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ON CONTINUITY AND GENERALIZED CONTINUITY WITH RESPECT TO TWO TOPOLOGIES

Abstract

Let τ_1 and τ_2 be topologies in X and let $\tau = \tau_1 \cap \tau_2$. Some conditions concerning the topologies τ, τ_1 and τ_2 and describing the relations between the τ-continuity (quasicontinuity) [cliquishness] and the τ_i-continuity (quasicontinuity) [cliquishness], $i = 1, 2$, of functions defined on X are considered.

Let $\mathcal{R}$ denote the set of all reals and let (X, τ) be a topological space.

A function $f : X \mapsto \mathcal{R}$ is called τ-quasicontinuous (τ-cliquish) at a point $x \in X$ ([3] if for every positive real η and for every set $U \in \tau$ containing x there is a nonempty set $V \in \tau$ such that $V \subset U$ and $|f(v) - f(x)| < \eta$ for all points $v \in V$ ($\operatorname{osc}_V f < \eta$, where $\operatorname{osc}_V f$ denotes the diameter of the set $f(V)$).

We will consider two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ in X. Let

$$\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2.$$

1 $\mathcal{T}$- and $\mathcal{T}_i$-continuity

Obviously, if a function $f : X \mapsto \mathcal{R}$ is $\mathcal{T}$-continuous at a point $x \in X$ then it is also $\mathcal{T}_i$-continuous at x for $i = 1, 2$. The converse implication need not be valid.

Example 1. Let

$$U = \bigcup_n \left[\frac{1}{4n+1}, \frac{1}{4n}\right] \cup \{0\}$$

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and

$$V = \bigcup_n \left[\frac{1}{4n+3}, \frac{1}{4n+2}\right] \cup \{0\}.$$

Denote by $\mathcal{T}_1$ the topology of subsets of $X = \mathcal{R}$ generated by the family

$$\{(-r, r) \cap U; r > 0\}$$

and by $\mathcal{T}_2$ the topology of subsets of X generated by the family

$$\{(-r, r) \cap V : r > 0\}.$$

For $n = 1, 2, \ldots$, let

$$I_n = \left[\frac{1}{4n+2}, \frac{1}{4n+1}\right]$$

and let $f_n : I_n \mapsto [0,1]$, be a continuous function (with respect to the Euclidean topology $\mathcal{T}_e$) such that

$$f_n(I_n) = [0,1] \quad \wedge \quad f_n\Big(\frac{1}{4n+2}\Big) = f_n\Big(\frac{1}{4n+1}\Big) = 0.$$

Put

$$f(x) = \begin{cases} f_n(x) & \text{for} \quad x \in I_n, \quad n = 1, 2, \ldots \\ 0 & \text{for} \quad x \in \mathcal{R} \setminus \bigcup_n I_n \end{cases}$$

and observe that the function f is $\mathcal{T}_i$-continuous at 0 for $i = 1, 2$. Since

$$\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2 = \{\mathcal{R}, \emptyset\},$$

the function f is not even $\mathcal{T}$-cliquish at 0.

Theorem 1. *Let $x \in X$ be a point. Suppose that the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy to the following condition*

$$\forall_{x \in A \in \mathcal{T}_1} \forall_{x \in B \in \mathcal{T}_2} \exists_{x \in E \in \mathcal{T}} E \subset A \cup B. \tag{1}$$

If a function $f : X \mapsto \mathcal{R}$ is $\mathcal{T}_i$-continuous at the point x for $i = 1, 2$ then f is also $\mathcal{T}$-continuous at x.

PROOF. Fix a positive real η. From $\mathcal{T}_i$-continuity of the function f at x, $i = 1, 2$, follows that there are sets $A_i \in \mathcal{T}_i$ such that $x \in A_i$ and for each point $t \in A_i$ the inequality

$$|f(t) - f(x)| < \eta$$

holds. By the condition (1) there is a set $E \in \mathcal{T}$ with

$$x \in E \subset A_1 \cup A_2.$$

Evidently,

$$|f(t) - f(x)| < \eta$$

for all points $t \in E$ and the proof is completed. □

We will show the necessity of condition (1) for Theorem 1 to hold.

Remark 1. *Let $x \in X$ be a point. If there are sets $A_i \in \mathcal{T}_i$, $i = 1, 2$, such that $x \in A_1 \cap A_2$ and*

$$\forall_{x \in A \in \mathcal{T}} A \setminus (A_1 \cup A_2) \neq \emptyset,$$

then there is a function $f : X \mapsto \mathcal{R}$ which is $\mathcal{T}_i$-continuous at x for $i = 1, 2$ and which is not $\mathcal{T}$-continuous at x.

Proof. Put

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in A_1 \cup A_2 \\ 1 & \text{for} \quad x \in X \setminus (A_1 \cup A_2) \end{cases}$$

and observe that f is $\mathcal{T}_i$-continuous at x for $i = 1, 2$. Since

$$\forall_{\emptyset \neq A \in \mathcal{T}} \operatorname{osc}_{A \cup \{x\}} f = 1,$$

the function f is not even $\mathcal{T}$-quasicontinuous at x. This completes the proof. □

2 Quasicontinuity

Example 2. Let $X = \mathcal{R}$ and let the sets U, V be the same as these from Example 1. Denote by $\mathcal{T}_1$ and $\mathcal{T}_2$ the topologies generated by family

$$\{U \cap (0, r); r > 0\}$$

and respectively by the family

$$\{V \cap (0, r); r > 0\}.$$

Let the function f be the same as that from Example 1. Then the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy to the condition (1) from Theorem 1, the function f is $\mathcal{T}_i$-quasicontinuous at the point 0 (even everywhere) for $i = 1, 2$, but f is not $\mathcal{T}$-quasicontinuos at 0 (it is not even $\mathcal{T}$-cliquish at 0).

So, Theorem 1 is not true for quasicontinuity.

For the case of quasicontinuity we recall the following notion:
A set $A \subset X$ is said τ-semiopen if

$$A \subset \mathrm{cl}_{\tau}(\mathrm{int}_{\tau}(A)),$$

where cl_{τ} and int_{τ} respectively denote the operations of the closure and the interior with respect to the topology τ.

Let $S(\tau)$ be the family of all τ-semiopen sets $A \subset X$.

It is well known ([3]) that a function $f : X \mapsto \mathcal{R}$ is τ-quasicontinuous at a point x if and only if for every positive real η there is a τ-semiopen set $A \subset X$ containing x such that for each point $t \in A$ the inequality

$$|f(t) - f(x)| < \eta$$

is true.

Theorem 2. *Let $x \in X$ be a point. Suppose that the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy to the following condition*

$$\forall_{x \in A \in S(\mathcal{T}_1)} \forall_{x \in B \in S(\mathcal{T}_2)} \exists_{\emptyset \neq E \in \mathcal{T}} E \subset A \cup B. \tag{2}$$

If a function $f : X \mapsto \mathcal{R}$ is $\mathcal{T}_i$-quasicontinuous at x for $i = 1, 2$, then f is also $\mathcal{T}$-quasicontinuous at x.

PROOF. Fix a positive real η and a set $A \in \mathcal{T}$ containing x. By the $\mathcal{T}_i$-quasicontinuity of the function f at x for $i = 1, 2$, there are nonempty sets

$$A_i \in S(\mathcal{T}_i), \quad i = 1, 2,$$

containing x and such that for all points $t \in A_i$, $i = 1, 2$, the inequality

$$|f(t) - f(x)| < \eta$$

holds. Observe that

$$A \cap A_i \in S(\mathcal{T}_i)$$

for $i = 1, 2$. By the condition (2) there is a nonempty $\mathcal{T}$-open set

$$B \subset A \cap (A_1 \cup A_2).$$

Evidently,

$$\forall_{t \in B} |f(t) - f(x)| < \eta.$$

So, the proof is completed. $\square$

Remark 2. *Let $x \in X$ be a point. Suppose that there are nonempty sets $A_i \in S(\mathcal{T}_i)$, $i = 1, 2$, containing x and such that*

$$\forall_{\emptyset \neq A \in \mathcal{T}} A \setminus (A_1 \cup A_2) \neq \emptyset.$$

Then there is a function $f : X \mapsto \mathcal{R}$ which is $\mathcal{T}_i$-quasicontinuous at the point x for $i = 1, 2$ and which is not $\mathcal{T}$-quasicontinuous at the point x.

Proof. The construction of such a function is the same as the construction of the function f in the proof of Remark 1. □

Observe that the case of quasicontinuity is different from the case of continuity. If a function f is $\mathcal{T}$-continuous at a point x then it is also $\mathcal{T}_i$-continuous at x for $i = 1, 2$. For quasicontinuity the situation is different.

Example 3. Let $\mathcal{T}_1 = \mathcal{T}_e^+$ and respectively $\mathcal{T}_2 = \mathcal{T}_e^-$ be the topologies generated by the families

$$\{[x, x + r); x \in \mathcal{R},\ \ r > 0\}$$

and

$$\{(x - r, x]; x \in \mathcal{R},\ \ r > 0\}.$$

Then

$$\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2 = \mathcal{T}_e$$

and the function

$$f(t) = \begin{cases} 0 & \text{for} \quad t \geq 0 \\ 1 & \text{for} \quad t < 0 \end{cases}$$

is $\mathcal{T}$-quasicontinuous at 0 and is not $\mathcal{T}_2$-quasicontinuous at 0. Analogously, the function

$$g(t) = f(-t), \ \ t \in \mathcal{R},$$

is $\mathcal{T}$-quasicontinuous at 0 and is not $\mathcal{T}_1$-quasicontinuous at 0.

Theorem 3. *Let $x \in X$ be a point. Then the following conditions are equivalent:*

$(2')$ *every function $f : X \mapsto \mathcal{R}$ which is $\mathcal{T}$-quasicontinuous at x is also $\mathcal{T}_i$-quasicontinuous at x for $i = 1, 2$;*

$(2'')$ *every $\mathcal{T}$-semiopen set containing x contains also $\mathcal{T}_i$-semiopen sets U_i, $i = 1, 2$, containing x.*

PROOF. $(2') \Rightarrow (2'')$. If $(2'')$ does not hold then there are a $\mathcal{T}$-semiopen set U containing x and an index $i \leq 2$ such that for every $\mathcal{T}_i$-semiopen set V containing x the relation

$$V \setminus U \neq \emptyset$$

holds. Let

$$f(t) = \begin{cases} 0 & \text{for} \quad t \in U \\ 1 & \text{for} \quad t \in X \setminus U. \end{cases}$$

Then f is $\mathcal{T}$-quasicontinuous at x, but it is not $\mathcal{T}_i$-quasicontinuous at x. So, we obtain a contradiction with $(2')$.

$(2'') \Rightarrow (2')$. Suppose that a function $f : X \mapsto \mathcal{R}$ is $\mathcal{T}$-quasicontinuous at x. Fix a positive real η and an index $i \leq 2$. From the $\mathcal{T}$-quasicontinuity of f at x follows the existence of an $\mathcal{T}$-semiopen set U containing x and such that

$$|f(t) - f(x)| < \eta$$

for each point $t \in U$. By $(2'')$ there is a $\mathcal{T}_i$-semiopen set $V \subset U$ containing x. Since

$$|f(t) - f(x)| < \eta$$

for all points $t \in V$, the function f is $\mathcal{T}_i$-quasicontinuous at x and the proof is completed. $\square$

3 Cliquishness

Remark 3. *Assume all hypotheses of Remark 2. Moreover suppose that for each nonempty set $A \in \mathcal{T}$ there are sets B, D which are dense (with respect to the topology $\mathcal{T}$) in A and such that*

$$B \cap D = \emptyset \quad \wedge \quad B \cup D = A.$$

Then there is a function $f : X \mapsto \mathcal{R}$ which is $\mathcal{T}_i$-quasicontinuous at the point x for $i = 1, 2$ and which is not $\mathcal{T}$-cliquish at x.

PROOF. If

$$E = \operatorname{int}_{\mathcal{T}}(X \setminus (A_1 \cup A_2)) \neq \emptyset$$

then let

$$E = B \cup D,$$

where the sets B, D are disjoint and dense (with respect to the topology $\mathcal{T}$) in E. If $E = \emptyset$ we put

$$B = X \setminus (A_1 \cup A_2).$$

Define

$$f(x) = \begin{cases} 1 & \text{for} \quad x \in B \\ 0 & \text{for} \quad x \in X \setminus B. \end{cases}$$

Then f is $\mathcal{T}_i$-quasicontinuous at x for $i = 1, 2$, but f is not $\mathcal{T}$-cliquish at x. □

The following example shows that all hypotheses of Remark 3 are essential.

Example 4. Let

$$X = \{0, 1, 2, 3\},$$
$$\mathcal{T}_1 = \{\emptyset, \{0, 1\}, \{3\}, \{0, 1, 3\}, X\}, \text{ and } \mathcal{T}_2 = \{\emptyset, \{0, 2\}, \{3\}, \{0, 2, 3\}, X\}.$$

Then

$$\mathcal{T} = \{\emptyset, \{3\}, X\}.$$

Let $f : X \mapsto \mathcal{R}$ be a function which is $\mathcal{T}_i$-quasicontinuous at the point 0 for $i = 1, 2$. Then the function f is also $\mathcal{T}_i$-continuous at 0 for $i = 1, 2$ and

$$f(0) = f(1) = f(2).$$

If $0 \in A \in \mathcal{T}$, then

$$A = X \quad \wedge \quad \{3\} \subset A$$

and $\operatorname{osc}_{\{3\}} f = 0$. So f is $\mathcal{T}$-cliquish at the point 0.

Moreover,

$$0 \in U = \{0, 1\} \in \mathcal{T}_1,$$
$$0 \in W = \{0, 2\} \in \mathcal{T}_2, \text{ and } \forall_{\emptyset \neq A \in \mathcal{T}} A \setminus (U \cup W) \neq \emptyset.$$

Theorem 4. *Let $x \in X$ be a point. Suppose that*

(3) *there is a set $A \in \mathcal{T}$ containing x and such that for all nonempty sets $A_i \in \mathcal{T}_i$, $i = 1, 2$, contained in A there exists a nonempty set $D \in \mathcal{T}$ contained in $A_1 \cup A_2$ and for all nonempty disjoint sets $B_i \in \mathcal{T}_i$, $i = 1, 2$, there are an index $i \leq 2$ and a nonempty set $E \in \mathcal{T}$ with*

$$E \subset B_1 \quad \vee \quad E \subset B_2.$$

If a function $f : X \mapsto \mathcal{R}$ is $\mathcal{T}_i$-cliquish at the point x for $i = 1, 2$, then f is also $\mathcal{T}$-cliquish at x.

PROOF. Fix a positive real η and a set U with

$$x \in U \in \mathcal{T}.$$

Let a set A satisfies to the condition (3). The $\mathcal{T}_i$-cliquishness of f at x implies the existence of a nonempty sets $B_i \subset U \cap A$, $i = 1, 2$, with

$$\operatorname{osc}_{B_i} f < \frac{\eta}{2} \ \wedge \ B_i \in \mathcal{T}_i, \ \ i = 1, 2.$$

Observe that

$$B_1 \cap B_2 \neq \emptyset \Rightarrow \operatorname{osc}_{B_1 \cup B_2} f < \eta.$$

By (3) there is a nonempty set $E \in \mathcal{T}$ such that:

$$B_1 \cap B_2 = \emptyset \Rightarrow \exists_{i \leq 2} E \subset B_i$$

or

$$B_1 \cap B_2 \neq \emptyset \Rightarrow E \subset B_1 \cup B_2.$$

Since in both cases $\operatorname{osc}_E f < \eta$, the proof is completed. □

Remark 4. *Let $x \in X$ be a point. Suppose that the topologies $\mathcal{T}$, $\mathcal{T}_1$ and $\mathcal{T}_2$ are such that for each point $t \neq x$ there are sets $U_1 \in \mathcal{T}_1$, $U_2 \in \mathcal{T}_2$ and $U_3 \in \mathcal{T}$ for which $U_1 \cap U_3 = \emptyset$, $U_2 \cap U_3 = \emptyset$, $x \in U_3$ and $t \in U_1 \cap U_2$. Moreover we assume that there is a countable base of neighborhoods of x for the topology $\mathcal{T}$ and that for which set $A \in \mathcal{T}$ containing $x \in X$ there are nonempty sets $A_i(A) \in \mathcal{T}_i$, $i = 1, 2$, such that:*

$$A_1(A) \cap A_2(A) \neq \emptyset \Rightarrow \forall_{\emptyset \neq B \in \mathcal{T}} B \setminus (A_1(A) \cup A_2(A)) \neq \emptyset,$$

$$A_1(A) \cap A_2(A) = \emptyset \Rightarrow \forall_{\emptyset \neq B \in \mathcal{T}} \forall_{i \leq 2} B \setminus A_i(A) \neq \emptyset,$$

and

$$G = X \setminus \bigcup \{A_1(A) \cup A_2(A); x \in A \in \mathcal{T}\} \setminus \{x\} = H \cup K,$$

where $H \cap K = \emptyset$ and H, K are dense in G with respect to the topology $\mathcal{T}$. Then there is a function $f : X \mapsto \mathcal{R}$ which is $\mathcal{T}_i$-cliquish at the point x for $i = 1, 2$ and which is not $\mathcal{T}$-cliquish at x.

PROOF. At the beginning we assume that $\{x\}$ is not in $\mathcal{T}_i$ for $i = 1, 2$. Let

$$W_1, \ldots, W_n, \ldots$$

be an enumeration of all elements of some basis of neighborhoods of x in the topology $\mathcal{T}$. Let

$$A_i(W_1) \in \mathcal{T}_i, \ \ i = 1, 2,$$

be nonempty sets satisfying our hypothesis for $A = W_1$. Since in this case there are points $t_i \in A_i(W_1)$, $i = 1, 2$, we can find nonempty sets $V_{1,i} \in \mathcal{T}_i$, $i \leq 2$, and $V_{1,3} \in \mathcal{T}$ with $t_i \in V_{1,i}$, $i \leq 2$, $x \in V_{1,3}$, and

$$V_{1,3} \cap (V_{1,1} \cup V_{1,2}) = \emptyset.$$

If $V_{1,1} \cap V_{1,2} = \emptyset$ we put

$$f(t) = (-1)^i, \ \ t \in V_{1,i}, \ \ i \leq 2.$$

In the contrary case, where $V_{1,1} \cap V_{1,2} \neq \emptyset$, we put

$$f(t) = 1, \ \ t \in V_{1,1} \cup V_{1,2}.$$

Analogously, for $n > 1$ we find nonempty sets $V_{n,i} \in \mathcal{T}_i$, $i \leq 2$, and $V_{n,3} \in \mathcal{T}$ such that:

$$x \in V_{n,3} \subset X \setminus (V_{n,1} \cup V_{n,2}),$$

$$V_{n,1} \cup V_{n,2} \subset W_n \cap V_{1,3} \cap \ldots \cap V_{n-1,3},$$

$$V_{n,1} \cap V_{n,2} \neq \emptyset \Rightarrow \forall_{\emptyset \neq B \in \mathcal{T}} B \setminus (V_{n,1} \cup V_{n,2}) \neq \emptyset,$$

and

$$V_{n,1} \cap V_{n,2} = \emptyset \Rightarrow \forall_{\emptyset \neq B \in \mathcal{T}} \forall_{i \leq 2} B \setminus V_{n,i} \neq \emptyset.$$

If $V_{n,1} \cap V_{n,2} = \emptyset$ then we put

$$f(t) = (-1)^i n, \ \ t \in V_{n,i}, \ \ i \leq 2.$$

In the contrary case, where $V_{n,1} \cap V_{n,2} \neq \emptyset$ we put

$$f(t) = n, \ \ t \in V_{n,1} \cup V_{n,2}.$$

Moreover, let

$$f(x) = 0,$$

$$f(t) = 0, \ \ t \in H, \text{ and } f(t) = \frac{1}{2}$$

at all other points of the space X.

Observe that the function f satisfies all requirements.

Evidently,

$$\neg(\{x\} \in \mathcal{T}_1 \cap \mathcal{T}_2).$$

In the case, where $\{x\} \in \mathcal{T}_i$ for $i = 1$ or $i = 2$, the construction of such a function f is simpler, since starting from some index n we find only one set V_{k,i_0} instead a pair $(V_{k,1}, V_{k,2})$. □

Theorem 5. *Let $x \in X$ be a point. Suppose that*

(a) all sets A with

$$x \in A \in \mathcal{T}_1 \cup \mathcal{T}_2$$

contains some sets $B \in \mathcal{T}$ with $x \in B$.

Then every function $f : X \mapsto \mathcal{R}$ which is $\mathcal{T}$-cliquish at x is also $\mathcal{T}_i$-cliquish at x for $i = 1, 2$.

Proof. Fix an index $i \leq 2$, a positive real η and a set $U \in \mathcal{T}_i$ with $x \in U$. By our hypothesis there is a set $B \in \mathcal{T}$ such that

$$x \in B \subset U.$$

Since f is $\mathcal{T}$-cliquish at x, there is a nonempty set $V \in \mathcal{T}$ such that

$$V \subset B \quad \wedge \quad \operatorname{osc}_V f < \eta.$$

So,

$$V \subset U \quad \wedge \quad V \in \mathcal{T}_i \quad \wedge \quad \operatorname{osc}_V f < \eta,$$

and the proof is completed. $\square$

The next example shows that hypothesis (a) from the last theorem is not necessary.

Example 5. Let

$$X = \{0, 1, 2, 3\},$$
$$\mathcal{T}_1 = \{\emptyset, X, \{1\}\}, \text{ and } \mathcal{T}_2 = \{\emptyset, X, \{2\}\}.$$

Then

$$\mathcal{T} = \{\emptyset, X\},$$

every function $f : X \mapsto \mathcal{R}$ which is $\mathcal{T}$-cliquish at x is constant and the condition (a) is not satisfied.

The proof of the next theorem gives a characterization of the cliquishness.

Remark 5. *Let τ be a topology of subsets of X and let $x \in X$ be a point. A function $f : X \mapsto \mathcal{R}$ is τ-cliquish at x if and only if for each positive real η there are τ-open sets A_s, $s \in S$, where S is a set of indexes, such that x belongs to the τ-closure $\operatorname{cl}_\tau(\bigcup_{s \in S} A_s)$ of the set $\bigcup_{s \in S} A_s$ and for every $s \in S$ the inequality $\operatorname{osc}_{A_s} f < \eta$ is true.*

Proof. Fix a positive real η. If f is τ-cliquish at x then for every set $U \in \tau$ containing x there is a nonempty τ-open set $B(U) \subset U$ with $\operatorname{osc}_{B(U)} f < \eta$. If

$$\{U_s; s \in S\}$$

is a directed family of all τ-open sets contained the point x, then the family

$$\{A_s = B(U_s); s \in S\}$$

satisfies all the required conditions.

The proof of the second implication is evident. □

Theorem 6. *Let $x \in X$ be a point. Suppose that the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the following condition*

(3′) *for each $\mathcal{T}$-open set A such that $x \in \operatorname{cl}_{\mathcal{T}}(A)$ and for each index $i \leq 2$ the point $x \in \operatorname{cl}_{\mathcal{T}_i}(A)$.*

If a function $f : X \mapsto \mathcal{R}$ is $\mathcal{T}$-cliquish at x then it is also $\mathcal{T}_i$-cliquish at x for $i = 1, 2$.

Proof. Fix a positive integer $i \leq 2$ and a positive real η. Since f is $\mathcal{T}$-cliquish at x, by the last Remark there are nonempty sets $A_s \in \mathcal{T}$, where S is a set of indexes, such that

$$x \in \operatorname{cl}_{\mathcal{T}}(\bigcup_{s \in S} A_s)$$

and for every $s \in S$ the inequality $\operatorname{osc}_{A_s} f < \eta$ is true.

From (3′) follows the existence of a $\mathcal{T}_i$-open set $D \subset \bigcup_{s \in S} A_s$ with

$$x \in \operatorname{cl}_{\mathcal{T}_i}(D).$$

For $s \in S$ the sets

$$D_s = D \cap A_s \in \mathcal{T}_i.$$

Let

$$S' = \{s \in S; D_s \neq \emptyset\}.$$

Since for $s \in S'$ the inequality

$$\operatorname{osc}_{D_s} f < \eta$$

is true and since

$$x \in \operatorname{cl}_{\mathcal{T}_i}(\bigcup_{s \in S'} D_s),$$

by the last Remark f is $\mathcal{T}_i$-cliquish at x. So, the proof is completed. □

Remark 6. *Let $x \in X$ be a point. Suppose that there are an index $i \leq 2$, a set $V \in \mathcal{T}_i$, disjoint sets $Y, Z \subset V$ and a set $U \in \mathcal{T}$ such that*

$$U \cap V = \emptyset \quad \wedge \quad x \in \operatorname{cl}_{\mathcal{T}}(U) \cap V,$$

and

$$x \in Y \quad \wedge \quad \operatorname{cl}_{\mathcal{T}_i}(Y) = \operatorname{cl}_{\mathcal{T}_i}(Z) = \operatorname{cl}_{\mathcal{T}_i}(V).$$

Then there is a function $f : X \mapsto \mathcal{R}$ which is $\mathcal{T}$-cliquish at x and which is not $\mathcal{T}_i$-cliquish at x.

PROOF. The function

$$f(t) = \begin{cases} 1 & \text{if} \quad t \in Z \\ 0 & \text{if} \quad t \in X \setminus Z \end{cases}$$

is $\mathcal{T}$-cliquish at x, but it is not $\mathcal{T}_i$-cliquish at x. So, the proof is completed. $\square$

The following example shows the importance of all hypothesis of the last remark.

Example 6. Let

$$X = \{0, 1, 2\},$$
$$\mathcal{T}_1 = \{\emptyset, X, \{1\}\}, \text{ and } \mathcal{T}_2 = 2^X.$$

Then $\mathcal{T} = \mathcal{T}_1$ and for

$$x = 0 \quad \wedge \quad A = \{1\} \quad \wedge \quad B = \{0\}$$

we have

$$A \in \mathcal{T} \quad \wedge \quad B \in \mathcal{T}_2 \quad \wedge \quad A \cap B = \emptyset$$

and

$$x \in B \quad \wedge \quad x \in \operatorname{cl}_{\mathcal{T}}(A).$$

Moreover, each function $f : X \mapsto \mathcal{R}$ is $\mathcal{T}_2$-continuous at the point 0, because $\{0\} \in \mathcal{T}_2$. So, every function $f : X \mapsto \mathcal{R}$ which is $\mathcal{T}$-cliquish at 0 is also $\mathcal{T}_2$-cliquish at 0.

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