac{\mathrm{w}(a_p, K_p)}{\sqrt{|f(a_p)|}} < \sqrt{|f(a_p)|} < \frac{\eta}{3\sqrt{2}}.$$

Hence if $x \in (\inf P, u_n) \setminus \mathcal{U}_0(g)$, then by definition of g on J_m, K_p, and $\widetilde{\mathcal{U}}_2(f)$, respectively, we have

$$
\begin{aligned}
|g(x)| \leq \max\Big\{ &\sup\Big\{ \sqrt{\sup|f|[\operatorname{bd} J_m]} + \frac{\varepsilon}{m+2}\,;\, m \geq 2\varepsilon/\eta \Big\},\\
&\sup\Big\{ \frac{\sqrt{2}\cdot \mathrm{w}(a_p, K_p)}{\sqrt{|f(a_p)|}} + (\sqrt{2}^{\,2} + \sqrt{2} - 2)\sqrt{|f(a_p)|} + \frac{\varepsilon}{p+2}\,;\\
&\qquad\qquad\qquad p \geq 3\varepsilon/\eta, K_p \subset (\inf P, u_n)\Big\},\\
&\sqrt{\sup|f|[(\inf P, u_n) \cap \widetilde{\mathcal{U}}_2(f)]}\ \Big\} < \eta.
\end{aligned}
$$

So, $\lim_{t\to\inf P,\, t\in P\setminus\mathcal{U}_0(g)} g(t) = 0 = g(\inf P)$.

Analogously we can prove that

$$\text{if } \sup P \in \mathbb{R} \setminus \textstyle\bigcup_{n\in\mathbb{N}} \operatorname{cl} L_n, \text{ then } \lim_{t\to\sup P,\, t\in P\setminus\mathcal{U}_0(g)} g(t) = 0 = g(\sup P),$$

and that $\lim_{t\to x,\, t\in P\setminus\mathcal{U}_0(h)} h(t) = 0 = h(x)$ for each $x \in \operatorname{bd} P$. $\triangleleft$

It is easy to verify that $|g| < 2\sqrt{2}\sqrt{\sup|f|[\operatorname{cl} P \cap \widetilde{\mathcal{U}}_2(f)]} + \varepsilon$ on $P \setminus \mathcal{U}_0(g)$ and $|h| < 2\sqrt{2}\sqrt{\sup|f|[\operatorname{cl} P \cap \widetilde{\mathcal{U}}_2(f)]} + \varepsilon$ on $P \setminus \mathcal{U}_0(h)$. $\square$

4 Proof of Theorem 2.1

ii) $\Rightarrow$ i).

Arrange all connected components of $\mathbb{R} \setminus F_0$ in a sequence $\langle P_n\,;\, n < \xi\rangle$, where $\xi \leq \omega_0$. For $x \in F_0$, define $g(x) \overset{\mathrm{df}}{=} 0$ and $h(x) \overset{\mathrm{df}}{=} 0$. For each $n < \xi$, use Lemma 3.7 to construct functions g and h on P_n so that

- $f = gh$ on P_n,
- $g{\restriction}\operatorname{cl} P_n \setminus \mathcal{U}_0(g)$ and $h{\restriction}\operatorname{cl} P_n \setminus \mathcal{U}_0(h)$ are continuous,
- $|g| < 2\sqrt{2}\sqrt{\sup|f|[\operatorname{cl} P_n \cap \widetilde{\mathcal{U}}_2(f)]} + 1/(n+1)$ on $P_n \setminus \mathcal{U}_0(g)$,
- $|h| < 2\sqrt{2}\sqrt{\sup|f|[\operatorname{cl} P_n \cap \widetilde{\mathcal{U}}_2(f)]} + 1/(n+1)$ on $P_n \setminus \mathcal{U}_0(h)$.

Then clearly $f = gh$ on $\mathbb{R}$. We will prove that $g, h \in \mathcal{S}_1$, showing that the restrictions $g{\restriction}\mathbb{R} \setminus \mathcal{U}_0(g)$ and $h{\restriction}\mathbb{R} \setminus \mathcal{U}_0(h)$ are continuous.

Take any sequence $(x_k) \subset \mathbb{R} \setminus \mathcal{U}_0(g)$ convergent to some $x \in \mathbb{R}$ such that $g(x_k) \to y \in [-\infty, \infty]$. If there is a subsequence $(x_{k_s}) \subset F_0$, then $x \in F_0$ as well and

$$y = \lim_{s\to\infty} g(x_{k_s}) = 0 = g(x).$$

So, assume that $(x_k) \subset \mathbb{R} \setminus F_0$. For each k, choose an $n_k < \xi$ such that $x_k \in P_{n_k}$. If there is an $n < \xi$ such that $x_k \in P_n$ for infinitely many k, then $x \in \operatorname{cl} P_n$ and $y = g(x)$ by continuity of g on $\operatorname{cl} P_n \setminus \mathcal{U}_0(g)$.

In the opposite case $n_k \to \infty$. For each k, choose a $t_k \in \operatorname{cl} P_{n_k} \cap \widetilde{\mathcal{U}}_2(f)$ such that

$$|g(x_k)| < 2\sqrt{2}\sqrt{|f(t_k)|} + 1/(n_k + 1).$$

Notice that in the present case $x \in F_0$. Since $t_k \to x$ and $f \restriction \widetilde{\mathcal{U}}_2(f)$ is continuous,

$$|y| = \lim_{k\to\infty} |g(x_k)| \le \overline{\lim_{k\to\infty}}\, 2\sqrt{2}\sqrt{|f(t_k)|} + \lim_{k\to\infty} 1/(n_k + 1) = 0 = g(x).$$

It follows that g is continuous at x. So, $\mathcal{U}_1(g) = \mathbb{R}$ and, by definition, $g \in \mathcal{S}_1$. Analogously we can show that $h \in \mathcal{S}_1$.

i) $\Rightarrow$ ii).

Proceeding as in the proof of [2, Theorem 7], we can prove that $f \in \mathcal{S}_2$.

Fix an $x \in [f \neq 0] \setminus \mathcal{U}_1(f)$. First we will prove that (2.1) holds.

Suppose toward a contradiction that there is a sequence

$$(x_k) \subset \big(\textstyle\bigcup_{I\in\mathcal{I}} \operatorname{bd} I\big)' \cap \mathcal{U}_1(f)$$

convergent to x. Fix a $k \in \mathbb{N}$.

Claim 4.1. $x_k \in \big([g = 0] \setminus \mathcal{U}_0(g)\big) \cup \big([h = 0] \setminus \mathcal{U}_0(h)\big)$.

Indeed, since x_k is an accumulation point of $\bigcup_{I\in\mathcal{I}} \operatorname{bd} I$, there is a one-to-one sequence $(I_{k,n}) \subset \mathcal{I}$ such that

$$x_k = \lim_{n\to\infty} \inf I_{k,n} = \lim_{n\to\infty} \sup I_{k,n}.$$

We may assume that $\operatorname{bd} I_{k,n} \subset \mathcal{U}_1(f)$ (in fact, $\operatorname{bd} I_{k,n} \subset \widetilde{\mathcal{U}}_1(f)$) for each n. (Recall that $x_k \in \mathcal{U}_1(f)$.) Since $I_{k,n} \in \mathcal{I}$, $f(\inf I_{k,n})f(\sup I_{k,n}) < 0$. Using the continuity of $f \restriction \widetilde{\mathcal{U}}_1(f)$ we conclude that

$$(f(x_k))^2 = \lim_{n\to\infty} f(\inf I_{k,n}) \cdot \lim_{n\to\infty} f(\sup I_{k,n}) \le 0$$

and finally $f(x_k) = 0$. So by assumption, $g(x_k) = 0$ or $h(x_k) = 0$. Without loss of generality we may assume that the first case holds.

If $x_k \notin \mathcal{U}_0(g)$, then we are done. So, assume that $x_k \in \mathcal{U}_0(g)$. Let $m \in \mathbb{N}$ be such that for all $n > m$, we have $\operatorname{cl} I_{k,n} \subset \mathcal{U}_0(g)$; then since $\operatorname{cl} I_{k,n} \subset [f \neq 0] \subset [g \neq 0]$, we obtain $g(\inf I_{k,n})g(\sup I_{k,n}) > 0$.

For all $n > m$, we have $\operatorname{bd} I_{k,n} \subset \mathbb{R} \setminus \mathcal{U}_0(f) \subset \widetilde{\mathcal{U}}_1(g) \cup \widetilde{\mathcal{U}}_1(h)$. Since $h \in \mathcal{S}_1$, the set $\widetilde{\mathcal{U}}_1(h)$ is closed. Hence $x_k = \lim_{n\to\infty} \inf I_{k,n} \in \widetilde{\mathcal{U}}_1(h)$. Using the continuity of $h\restriction\widetilde{\mathcal{U}}_1(h)$ and the definition of $\mathcal{I}$, we conclude that

$$(h(x_k))^2 = \lim_{n\to\infty} \big(h(\inf I_{k,n})h(\sup I_{k,n})\big) = \lim_{n\to\infty} \frac{f(\inf I_{k,n})f(\sup I_{k,n})}{g(\inf I_{k,n})g(\sup I_{k,n})} \le 0,$$

whence $h(x_k) = 0$. $\lhd$

By Claim 4.1, we may assume that there exists a subsequence (x_{k_s}) such that $x_{k_s} \in [g = 0]\setminus\mathcal{U}_0(g)$ for each s. However this implies that $x \in \widetilde{\mathcal{U}}_1(g)$ (since $\widetilde{\mathcal{U}}_1(g)$ is closed). Hence $g(x) = \lim_{s\to\infty} g(x_{k_s}) = 0$ and $f(x) = g(x)h(x) = 0$, a contradiction. This proves (2.1).

Now we will prove that (2.2) holds.

Fix an $\varepsilon > 0$. Since $x \notin \mathcal{U}_1(f)$ and $g, h \in \mathcal{S}_1$, we have $x \notin \mathcal{U}_0(g) \cup \mathcal{U}_0(h)$. Choose a $\delta > 0$ such that

$$\big(\forall t \in [x-\delta, x+\delta] \setminus \mathcal{U}_0(g)\big)\ |g(t) - g(x)| < \min\Big\{\frac{\varepsilon}{3|h(x)|+1}, |g(x)|\Big\}, \tag{4.1}$$

$$\big(\forall t \in [x-\delta, x+\delta] \setminus \mathcal{U}_0(h)\big)\ |h(t) - h(x)| < \min\Big\{\frac{\varepsilon}{3|g(x)|+1}, |h(x)|\Big\}. \tag{4.2}$$

(We can find such a δ since $g\restriction\mathbb{R}\setminus\mathcal{U}_0(g)$ and $h\restriction\mathbb{R}\setminus\mathcal{U}_0(h)$ are continuous.) Take $I = (a, b) \in \mathcal{I}$ such that $I \subset [f \cdot f(x) > 0] \cap (x - \delta, x + \delta)$. Assume that

$$f(a)f(x) < 0 < f(b)f(x). \tag{4.3}$$

(The other case is analogous.) Since $f(a)f(x) = g(a)g(x)h(a)h(x) < 0$, we may assume that $g(a)g(x) < 0$ and $h(a)h(x) > 0$. By (4.1), we have $a \in \mathcal{U}_0(g)$.

Let C be the connected component of $\mathcal{U}_0(g)$ to which a belongs, and let $c \overset{\mathrm{df}}{=} \sup(C \cap [a, b])$. Clearly $c > a$. Notice that since $[a, b] \subset [f \ne 0] \subset [g \ne 0]$, we have $[a, c) \subset [g \cdot g(x) < 0]$.

Claim 4.2. $c \notin \mathcal{U}_0(g) \cup \mathcal{U}_0(h)$.

First assume toward a contradiction that $c \in \mathcal{U}_0(g)$. Then $c = b$. Hence $b \notin \mathcal{U}_0(h)$ (otherwise $b \in \mathcal{U}_0(f)$). Since $f(b) \ne 0$, we have $g(b)g(x) < 0$ and by (4.3),

$$h(b)h(x) = \frac{f(b)f(x)}{g(b)g(x)} < 0,$$

contrary to (4.2). So, $c \notin \mathcal{U}_0(g)$ and by (4.1), $g(c)g(x) > 0$.

Since $(a,c) \subset [f \cdot f(x) > 0] \cap [g \cdot g(x) < 0]$, we obtain $(a,c) \subset [h \cdot h(x) < 0]$. Notice that $f(c)f(x) > 0$: either by assumption (if $c < b$) or by (4.3) (if $c = b$). So,

$$h(c)h(x) = \frac{f(c)f(x)}{g(c)g(x)} > 0,$$

whence $c \notin \mathcal{U}_0(h)$. ◁

By Claim 4.2, (4.1), and (4.2), we finally conclude that

$$\begin{aligned}\varrho(f(x), f[\operatorname{cl} I]) &\leq |f(x) - f(c)| = |g(x)h(x) - g(c)h(c)| \\ &\leq |g(x)||h(x) - h(c)| + |h(c)||g(x) - g(c)| \\ &< |g(x)| \frac{\varepsilon}{3|g(x)| + 1} + 2|h(x)| \frac{\varepsilon}{3|h(x)| + 1} < \varepsilon.\end{aligned}$$

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